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# ON REPRESENTABILITY OF QUASI QUADRATIC FUNCTIONALS BY SESQUILINEAR FUNCTIONALS

MEHMET AÇIKGÖZ AND ALIAKBAR GOSHABULAGHI

**ABSTRACT.** In this paper, the representability of quasi-quadratic functionals by sesquilinear ones over real and complex  $*$ -Banach algebras will be determined. Moreover, we will give some relative results for modules over general  $*$ -rings.

## 1. INTRODUCTION

Let  $R$  be a  $*$ -ring with identity such that 2 is a unit in  $R$  and  $M$  be left  $R$ -module. The mapping  $Q : M \rightarrow R$  is said to be quasi quadratic functional if for any  $x, y \in M$  and  $\alpha \in R$  the parallelogram law

$$(1.1) \quad Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

and the homogeneity equation

$$(1.2) \quad Q(ax) = aQ(x)a^*$$

holds. A biadditive mapping  $S : M \times M \rightarrow R$  satisfying

$$(1.3) \quad S(a_1x_1 + a_2x_2, y) = a_1S(x_1, y) + a_2S(x_2, y) \quad (a_1, a_2 \in R, x_1, x_2, y \in M),$$

$$(1.4) \quad S(x, a_1y_1 + a_2y_2) = S(x, y_1)a_1^* + S(x, y_2)a_2^* \quad (a_1, a_2 \in R, x, y_1, y_2 \in M)$$

is a sesquilinear functional.

Over  $*$ -algebra  $A$  an element with the property  $h^* = h$  is said to be hermitian. An element  $a \in A$  will be called normal if  $a^*a = aa^*$ . It is seen that over a complex  $*$ -algebra  $A$  each element  $a \in A$  has a unique decomposition  $a = h + ik$  with hermitian  $h$  and  $k$ . A  $*$ -algebra being a Banach algebra is said to be Banach  $*$ -algebra. A Banach  $*$ -algebra is called hermitian if each hermitian element has real spectrum. If  $A$  be a hermitian Banach  $*$ -algebra and  $h \in A$  be a hermitian element then it is convenient to write  $h > 0$  if the spectrum of  $h$  be positive.

Now let  $M$  be a left  $R$ -module over a  $*$ -ring  $R$  and  $Q : M \rightarrow R$  be a quasi quadratic functional. It is interesting to know that is there a sesquilinear functional  $S$  such that  $Q(x) = S(x, x)$  for any  $x \in M$ ?

In 1984 Vukman [10] posed the problem of representability of quasi quadratic functionals by sesquilinear ones over complex  $*$ -algebras. The complete solution was given in [2]. In our present work [1] we gave a solution to a general case, modules over special  $*$ -rings including complex  $*$ -algebras. However here we will give a solution of the problem of representability of some quasi quadratic functionals

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with sesquilinear functionals for ones over modules on general  $*$ -rings including commutative  $*$ -rings with trivial involution.

## 2. QUASI-QUADRATIC FUNCTIONALS OVER REAL BANACH $*$ -ALGEBRAS

**Lemma 1.** *Let  $X$  be a real vector space and  $Q : X \rightarrow R$  be additive functional. Then the functional*

$$m(x, y) = \frac{1}{4} [Q(x + y) - Q(x - y)]$$

*is symmetric and biadditive.*

*Proof.* See [10]. □

**Theorem 1.** *Let  $A$  be a commutative Banach algebra and  $X$  be a left  $A$ -module. Moreover  $Q : X \rightarrow A$  be a mapping such that each invertible  $a \in A$  and  $x, y \in X$  satisfies:*

- 1)  $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$
  - 2)  $Q(ax) = a^2Q(x)$
- then for the functional  $S : X \times X \rightarrow A$  defined by*

$$S(x, y) = \frac{1}{4} [Q(x + y) - Q(x - y)]$$

*the following conditions hold.*

- 1)  $S$  is biadditive;
- 2) for any  $x, y \in X$ ,  $S(x, x) = Q(x)$ ;
- 3) for any  $x, y \in X$  and  $a \in A$ ,

$$S(ax, y) + S(x, ay) = 2aS(x, y).$$

*Proof.* (1) and (2) are clear. To prove (3), let  $x, y \in X$  be fixed and the mapping  $f : A \rightarrow A$  be defined in the way

$$f(a) = S(ax, y) + S(x, ay).$$

Since  $S$  is biadditive so  $f$  is additive. It is seen that for any invertible  $a \in A$ ,  $f(a) = a^2f(a^{-1})$ . For any  $a \in A$  the identity  $f(a) = af(e)$  holds (See [7]). So for any  $a \in A$  and  $x, y \in X$ ;

$$\begin{aligned} S(ax, y) + S(x, ay) &= a[S(ex, y) + S(x, ey)] \\ &= a[S(x, y) + S(x, y)] \\ &= 2aS(x, y). \end{aligned}$$

□

**Theorem 2.** *Let  $A$  be a commutative Banach algebra with identity and  $X$  be a unitary left  $A$ -module and let  $Q : X \rightarrow A$  be a such mapping that for any invertible  $a \in A$  and  $x, y \in X$ ,*

- i)  $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$ ;
- ii)  $Q(ax) = a^2Q(x)$ ;

*iii) Let  $Q : X \rightarrow A$  be a functional such that  $a \rightarrow Q(ax + y)$  be continuous for any fixed  $x, y \in X$ .*

## ON QUASI QUADRATIC FUNCTIONALS BY SESQUILINEAR FUNCTIONALS

Then then the functional  $S : X \times X \rightarrow A$  defined by

$$S(x, y) = \frac{1}{4} [Q(x + y) - Q(x - y)]$$

is a bilinear form and for any  $x \in X$ ,  $S(x, x) = Q(x)$ .

*Proof.* From Theorem 1, it suffices to prove

$$S(ax, y) = aS(x, y), \quad S(x, ay) = aS(x, y).$$

For fixed  $x, y \in X$ , define  $f : A \rightarrow A$  such as

$$f(a) = S(ax, y) - S(x, ay),$$

$f$  is additive since  $S$  is biadditive. From (iii)  $f$  is continuous and for any invertible  $a \in A$ ,  $f(a) = -a^2 f(a^{-1})$  holds hence from [10], Corollary 1 (3), for any  $a \in A$ ,  $f(a) = 0$  (See [6], [7]), that yields;

$$S(ax, y) = S(x, ay),$$

and from Theorem 1, (3), the proof completes.  $\square$

**Theorem 3.** Let  $X$  be a vector space over a complex  $*$ -algebra  $A$  having an identity element 1. For any quasi-quadratic functional  $Q : X \rightarrow A$  define  $S : X \times X \rightarrow A$ ;

$$S(x, y) = \frac{1}{4} [Q(x + y) - Q(x - y)] + \frac{i}{4} [Q(x + iy) - Q(x - iy)].$$

The following conditions hold.

- (a)  $S$  is biadditive;
- (b) For  $x, y \in X$  and  $a \in A$ ,

$$S(ax, y) = aS(x, y), \quad S(x, ay) = S(x, y) a^*;$$

- (c) For any  $x \in X$ ,  $Q(x) = S(x, x)$ .

*Proof.* See [10].  $\square$

3. MODULES OVER  $*$ -RINGS

**Theorem 4.** Let  $R$  be a  $*$ -ring with identity and there exist  $a_0 \in R$  satisfying

$$(3.1) \quad a_0 + a_0^* = 0, \quad a_0 a_0^* = 1$$

$$(3.2) \quad a_0 a = a a_0, \quad (a \in R).$$

then for any quasi quadratic functional  $Q : M \rightarrow R$ , the functional  $S : M \times M \rightarrow R$  defined by

$$S(x, y) = \frac{1}{4} (Q(x + y) - Q(x - y)) + \frac{a_0}{4} (Q(x + a_0 y) - Q(x - a_0 y))$$

is the unique sesquilinear functional satisfying  $Q(x) = S(x, x)$  for any  $x \in M$ .

*Proof.* From, Theorem 1 of [1],  $S$  is sesquilinear and for any  $x \in M$ ,  $S(x, x) = Q(x)$ . The only fact remains to prove is uniqueness. Let  $T$  be an other sesquilinear functional satisfying the condition of theorem. Now for  $x, y \in M$ ;

$$\begin{aligned} S(x, x) + S(x, y) + S(y, x) + S(y, y) &= S(x + y, x + y) = Q(x + y) \\ &= T(x + y, x + y) \\ &= T(x, x) + T(x, y) + T(y, x) + T(y, y) \end{aligned}$$

So

$$(3.3) \quad S(x, y) + S(y, x) = T(x, y) + T(y, x)$$

Now by replacing  $x$  with  $a_0x$  we conclude

$$S(x, y) - S(y, x) = T(x, y) - T(y, x)$$

This with (3.3) yields  $S = T$ .  $\square$

**Proposition 1.** *Let  $R$  be a  $*$ -ring with identity then  $R \oplus R$  is a  $*$ -ring with the addition  $(a, b) + (c, d) = (a + c, b + d)$ , product  $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$  and involution  $(a, b)^* = (a^*, -b^*)$ .*

*Proof.* Since

$$\begin{aligned} (c, d)^* \cdot (a, b)^* &= (c^*, -d^*) \cdot (a^*, -b^*) = (c^* a^* - d^* b^*, -c^* b^* - d^* a^*) \\ &= (ac - bd, ad + bc)^* \\ &= ((a, b) \cdot (c, d))^*. \end{aligned}$$

Reminder of proof is clear.  $\square$

**Proposition 2.** *Let  $M$  be a left  $R$ -module then  $M \oplus M$  is a left module over  $R \oplus R$  with the module action defined by*

$$(a, b) \cdot (x, y) = (ax - by, ay + bx) \quad (a, b \in R, x, y \in M).$$

*Proof.* For any  $a, b, c, d \in R, x, y \in M$  we have;

$$\begin{aligned} (a, b) \cdot [(c, d) \cdot (x, y)] &= (a, b) \cdot (cx - dy, cy + dx) \\ &= (acx - ady - bcy - bdx, acy + adx + bcx - bdy) \\ &= ((ac - bd)x - (ad + bc)y, (ac - bd)y + (ad + bc)x) \\ &= (ac - bd, ad + bc) \cdot (x, y) \\ &= ((a, b)(c, d)) \cdot (x, y). \end{aligned}$$

One can easily check other module action properties.  $\square$

**Proposition 3.** *i)  $R \cong R \oplus 0 \subseteq R \oplus R$ ,*

*ii)  $(1, 0)$  is the identity of  $R \oplus R$  and  $a_0 = (0, 1)$  satisfies (3.1), (3.2).*

**Theorem 5.** *Let  $M$  be a left module over  $*$ -ring  $R$  with identity such that 2 be a unit in  $R$ . Moreover, let  $q : M \rightarrow R$  be a functional such that for any  $a, b \in R, x, y \in M$ ;*

$$(3.4) \quad q(ax - by) + q(ay + bx) = a(q(x) + q(y))a^* + b(q(x) + q(y))b^*,$$

*and*

$$(3.5) \quad aq(x)b^* = bq(x)a^*$$

*then the functional  $S : M \times M \rightarrow R$  defined by*

$$S(x, y) = \frac{1}{4} (q(x + y) - q(x - y)),$$

*is sesquilinear and  $S(x, x) = q(x)$  for any  $x \in M$ .*

**Notice 1.** By giving  $a = b = 1$  and  $a = b, y = 0$  in (3.4) we get parallelogram law and homogeneity equation, respectively.

**Notice 2.** In the case of commutative  $*$ -rings with trivial involution (3.5) holds.

**Proof of Theorem 5.** Consider  $Q : M \oplus M \rightarrow R \subseteq R \oplus R$  as  $Q((x, y)) = (q(x) + q(y), 0)$  then

$$\begin{aligned}
 & Q((x_1, y_1) + (x_2, y_2)) + Q((x_1, y_1) - (x_2, y_2)) \\
 = & Q((x_1 + x_2, y_1 + y_2)) + Q((x_1 - x_2, y_1 - y_2)) \\
 = & (q(x_1 + x_2) + q(y_1 + y_2) + q(x_1 - x_2) + q(y_1 - y_2), 0) \\
 = & (2q(x_1) + 2q(x_2) + 2q(y_1) + 2q(y_2), 0) \\
 = & 2(q(x_1) + q(y_1), 0) + 2(q(x_2) + q(y_2), 0) \\
 = & 2Q((x_1, y_1)) + 2Q((x_2, y_2)),
 \end{aligned}$$

also

$$\begin{aligned}
 (a, b) Q((x, y)) (a, b)^* &= (a, b) (q(x) + q(y), 0) (a^*, -b^*) \\
 &= (aq(x) + aq(y), bq(x) + bq(y)) (a^*, -b^*) \\
 &= (aq(x)a^* + aq(y)a^* + bq(x)b^* + bq(y)b^*, bq(x)a^* \\
 &\quad + bq(y)a^* - aq(x)b^* - aq(y)b^*) \\
 &= (q(ax - by) + q(ay + bx), 0) \\
 &= Q((ax - by, ay + bx)) \\
 &= Q((a, b) \cdot (x, y))
 \end{aligned}$$

So  $Q$  is quasi quadratic, now by theorem 4, the mapping  $S : (M \oplus M) \times (M \oplus M) \rightarrow R \oplus R$  determined by

$$\begin{aligned}
 & T((x_1, y_1), (x_2, y_2)) \\
 = & \frac{1}{4} (Q((x_1 + x_2, y_1 + y_2)) - Q((x_1 - x_2, y_1 - y_2))) \\
 & + \frac{(0, 1)}{4} (Q((x_1, y_1) + (0, 1) \cdot (x_2, y_2)) - Q((x_1, y_1) - (0, 1) \cdot (x_2, y_2))) \\
 = & \frac{1}{4} (Q((x_1 + x_2, y_1 + y_2)) - Q((x_1 - x_2, y_1 - y_2))) \\
 & + \frac{(0, 1)}{4} (Q((x_1 - y_2, y_1 + y_2)) - Q((x_1 + y_2, y_1 - x_2))) \\
 = & \frac{1}{4} (q(x_1 + x_2) + q(y_1 + y_2) - q(x_1 - x_2) - q(y_1 - y_2), 0) \\
 & + \frac{(0, 1)}{4} (q(x_1 - y_2) + q(y_1 + x_2) - q(x_1 + y_2) - q(y_1 - x_2), 0) \\
 = & \frac{1}{4} (q(x_1 + x_2) + q(y_1 + y_2) - q(x_1 - x_2) - q(y_1 - y_2), \\
 & q(x_1 - y_2) + q(y_1 + x_2) - q(x_1 + y_2) - q(y_1 - y_2))
 \end{aligned}$$

is sesquilinear and  $T((x, y), (x, y)) = Q((x, y)) = (q(x) + q(y), 0)$ . Now let  $B(x, y) = T((x, 0), (y, 0))$  then clearly  $B : M \times M \rightarrow R \oplus R$  is biadditive. Also for  $a, b \in R$ ,

$x, y \in M$ ;

$$\begin{aligned}
 B(ax, by) &= S((ax, 0), (by, 0)) \\
 &= S((a, 0) \cdot (x, 0), (b, 0)(y, 0)) \\
 &= (a, 0) S((x, 0), (y, 0)) (b^*, 0) \\
 &= a S((x, 0), (y, 0)) b^* \\
 &= a B(x, y) b^*.
 \end{aligned}$$

On the other hand  $B(x, y) = \frac{1}{4} (q(x+y) - q(x-y), 0)$  so by giving

$$S(x, y) = \frac{1}{4} (q(x+y) - q(x-y))$$

the proof completes.

**Corollary 1.** *Let  $M$  be a module over a commutative  $*$ -ring  $R$  with trivial involution and identity element such that  $2$  be a unit in  $R$ . Moreover, let  $q : M \rightarrow R$  be a functional such that for any  $a, b \in R, x, y \in M$ ;*

$$(3.6) \quad q(ax - by) + q(ay + bx) = a^2 (q(x) + q(y)) + b^2 (q(x) + q(y)),$$

*then the functional  $S : M \times M \rightarrow R$  defined by*

$$S(x, y) = \frac{1}{4} (q(x+y) - q(x-y)),$$

*is sesquilinear and  $S(x, x) = q(x)$  for any  $x \in M$ .*

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# SOME INEQUALITIES EQUIVALENT TO TRIANGULAR INEQUALITY IN NORMED AND 2-NORMED SPACES

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ABSTRACT. We give an equivalent definition of semi norm and 2-norm extending the notion of  $q$ -norm to  $q \in R - \{0\}$ .

## 1. INTRODUCTION

It is known that the parallelogram equation

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

characterizes Hilbert spaces among normed spaces. S. Saitoh [Sa] noted that the inequality  $\|x + y\|^2 \leq 2(\|x\|^2 + \|y\|^2)$  may be more suitable than the usual triangular inequality. He used this inequality to the setting of a natural sum Hilbert space for two arbitrary Hilbert spaces. Obviously the triangle inequality implies the above inequality so one motivates to consider an extension of the triangle inequality. In [Mos], it was shown that every  $q$ -norm which is defined by replacing the triangle inequality by  $\|x + y\|^q \leq 2^{q-1}(\|x\|^q + \|y\|^q)$ , where  $q \geq 1$ , is a norm in usual sense. Here we will give an extension of the notion of  $q$ -norm by determining some results. Finally, we will assert the previous results for 2-normed spaces, a notion which is defined naturally different of  $q$ -normed spaces.

## 2. NORMED SPACES

**Definition 1.** Let  $X$  be a real or complex vector space and  $q \in [1, \infty)$ . A mapping  $\|\cdot\| : X \rightarrow R$  is called a  $q$ -norm on  $X$  if it satisfies the following conditions:

- (i)  $\|\cdot\| = 0 \Leftrightarrow x = 0$ ,
- (ii)  $\|\lambda x\| = |\lambda| \|x\|$  for all  $x \in X$  and scalar  $\lambda$ .
- (iii)  $\|x + y\|^q \leq 2^{q-1}(\|x\|^q + \|y\|^q)$  for all  $x, y \in X$ .

**Theorem 1.**  $\|\cdot\|$  is a norm iff it is a  $q$ -norm for any  $q \in [1, \infty)$ .

*Proof.* see [2]. □

Now we obtain the following theorems.

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**Theorem 2.** *Let  $q < 0$  be real. Then the triangular inequality holds iff the inequality*

$$2^{q-1} (\|x\|^q + \|y\|^q) \leq \|x + y\|^q \quad (x, y \in X)$$

*holds.*

*Proof.* ( $\Rightarrow$ ) Consider the function  $f(t) = \frac{1+t^q}{2} - \left(\frac{1+t}{2}\right)^q$  on  $(0, \infty)$ . It is seen that  $f$  has a negative derivative on  $[1, \infty)$  and so for any  $0 < t \leq 1$ ,  $f(t) \leq 0$ . Now let  $\|x\| \geq \|y\|$  then

$$\frac{1 + \left(\frac{\|y\|}{\|x\|}\right)^q}{2} \leq \left(\frac{1 + \frac{\|y\|}{\|x\|}}{2}\right)^q,$$

Hence

$$\frac{\|x\|^q + \|y\|^q}{2} \leq \left(\frac{\|x\| + \|y\|}{2}\right)^q \leq \left(\frac{\|x + y\|}{2}\right)^q,$$

so

$$2^{q-1} (\|x\|^q + \|y\|^q) \leq \|x + y\|^q.$$

( $\Leftarrow$ ) conversely, the function  $f(t) = 2^{\frac{q-1}{q}} (t^q + 1)^{\frac{1}{q}} - t - 1$  has positive derivative on  $(0, 1]$ . Then for  $t = \frac{\|y\|}{\|x\|}$  with  $\|y\| \leq \|x\|$ ,  $f(t) \leq 0$ . So

$$2^{\frac{q-1}{q}} (\|x\|^q + \|y\|^q)^{\frac{1}{q}} \leq \|x\| + \|y\|,$$

Hence

$$(\|x\| + \|y\|)^q \leq 2^{q-1} (\|x\|^q + \|y\|^q) \leq \|x + y\|^q$$

which yields the triangular inequality.  $\square$

**Theorem 3.** *Let  $q \in (0, 1]$  and  $\|\cdot\| : X \rightarrow R$  satisfies  $\|\lambda x\| = |\lambda| \|x\|$  for all  $x \in X$  and scalar  $\lambda$ . Then triangular inequality holds iff the inequality*

$$\|x + y\|^q \leq \|x\|^q + 2^{q-1} \|y\|^q, \quad (x, y \in X, \quad \|y\| \leq \|x\|)$$

*holds.*

*Proof.* ( $\Rightarrow$ ) The function  $f(t) = \frac{1+t^q}{2} - \left(\frac{1+t}{2}\right)^q$  is increasing on  $[0, 1]$  since the derivative function of  $f$  is nonnegative on  $(0, 1)$ . So for  $t = \frac{\|y\|}{\|x\|}$  with  $\|y\| \leq \|x\|$ ,  $f(t) \geq \frac{1}{2} - \frac{1}{2^q}$  that is

$$\frac{1 + \left(\frac{\|y\|}{\|x\|}\right)^q}{2} + \frac{1}{2^q} - \frac{1}{2} \geq \left(\frac{1 + \frac{\|y\|}{\|x\|}}{2}\right)^q \geq \left(\frac{\|x + y\|}{2\|x\|}\right)^q,$$

Hence

$$\|x + y\|^q \leq 2^{q-1} (\|x\|^q + \|y\|^q) + (1 - 2^{q-1}) \|x\|^q = \|x\|^q + 2^{q-1} \|y\|^q.$$

( $\Leftarrow$ ) Conversely, the function  $f(t) = (t^q + 2^{q-1})^{\frac{1}{q}} - t - 1$  is increasing on  $[0, \infty)$  and  $f(0) f\left(2^{\frac{q-1}{q}}\right) < 0$  so there exists  $c \in \left(0, 2^{\frac{q-1}{q}}\right)$  such that  $f(c) = 0$  and for any  $t \in [0, c]$ ,  $f(t) \leq f(c) = 0$ . Now for  $t = \frac{\|y\|}{\|x\|}$  where  $\|y\| \leq c\|x\|$ ;

$$\begin{aligned} \left(\left(\frac{\|y\|}{\|x\|}\right)^q + 2^{q-1}\right)^{\frac{1}{q}} &\leq \frac{\|y\|}{\|x\|} + 1 \Rightarrow (\|y\|^q + 2^{q-1} \|x\|^q)^{\frac{1}{q}} \leq \|x\| + \|y\| \\ &\Rightarrow \|x + y\| \leq \left((\|y\|^q + 2^{q-1} \|x\|^q)^{\frac{1}{q}}\right) \leq \|x\| + \|y\|. \end{aligned}$$

## SOME INEQUALITIES EQUIVALENT TO TRIANGULAR INEQUALITY

Similarly, for  $x, y \in X$  which  $\|x\| \leq c \|y\|$ ,  $\|x + y\| \leq \|x\| + \|y\|$ .

Now let  $c \|x\| < \|y\| < \frac{1}{c} \|x\|$ , then

$$(2.1) \quad \begin{aligned} \left\| \frac{x}{c} \right\| &> \frac{1}{c} \|cy\| \\ \left\| \frac{x}{c} + cy \right\| &< \frac{1}{c} \|x\| + c \|y\| \\ \|x + c^2 y\| &< \|x\| + c^2 \|y\|, \end{aligned}$$

and

$$(2.2) \quad \begin{aligned} \|x\| &< c \left\| \frac{y}{c^2} \right\| \\ \left\| x + \frac{y}{c^2} \right\| &< \|x\| + \frac{1}{c^2} \|y\| \\ \|c^2 x + y\| &< c^2 \|x\| + \|y\|. \end{aligned}$$

Now if both  $c \|x + c^2 y\| > \|c^2 x + y\|$  and  $c \|c^2 x + y\| > \|x + c^2 y\|$  hold then

$$c (\|x + c^2 y\| + \|c^2 x + y\|) > \|c^2 x + y\| + \|x + c^2 y\| \Rightarrow c > 1 \text{ or } 0 > 0$$

which is a contradiction. Now let  $c \|x + c^2 y\| \leq \|c^2 x + y\|$  then by (2.1) and (2.2),

$$(1 + c^2) \|x + y\| = \|x + c^2 y + c^2 x + y\| \leq \|x + c^2 y\| + \|c^2 x + y\| < (c^2 + 1) (\|x\| + \|y\|),$$

so the triangular inequality holds.  $\square$

Theorems (1), (2) and (3) show that the definition of a semi-norm is equivalent to the following:

**Corollary 1.** *Let  $X$  be a real or complex vector space. A mapping  $\|\cdot\| : X \rightarrow R$  which satisfies  $\|\lambda x\| = |\lambda| \cdot \|x\|$  for all  $x \in X$  and scalar  $\lambda$ , then the following conditions are equivalent:*

- 1)  $\|x + y\| \leq \|x\| + \|y\|$  for all  $x, y \in X$ .
- 2) For any  $q \in [1, \infty)$ ,  $\|x + y\|^q \leq (\|x\|^q + \|y\|^q)$  for all  $x, y \in X$ .
- 3) For any  $q \in (0, 1)$ ,  $\|x + y\|^q \leq \|x\|^q + 2^{q-1} \|y\|^q$  where  $x, y \in X$ ,  $\|y\| \leq \|x\|$ .
- 4) For any  $q < 0$ ,  $2^{q-1} (\|x\|^q + \|y\|^q) \leq \|x + y\|^q$  for all  $x, y \in X$ .

If one of the above four conditions satisfied then the function  $\|\cdot\|$  is a semi norm on  $X$ .

It is known that the parallelogram equality

$$\|x + y\|^2 + \|x - y\|^2 = 2 (\|x\|^2 + \|y\|^2),$$

characterizes Hilbert spaces among normed spaces. Now for the equation

$$(2.3) \quad \|x + y\|^q + \|x - y\|^q = 2^{q-1} (\|x\|^q + \|y\|^q), \quad q \neq 2.$$

we state the following proposition.

**Proposition 1.** *For any  $q \neq 2$  the equation (2.3) yields  $\|\cdot\| = 0$ .*

*Proof.* First give  $x = y = 0$  and then give  $y = 0$  in (2.3). Note that the limitation  $\|0\| = 0$  is necessary in case  $q = 1$ .  $\square$

## 3. 2-NORMED SPACES

**Definition 2.** Let  $X$  be a vector space of dimension greater than one over  $F$ , where  $F$  is the real or complex numbers field. Suppose  $\|.,.\|$  be a non-negative real function on  $X \times X$  satisfies the following conditions:

- i)  $\|x, y\| = 0$  if and only if  $x$  and  $y$  are linearly dependent vectors,
- ii)  $\|x, y\| = \|y, x\|$  for all  $x, y \in X$ ,
- iii)  $\|\lambda x, y\| = |\lambda| \|x, y\|$  for all  $\lambda \in F$  and  $x, y \in X$ ,
- iv)  $\|x + y, z\| \leq \|x, z\| + \|y, z\|$  for all  $x, y, z \in X$ ,

Then  $\|.,.\|$  is called a 2-normed space.

Clearly, for any 2-norm  $\|.,.\|$  and each  $y \in X$  the mapping  $\|.,.\|$  defined by  $\|x\| = \|x, y\|$  is a semi-norm. So we can obtain the results of section II to the 2-normed spaces.

**Theorem 4.** Let  $X$  be a vector space of dimension greater than one over  $F$ , where  $F$  is the real or complex numbers field. Suppose  $\|.,.\|$  be a non-negative real function on  $X \times X$  satisfies the following conditions:

- i)  $\|x, y\| = 0$  if and only if  $x$  and  $y$  are linearly dependent vectors,
- ii)  $\|x, y\| = \|y, x\|$  for all  $x, y \in X$ ,
- iii)  $\|\lambda x, y\| = |\lambda| \|x, y\|$  for all  $\lambda \in F$  and  $x, y \in X$ ,

Then the following statements are equivalent:

- 1)  $\|x + y, z\| \leq \|x, z\| + \|y, z\|$  for all  $x, y, z \in X$ .
- 2) For any  $q \in [1, +\infty)$ ,  $\|x + y, z\|^q \leq 2^{q-1} (\|x, z\|^q + \|y, z\|^q)$  for all  $x, y, z \in X$ .
- 3) For any  $q \in (0, 1)$ ,  $\|x + y, z\|^q \leq \|x, z\|^q + 2^{q-1} \|y, z\|^q$  where  $x, y, z \in X$ ,  $\|y, z\| \leq \|x, z\|$ .
- 4) For any  $q < 0$ ,  $2^{q-1} (\|x, z\|^q + \|y, z\|^q) \leq (\|x + y, z\|^q)$  for all  $x, y, z \in X$ .

If one of the above four conditions satisfied then the function  $\|.,.\|$  is a 2-norm on  $X$ .

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## On Distributions of Order Statistics of Random Vectors

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**Abstract.** In this study, the probability density and distribution functions of order statistics of the *innid* random vectors expressed as the probability density and distribution functions of order statistics of the *iid* random vectors using permanents. Moreover, we collect some results concerning the probability density and distribution functions of order statistics for the specialized cases.

**Keywords.** Order statistics, permanent, joint probability density function, joint distribution function, *iid* random variables, *innid* random variables, recurrence relations.

**MSC 2000:** 62G30, 62E15.

### 1. Introduction

Balakrishnan[1] considered recent developments on order statistics arising from independent and not necessarily identically distributed(*innid*) random variables based primarily on the theory of permanents.

Balasubramanian et al.[2] established identities satisfied by distributions of order statistics from non-independent non-identical variables through operator methods based on difference and differential operators.

Beg[3] obtained several recurrence relations and identities for product moments of order statistics of *innid* random variables using permanents.

Childs and Balakrishnan[4] obtained, using multinomial arguments, the probability density function(*pdf*) of  $X_{r:n+1}$  ( $1 \leq r \leq n+1$ ) if another independent random variable with distribution function(*df*)  $F_i$  and *pdf*  $f_i$  ( $i=1,2,...,n$ ) is added to the original  $n$  variables  $X_1, X_2, ..., X_n$ .

David[5] considered the fundamental distribution theory of order statistics.

Gan and Bain[6] obtained the joint probability function(*pf*) of any  $k$  order statistics and also conditional distributions of discrete order statistics from a general discrete parent by “tie-runs”.

Guilbaud[7] expressed probability of the functions of *innid* random vectors as a linear

combination of probabilities of the functions of independent and identically distributed(*iid*) random vectors and thus also for order statistics of random variables.

Khatri[8] examined the *pf* and *df* of a single order statistics, the joint *pf* and *df* of any two order statistics and joint *df* of any three order statistics of *iid* random variables from a discrete parent.

Reiss[9] considered the joint *pdf*, marginal *pdf* and *df* of any *k* order statistics of *iid* random variables under a continuous *df* and discontinuous *df*. He also considered *pdf* of bivariate order statistics by marginal ordering of bivariate *iid* random vectors with a continuous *df* by means of multinomial probabilities of appropriate “cell frequency vectors”, defining multivariate order statistics by marginal ordering of *iid* random vectors with a continuous *df*.

Vaughan and Venables[10] denoted the joint *pdf* and marginal *pdf* of order statistics of *innid* random variables by means of permanents.

If  $a_1, a_2, \dots$  are defined as column vectors, then the matrix obtained by taking  $m_1$  copies  $a_1$ ,  $m_2$  copies  $a_2, \dots$  can be denoted as

$$\begin{bmatrix} a_1 & a_2 & \dots \\ m_1 & m_2 & \dots \end{bmatrix}$$

and *perA* denotes the permanent of a square matrix *A*, which is defined as similar to determinants except that all terms in the expansion have a positive sign.

Consider  $x = (x^{(1)}, x^{(2)}, \dots, x^{(b)})$  and  $y = (y^{(1)}, y^{(2)}, \dots, y^{(b)})$ , then it can be written as  $x \leq y$  if  $x^{(v)} \leq y^{(v)}$ ,  $v=1, 2, \dots, b$  and  $x + y = (x^{(1)} + y^{(1)}, x^{(2)} + y^{(2)}, \dots, x^{(b)} + y^{(b)})$ .

Let  $\xi_i = (\xi_i^{(1)}, \xi_i^{(2)}, \dots, \xi_i^{(b)})$ ,  $i=1, 2, \dots, n$  be *n innid* random vectors which components of  $\xi_i$  are independent. The expression

$$X_{r:n}^{(v)} = Z_{r:n}(\xi_1^{(v)}, \xi_2^{(v)}, \dots, \xi_n^{(v)}) \quad (1)$$

is stated as the *r*th order statistic of the *v*th components of  $\xi_1, \xi_2, \dots, \xi_n$ .

From (1), the ordered values of the *v*th components of  $\xi_1, \xi_2, \dots, \xi_n$  are expressed as

$$X_{1:n}^{(v)} \leq X_{2:n}^{(v)} \leq \dots \leq X_{n:n}^{(v)}. \quad (2)$$

From (2), we can write

$$X_{r:n} = (X_{r:n}^{(1)}, X_{r:n}^{(2)}, \dots, X_{r:n}^{(b)}), \quad 1 \leq r \leq n.$$

Let  $F_i$  be *df* of  $\xi_i$  and be continuous at any  $x_l = (x_l^{(1)}, x_l^{(2)}, \dots, x_l^{(b)})$ ,  $l=1, 2, \dots, d$ ;  $d=1, 2, \dots, n$ .

In this study, the *df* and *pdf* of  $X_{r_1:n}, X_{r_2:n}, \dots, X_{r_d:n}$ ,  $1 \leq r_1 < r_2 < \dots < r_d \leq n$  will be given in Theorem 1 and Theorem 2, respectively. Let  $X^{(v)} = (X_{r_1:n}^{(v)}, X_{r_2:n}^{(v)}, \dots, X_{r_d:n}^{(v)})$  and  $x^{(v)} = (x_1^{(v)}, x_2^{(v)}, \dots, x_d^{(v)})$ . For notational convenience we write  $\sum \sum$  and  $\sum_{m_1, m_2, \dots, m_d}$  instead of

$\sum_{k=1}^n (-1)^{n-k} \frac{k^n}{n!} \sum_{n_s=k}$  and  $\sum_{m_d=r_d}^n \dots \sum_{m_2=r_2}^{m_3} \sum_{m_1=r_1}^{m_2}$  in the expressions below, respectively. Moreover,

$X^{(v),s} = (X_{r_1:n}^{(v),s}, X_{r_2:n}^{(v),s}, \dots, X_{r_d:n}^{(v),s})$  are random vectors with common *df*  $F^s$  and *pdf*  $f^s$ , respectively, defined by

$$F^s(x_a^{(v)}) = \frac{1}{n_s} \sum_{i \in s} F_i(x_a^{(v)}), \quad a=1, 2, \dots, d. \quad (3)$$

and

$$f^s(x_a^{(v)}) = \frac{1}{n_s} \sum_{i \in s} f_i(x_a^{(v)}), \quad a=1, 2, \dots, d. \quad (4)$$

Here,  $s$  is a non-empty subset of the integers  $\{1, 2, \dots, n\}$  with  $n_s \geq 1$  elements.

The following theorem connects the *df* of order statistics of *innid* random vectors to that of order statistics of *iid* random vectors using (3).

**Theorem 1.**

$$F_{r_1, r_2, \dots, r_d:n}(x_1, x_2, \dots, x_d) = \prod_{v=1}^b \left\{ \sum \sum \sum_{m_1, m_2, \dots, m_d} n! \prod_{a=1}^{d+1} \frac{[F^s(x_a^{(v)}) - F^s(x_{a-1}^{(v)})]^{(m_a - m_{a-1})}}{(m_a - m_{a-1})!} \right\},$$

$$x_1 < x_2 < \dots < x_d, \quad m_0 = 0, \quad m_{d+1} = n, \quad F^s(x_0^{(v)}) = 0 \quad \text{and} \quad F^s(x_{d+1}^{(v)}) = 1.$$

**Proof.** It can be written

$$\begin{aligned} F_{r_1, r_2, \dots, r_d:n}(x_1, x_2, \dots, x_d) &= P\{X_{r_1:n} \leq x_1, X_{r_2:n} \leq x_2, \dots, X_{r_d:n} \leq x_d\} \\ &= P\{X^{(1)} \leq x^{(1)}, X^{(2)} \leq x^{(2)}, \dots, X^{(b)} \leq x^{(b)}\} \\ &= \prod_{v=1}^b P\{X^{(v)} \leq x^{(v)}\}. \end{aligned} \quad (5)$$

(5) can be expressed as

$$\begin{aligned} \prod_{v=1}^b [\sum \sum P\{X_{r_1:n}^{(v),s} \leq x_1^{(v)}, X_{r_2:n}^{(v),s} \leq x_2^{(v)}, \dots, X_{r_d:n}^{(v),s} \leq x_d^{(v)}\}] \\ = \prod_{v=1}^b \left\{ \sum \sum \sum_{m_1, m_2, \dots, m_d} n! \prod_{a=1}^{d+1} \frac{[F^s(x_a^{(v)}) - F^s(x_{a-1}^{(v)})]^{(m_a - m_{a-1})}}{(m_a - m_{a-1})!} \right\}. \end{aligned}$$

The proof is complete.

Theorem 1 will be specialized to the following results.

**Result 1.** From Theorem 1, we can write

$$F_{r_1, r_2, \dots, r_d; n}(x_1, x_2, \dots, x_d) = \prod_{v=1}^b \sum_{m_1, m_2, \dots, m_d} C_{per} A$$

$$= \prod_{v=1}^b \left\{ \sum \sum \sum_{m_1, m_2, \dots, m_d} n! \prod_{a=1}^{d+1} \frac{[F^s(x_a^{(v)}) - F^s(x_{a-1}^{(v)})]^{(m_a - m_{a-1})}}{(m_a - m_{a-1})!} \right\}, \quad (6)$$

where  $C = [m_1!(m_2 - m_1)! \dots (n - m_d)!]^{-1}$ ,

$A = [F(x_1^{(v)}) \quad F(x_2^{(v)}) - F(x_1^{(v)}) \quad \dots \quad 1 - F(x_d^{(v)})]$  is matrix,

$$F(x_t^{(v)}) - F(x_{t-1}^{(v)}) = (F_1(x_t^{(v)}) - F_1(x_{t-1}^{(v)}), F_2(x_t^{(v)}) - F_2(x_{t-1}^{(v)}), \dots, F_n(x_t^{(v)}) - F_n(x_{t-1}^{(v)}))',$$

$t=1, 2, \dots, d+1$  is column vector,  $F(x_{d+1}^{(v)}) = 1$  and  $F(x_0^{(v)}) = 0$ .

**Result 2.**

$$\sum_{m=r}^n \frac{1}{m!(n-m)!} \sum_{l=0}^{n-m} (-1)^{n-m-l} \binom{n-m}{l} l! \sum_{n_s=n-l} per[F(x)] [s/.] = \sum \sum \sum_{m=r}^n \binom{n}{m} \sum_{l=0}^{n-m} (-1)^{n-m-l} \binom{n-m}{l} (F^s(x))^{n-l}. \quad (7)$$

**Proof.** In (6), if  $b=1$ ,  $d=1$  and using properties of permanent and binomial expansion, we can write

$$F_{r;n}(x) = \sum_{m=r}^n \frac{1}{m!(n-m)!} per[F(x) \quad 1 - F(x)]$$

$$= \sum_{m=r}^n \frac{1}{m!(n-m)!} \sum_{l=0}^{n-m} (-1)^{n-m-l} \binom{n-m}{l} per[F(x) \quad 1]$$

$$= \sum_{m=r}^n \frac{1}{m!(n-m)!} \sum_{l=0}^{n-m} (-1)^{n-m-l} \binom{n-m}{l} l! \sum_{n_s=n-l} per[F(x)] [s/.], \quad (8)$$

where  $F(x) = (F_1(x), F_2(x), \dots, F_n(x))'$ ,  $1 - F(x) = (1 - F_1(x), 1 - F_2(x), \dots, 1 - F_n(x))'$ ,  $1 = (1, 1, \dots, 1)'$  are column vectors and  $A[s/.]$  is the matrix obtained from A by taking rows whose indices are in s.

Furthermore,

$$F_{r;n}(x) = \sum \sum \sum_{m=r}^n \binom{n}{m} (F^s(x))^m (1 - F^s(x))^{n-m}$$



$$= \sum \sum_{m=r}^n \sum_{l=0}^{n-m} \binom{n}{m} (-1)^{n-m-l} \binom{n-m}{l} (F^s(x))^{n-l}. \quad (9)$$

(7) is immediate from (8) and (9). The proof is complete.

In addition, if  $\sum_P$  denotes the sum over all  $n!$  permutations  $(i_1, i_2, \dots, i_n)$  of  $(1, 2, \dots, n)$ ,

$$\begin{aligned} F_{r:n}(x) &= \sum_{m=r}^n \frac{1}{m!(n-m)!} \sum_P F_{i_1}(x) F_{i_2}(x) \dots F_{i_m}(x) (1 - F_{i_{m+1}}(x)) \dots (1 - F_{i_n}(x)) \\ &= \sum_{m=r}^n \frac{1}{m!(n-m)!} \sum_P F_{i_1}(x) F_{i_2}(x) \dots F_{i_m}(x) \sum_{j=0}^{n-m} (-1)^j \sum_{n_s=j} F_{l_1}(x) \dots F_{l_j}(x), \end{aligned}$$

where  $\sum_{n_s=j}$  denotes the sum over all  $\binom{n-m}{j}$  subsets  $s = \{l_1, l_2, \dots, l_j\}$  of  $\{i_{m+1}, i_{m+2}, \dots, i_n\}$ .

**Result 3.** In (7), if  $r = 1$ ,

$$F_{1:n}(x) = 1 - \frac{1}{n!} \text{per}[1 - F(x)] = \sum \sum [1 - (1 - F^s(x))^n].$$

**Result 4.** In (7), if  $r = n$ ,

$$F_{n:n}(x) = \frac{1}{n!} \text{per}[F(x)] = \sum \sum (F^s(x))^n.$$

The following theorem connects the *pdf* of order statistics of *innid* random vectors to that of order statistics of *iid* random vectors using (4).

**Theorem 2.**

$$f_{r_1, r_2, \dots, r_d:n}(x_1, x_2, \dots, x_d) = \prod_{v=1}^b \left\{ \sum \sum n! \left( \prod_{a=1}^d f^s(x_a^{(v)}) \right) \prod_{a=1}^{d+1} \frac{[F^s(x_a^{(v)}) - F^s(x_{a-1}^{(v)})]^{(r_a - r_{a-1} - 1)}}{(r_a - r_{a-1} - 1)!} \right\},$$

where  $r_0 = 0$  and  $r_{d+1} = n + 1$ .

**Proof.** Omitted.

Theorem 2 will be specialized to the following results.

**Result 5.** From Theorem 2, we can write

$$f_{r_1, r_2, \dots, r_d:n}(x_1, x_2, \dots, x_d) = \prod_{v=1}^b D_{\text{per}B}$$

$$= \prod_{v=1}^b \left\{ \sum \sum n! \left( \prod_{a=1}^d f^s(x_a^{(v)}) \right) \prod_{a=1}^{d+1} \frac{[F^s(x_a^{(v)}) - F^s(x_{a-1}^{(v)})]^{(r_a - r_{a-1} - 1)}}{(r_a - r_{a-1} - 1)!} \right\}, \quad (10)$$

where  $D = [(r_1 - 1)!(r_2 - r_1 - 1)! \dots (n - r_d)!]^{-1}$ ,

$B = [F(x_1^{(v)}) \quad f(x_1^{(v)}) \quad F(x_2^{(v)}) - F(x_1^{(v)}) \quad f(x_2^{(v)}) \quad \dots \quad f(x_d^{(v)}) \quad 1 - F(x_d^{(v)})]$  is

matrix and  $f(x_t^{(v)}) = (f_1(x_t^{(v)}), f_2(x_t^{(v)}), \dots, f_n(x_t^{(v)}))'$ ,  $t=1, 2, \dots, d$  is column vector.

**Result 6.**

$$\begin{aligned} & \frac{1}{(r-1)!(n-r)!} \sum_{l=0}^{n-r} (-1)^{n-r-l} \binom{n-r}{l} l! \sum_{n_s=n-l} \text{per}[F(x) \quad f(x)] [s/.] \\ & = \sum \sum r \binom{n}{r} f^s(x) \sum_{l=0}^{n-r} (-1)^{n-r-l} \binom{n-r}{l} [F^s(x)]^{n-l-1}. \end{aligned} \quad (11)$$

**Proof.** In (10), if  $b=1$ ,  $d=1$  and using properties of permanent and binomial expansion, we can write

$$\begin{aligned} f_{r,n}(x) &= \frac{1}{(r-1)!(n-r)!} \text{per}[F(x) \quad f(x) \quad 1 - F(x)] \\ &= \frac{1}{(r-1)!(n-r)!} \sum_{l=0}^{n-r} (-1)^{n-r-l} \binom{n-r}{l} \text{per}[F(x) \quad f(x) \quad 1] \\ &= \frac{1}{(r-1)!(n-r)!} \sum_{l=0}^{n-r} (-1)^{n-r-l} \binom{n-r}{l} l! \sum_{n_s=n-l} \text{per}[F(x) \quad f(x)] [s/.]. \end{aligned} \quad (12)$$

Furthermore,

$$\begin{aligned} f_{r,n}(x) &= \sum \sum r \binom{n}{r} [F^s(x)]^{r-1} f^s(x) [1 - F^s(x)]^{n-r} \\ &= \sum \sum r \binom{n}{r} f^s(x) \sum_{l=0}^{n-r} (-1)^{n-r-l} \binom{n-r}{l} [F^s(x)]^{n-l-1}. \end{aligned} \quad (13)$$

(11) is immediate from (12) and (13). The proof is complete.

**Result 7.** In (11), if  $r=1$ ,

$$f_{1:n}(x) = \frac{1}{(n-1)!} \text{per} \begin{bmatrix} f(x) & 1-F(x) \\ 1 & n-1 \end{bmatrix} = \sum \sum n f^s(x) [1-F^s(x)]^{n-1}.$$

**Result 8.** In (11), if  $r = n$ ,

$$f_{n:n}(x) = \frac{1}{(n-1)!} \text{per} \begin{bmatrix} F(x) & f(x) \\ n-1 & 1 \end{bmatrix} = \sum \sum n f^s(x) [F^s(x)]^{n-1}.$$

**Result 9:** In (10), if  $b = 1$ ,  $d = 2$  and  $r_1 = 1$ ,  $r_2 = n$ ,

$$f_{1,n:n}(x_1, x_2) = \sum \sum n(n-1) f^s(x_1) f^s(x_2) [F^s(x_2) - F^s(x_1)]^{n-2}, \quad x_1 < x_2.$$

**Result 10:** In (10), if  $r_1 = 1$ ,  $r_2 = 2, \dots, r_k = k$ ,

$$f_{1,2,\dots,k:n}(x_1, x_2, \dots, x_k) = \prod_{v=1}^b \left\{ \sum \sum n! \left( \prod_{a=1}^k f^s(x_a^{(v)}) \right) \frac{[1 - F^s(x_k^{(v)})]^{n-k}}{(n-k)!} \right\}, \quad x_1 < x_2 < \dots < x_k.$$

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# A-Equistatistical Convergence of Positive Linear Operators

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## Abstract

Recently, the concept of equistatistical convergence which is stronger than statistical uniform convergence was introduced by Balcerzac, Dems and Komisarski. In the present paper we introduce the concept of A-equistatistical convergence which extends equi-statistical convergence. Moreover, we construct examples to show that A-equistatistical convergence lies between A-statistically pointwise and uniform convergence. We also obtain A-equistatistical case of Korovkin result and finally we compute the rates of A-equistatistical convergence of sequences of positive linear operators.

*Key words:* Statistical convergence, A-statistical convergence, equi-statistical convergence, Korovkin type approximation theorem, modulus of continuity, Bernstein polynomials

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## 1 Introduction

The concept of statistical convergence has been initiated by Fast [11] and so far various kinds of generalization of statistical convergence and their applications have been studied by different researchers (see [4],[7],[12],[13],[18]). Recently, Balcerzak et al. [3] have introduced the notion of equi-statistical convergence which is stronger than the statistical uniform convergence. Then Korovkin type approximation theorem via equi-statistical convergence is considered in [15]. In the present paper our main interest is to introduce the

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notion of  $A$ -equistatistical convergence which extends equi-statistical convergence and to prove a Korovkin type approximation theorem by means of  $A$ -equistatistical convergence.

Recall that for an infinite summability matrix  $A = (a_{mk})$   $m, k = 1, 2, \dots$  and a sequence  $x = (x_j)_{j \in \mathbb{N}}$ , the  $A$ -transform of  $x$  is denoted by  $Ax = (Ax)_m$  and defined as;

$$(Ax)_m = \sum_{k=1}^{\infty} a_{mk} x_k$$

provided that the series converges for each  $m \in \mathbb{N}$ . The matrix  $A$  is said to be regular if  $A$ -transform of  $x$  preserves the limit of  $x$ . Suppose that  $A$  is a non-negative regular summability matrix and  $K \subset \mathbb{N}$ , then

$$\delta_A^m(K) := \sum_{k=1}^{\infty} a_{mk} \chi_K(k)$$

is called the  $m$ th partial  $A$ -density of  $K$  where  $\chi_K$  denotes the characteristic function of  $K$  [12]. The  $A$ -density of  $K$  is defined as  $\delta_A(K) := \lim_{m \rightarrow \infty} \delta_A^m(K)$  provided that limit exists. A sequence  $x := (x_j)$  is called  $A$ -statistically convergent to  $L$  and denoted by  $st_A - \lim_{n \rightarrow \infty} x_n = L$ , if for every  $\varepsilon > 0$ ,  $\delta_A \{n \in \mathbb{N}: |x_n - L| \geq \varepsilon\} = 0$ . For the case  $A = C_1$ , the Cesàro matrix of order one,  $A$ -statistical convergence reduces to statistical convergence. Also, taking  $A = I$ , the identity matrix,  $A$ -statistical convergence coincides with the ordinary convergence. We also note that if a non-negative regular summability matrix  $A = (a_{mk})$  satisfies the condition  $\lim_m \max_k \{a_{mk}\} = 0$ , then  $A$ -statistical convergence is stronger than ordinary convergence [16].

## 2 Types of convergence for sequences of functions

In this section we introduce  $A$ -equistatistical convergence which is lying between  $A$ -statistical pointwise and  $A$ -statistical uniform convergence, for sequences of real valued functions.

Let  $X$  be a subset of  $\mathbb{R}$  and assume  $f, f_n : X \longrightarrow \mathbb{R}$ , for all  $n \in \mathbb{N}$ , then we have the following well known definitions.

**Definition 1** Let  $A = (a_{mk})$  be a non-negative regular summability matrix then  $(f_n)_{n \in \mathbb{N}}$  is said to be  $A$ -statistically pointwise convergent to  $f$  on  $X$  and denoted by  $f_n \longrightarrow_A f$  if

$$st_A - \lim_{n \rightarrow \infty} f_n(x) = f(x)$$

for each  $x \in X$ , i.e. for every  $\varepsilon > 0$  and for each  $x \in X$ ,

$$\delta_A(\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

**Definition 2** Let  $A = (a_{mk})$  be a non-negative regular summability matrix then  $(f_n)_{n \in \mathbb{N}}$  is said to be  $A$ -statistically uniform convergent to  $f$  on  $X$  and denoted by  $f_n \Rightarrow_A f$  if

$$st_A - \lim_{n \rightarrow \infty} \|f_n - f\|_{C(X)} = 0$$

i.e. for every  $\varepsilon > 0$ ,

$$\delta_A(\{n \in \mathbb{N} : \|f_n(x) - f(x)\|_{C(X)} \geq \varepsilon\}) = 0.$$

In [3], the concept of equi-statistical convergence was based on the Cesàro matrix of order one. In the following definition we extend this idea for an arbitrary non-negative regular summability matrix  $A$ .

**Definition 3** Let  $A = (a_{mk})$  be a non-negative regular summability matrix then  $(f_n)_{n \in \mathbb{N}}$  is said to be  $A$ -equistatistically convergent to  $f$  on  $X$  and denoted by  $f_n \twoheadrightarrow_A f$  if for every  $\varepsilon > 0$ , the sequence of real valued functions  $(h_{m,\varepsilon})_{m \in \mathbb{N}}$  where

$$h_{m,\varepsilon}(x) = \delta_A^m(\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon\}), \quad x \in X \quad (2.1)$$

uniformly converges to the zero function on  $X$ . i.e.  $\lim_{m \rightarrow \infty} \|h_{m,\varepsilon}(\cdot)\|_{C(X)} = 0$ .

Choosing  $A = C_1$  then above definition reduces to the one considered in [3] and [15]. As a direct consequence of the definitions, we can state the following lemma.

**Lemma 2.1** Let  $X$  be a subset of  $\mathbb{R}$  and assume  $f, f_n : X \rightarrow \mathbb{R}$ , for all  $n \in \mathbb{N}$  then

- i)  $f_n \Rightarrow_A f$  on  $X \implies f_n \twoheadrightarrow_A f$  on  $X$
- ii)  $f_n \twoheadrightarrow_A f$  on  $X \implies f_n \longrightarrow_A f$  on  $X$ .

The following intriguing examples guarantee that, in general the inverse implications of (i) and (ii) does not hold.

**Example 1** Consider the sequence of continuous functions  $f_n : [0, 1] \rightarrow \mathbb{R}$ ,  $n \in \mathbb{N}$ , defined as

$$f_n(x) = \begin{cases} -4n^2(n+1)^2 \left(x - \frac{1}{n}\right) \left(x - \frac{1}{n+1}\right) & , \text{ if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right] \\ 0 & , \text{ otherwise} \end{cases}$$

and let  $A = (a_{mk})$  be the non-negative regular summability matrix such that

$$a_{mk} \leq b_m, \quad k = 1, 2, \dots \text{ and } \lim_{m \rightarrow \infty} b_m = 0$$

then  $f_n \rightarrow_A 0$  but  $f_n \rightrightarrows_A 0$  does not hold. Indeed, for each  $\varepsilon > 0$  and for all  $x \in [0, 1]$ , the set  $\{n \in \mathbb{N} : |f_n(x)| \geq \varepsilon\}$  has cardinality at most one. Thus for every  $\varepsilon > 0$  and  $x \in [0, 1]$

$$h_{m,\varepsilon}(x) = \delta_A^m(\{n \in \mathbb{N} : |f_n(x)| \geq \varepsilon\}) \leq b_m.$$

Taking limit from both sides as  $m \rightarrow \infty$ , we have

$$\lim_{m \rightarrow \infty} \|h_{m,\varepsilon}(\cdot)\|_{C[0,1]} = 0.$$

Hence  $f_n \rightarrow_A 0$ . Finally, choosing  $\varepsilon = 1$  and taking into account that

$$\|f_n\|_{C[0,1]} = \sup_{x \in [0,1]} |f_n(x)| = 1 \text{ for all } n \in \mathbb{N}$$

we have

$$\delta_A \{n \in \mathbb{N} : \|f_n\|_{C[0,1]} \geq 1\} = \delta_A \{\mathbb{N}\} = 1 \neq 0.$$

**Example 2** Consider the sequence of functions  $f_n : [0, 1] \rightarrow \mathbb{R}$ ,  $n \in \mathbb{N}$ , defined by  $f_n(x) := \chi_{\{\frac{1}{2n}\}}$  and let  $A = (a_{nk})$  be the non-negative regular summability matrix, where

$$a_{nk} = \begin{cases} \frac{1}{2n}, & n \leq k \leq 3n - 1 \\ 0, & \text{otherwise} \end{cases}$$

then for each  $\varepsilon > 0$  and for every  $x \in [0, 1]$ ,

$$h_{m,\varepsilon}(x) = \delta_A^m(\{n \in \mathbb{N} : |f_n(x)| \geq \varepsilon\}) \leq \frac{1}{2m}$$

thus  $f_n \rightarrow_A 0$ . But  $f_n \rightrightarrows_A 0$  does not hold.

**Example 3** Consider the sequence of functions  $f_n : [0, 1] \rightarrow \mathbb{R}$ ,  $n \in \mathbb{N}$ , defined by  $f_n(x) = x^n$  and let  $C_1 = (c_{nk})$  be the Cesàro matrix of order one, i.e.

$$c_{nk} = \begin{cases} \frac{1}{n}, & 1 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$$

and take  $\varepsilon = \frac{1}{4}$ , then  $\forall n \in \mathbb{N}$ ,  $\exists m \geq n$  such that for any  $x \in \left(\sqrt[m]{\frac{1}{4}}, 1\right)$ ,

$$\{1, 2, \dots, m\} \subset \left\{n \in \mathbb{N} : |f_n(x)| \geq \frac{1}{4}\right\}$$

which implies that

$$1 = \delta_{C_1}^m(\{1, 2, \dots, m\}) \leq \delta_{C_1}^m\left(\left\{n \in \mathbb{N} : |f_n(x)| \geq \frac{1}{4}\right\}\right)$$

it follows from (2.1) that  $f_n$  is not equi-statistically convergent to the ordinary limit function.

### 3 Korovkin type approximation theorem

As it is well known, many researchers obtained many Korovkin type approximation theorems via  $A$ -statistical convergence (see [1],[2],[4], [6], [8], [9], [10] and [14]). Recently, a Korovkin type approximation theorem by means of equi-statistical convergence was given in [15]. The primary goal of this section is to prove a Korovkin type approximation theorem via  $A$ -equi-statistical convergence.

**Theorem 3.1** *Let  $A = (a_{mk})$  be a non-negative regular summability matrix and  $X$  be a compact subset of  $\mathbb{R}$ . Suppose that  $\{L_n\}$  is a sequence of positive linear operators defined on  $C(X^r)$ , the space of all continuous real valued functions on  $X^r$  into itself, where  $X^r = X \times \dots \times X$ . Then for all  $f \in C(X^r)$ ,*

$$L_n(f) \rightarrow_A f$$

is satisfied if the following holds;

$$L_n(f_\nu) \rightarrow_A f_\nu, \nu = 0, 1, \dots, r+1 \quad (3.1)$$

where, for  $\vec{y} = (y_1, \dots, y_r) \in X^r$ ,  $f_0(\vec{y}) = 1$ ,  $f_\nu(\vec{y}) = y_\nu$ ,  $\nu = 1, 2, \dots, r$ , and  $f_{r+1}(\vec{y}) = \sum_{\nu=1}^r y_\nu^2$ .

**Proof.** Let  $f$  be a continuous function on  $X^r$  and let  $\vec{x} = (x_1, x_2, \dots, x_r)$  be a fixed point in  $X^r$ . For every  $\varepsilon > 0$  there exists a real number  $\delta > 0$  such that  $|f(\vec{y}) - f(\vec{x})| < \varepsilon$  for all  $\vec{y} = (y_1, y_2, \dots, y_r) \in X^r$  satisfying  $|y_\nu - x_\nu| < \delta$ ,  $\nu = 1, 2, \dots, r$ . If  $X_\delta^r := X^r \cap K_\delta^r$  where  $K_\delta^r := \{\vec{y} \in \mathbb{R}^r : |y_\nu - x_\nu| < \delta\}$ , then

$$\begin{aligned} |f(\vec{y}) - f(\vec{x})| &\leq |f(\vec{y}) - f(\vec{x})| \chi_{X_\delta^r}(\vec{y}) \\ &\quad + |f(\vec{y}) - f(\vec{x})| \chi_{X^r \setminus X_\delta^r}(\vec{y}) \\ &\leq \varepsilon + 2M \chi_{X^r \setminus X_\delta^r}(\vec{y}), \end{aligned}$$



where  $M := \|f\|_{C(X^r)}$ . After some simple calculations we may write that

$$\chi_{X^r \setminus X_\delta^r}(\vec{y}) \leq \frac{1}{\delta^2} \sum_{\nu=1}^r (y_\nu - x_\nu)^2.$$

Therefore, we get

$$|f(\vec{y}) - f(\vec{x})| \leq \varepsilon + \frac{2M}{\delta^2} \sum_{\nu=1}^r (y_\nu - x_\nu)^2,$$

for all  $\vec{y} \in X^r$ . By linearity and positivity of the operators  $\{L_n\}$ , we have

$$\begin{aligned} & |L_n(f; \vec{x}) - f(\vec{x})| \\ & \leq L_n(|f(\vec{y}) - f(\vec{x})|; \vec{x}) + |f(\vec{x})| |L_n(f_0; \vec{x}) - f_0(\vec{x})| \\ & \leq \varepsilon L_n(f_0; \vec{x}) + \frac{2M}{\delta^2} \left\{ \sum_{\nu=1}^r L_n((y_\nu - x_\nu)^2; \vec{x}) \right\} \\ & \quad + M |L_n(f_0; \vec{x}) - f_0(\vec{x})| \\ & \leq \varepsilon + (\varepsilon + M) |L_n(f_0; \vec{x}) - f_0(\vec{x})| \\ & \quad + \frac{2M}{\delta^2} |L_n(f_{r+1}; \vec{x}) - f_{r+1}(\vec{x})| \\ & \quad + \sum_{\nu=1}^r \frac{4M}{\delta^2} |x_\nu| |L_n((f_\nu; \vec{x}) - f_\nu(\vec{x}))| \\ & \quad + \frac{2M}{\delta^2} \left[ \sum_{\nu=1}^r x_\nu^2 \right] |L_n(f_0; \vec{x}) - f_0(\vec{x})| \\ & \leq \varepsilon + \left( \varepsilon + M + \frac{2rM \|\vec{x}\|_{C(X^r)}^2}{\delta^2} \right) |L_n(f_0; \vec{x}) - f_0(\vec{x})| \\ & \quad + \frac{4rM \|\vec{x}\|_{C(X^r)}}{\delta^2} \left\{ \sum_{\nu=1}^r |L_n(f_i; \vec{x}) - f_i(\vec{x})| \right\} \\ & \quad + \frac{2M}{\delta^2} |L_n(f_{r+1}; \vec{x}) - f_{r+1}(\vec{x})| \\ & \leq \varepsilon + B \sum_{\nu=0}^{r+1} |L_n(f_\nu; \vec{x}) - f_\nu(\vec{x})| \end{aligned} \tag{3.2}$$

where  $B := \varepsilon + M + \frac{4rM}{\delta^2} (\|\vec{x}\|_{C(X^r)}^2 + \|\vec{x}\|_{C(X^r)} + 1)$ .

For a given  $s > 0$ , choose  $0 < \varepsilon < s$  and define the following sets:

$$\begin{aligned} D_s(\vec{x}) &:= \{n \in \mathbb{N} : |L_n(f, \vec{x}) - f(\vec{x})| \geq s\} \\ D_s^\nu(\vec{x}) &:= \left\{ n \in \mathbb{N} : |L_n(f_\nu, \vec{x}) - f_\nu(\vec{x})| \geq \frac{s - \varepsilon}{(r+2)B} \right\} \end{aligned}$$

$\nu = 0, 1, \dots, r+1$ . Then from (3.2), we clearly have

$$D_s(\vec{x}) \subset \bigcup_{\nu=0}^{r+1} D_s^\nu(\vec{x}). \quad (3.3)$$

Also define the following real valued functions

$$p_{m,s}(\vec{x}) := \delta_A^m(\{n \in \mathbb{N} : |L_n(f, \vec{x}) - f(\vec{x})| \geq s\})$$

and

$$p_{m,s}^\nu(\vec{x}) := \delta_A^m\left(\left\{n \in \mathbb{N} : |L_n(f_\nu, \vec{x}) - f_\nu(\vec{x})| \geq \frac{s - \varepsilon}{(r+2)B}\right\}\right)$$

$\nu = 0, 1, \dots, r+1$ , then by the monotonicity of the operators  $\delta_A^m$  and (3.3), we find

$$p_{m,s}(\vec{x}) \leq \sum_{\nu=0}^{r+1} p_{m,s}^\nu(\vec{x})$$

for all  $\vec{x} \in X^r$ . This gives the inequality

$$\|p_{m,s}(\cdot)\|_{C(X^r)} \leq \sum_{\nu=0}^{r+1} \|p_{m,s}^\nu(\cdot)\|_{C(X^r)}. \quad (3.4)$$

Taking limit in (3.4) as  $m \rightarrow \infty$  and using (3.1) we have

$$\lim_m \|p_{m,s}(\cdot)\|_{C(X^r)} = 0$$

which completes the proof.

**Corollary 3.2** *Let  $A = (a_{mk})$  be a non-negative regular summability matrix. Suppose that  $\{L_n\}$  is a sequence of positive linear operators from  $C(X)$  into itself, where  $X$  is a compact subset of  $\mathbb{R}$ . Then for all  $f \in C(X)$ ,*

$$L_n(f) \rightarrow_A f$$

*is satisfied if the following holds;*

$$L_n(e_i) \rightarrow_A e_i, i = 0, 1, 2$$

*where  $e_i(y) = y^i$ .*

**Remark 1** *Let  $A = C_1$ , and let  $r = 1$ , then the Theorem 3.1 reduces to the Theorem 2.1 of [15].*

#### 4 Rates of A-equistatistical convergence

Although, there is no standard definition for the rates of  $A$ -statistical convergence, in [4] Duman et al. defined this rates in four different ways. After these definitions, rates of  $A$ -statistical convergence of various classes of linear positive operators has been computed in several articles (see [5], [6],[7],[19], [20]).

For the rates of  $A$ -equistatistical convergence we use the idea which was borrowed from the concept of convergence in measure:

**Definition 4** Let  $A = (a_{mk})$  be a non-negative regular summability matrix and let  $(a_n)$  be a positive, non-increasing and real valued sequence. We say that the sequence  $(f_n)_{n \in \mathbb{N}}$  of functions is  $A$ -equistatistically convergent to  $f$  with the rate of  $o(a_n)$ , if for every  $\varepsilon > 0$ ,

$$h_{m,\varepsilon}(x) = \delta_A^m(\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon a_n\})$$

converges uniformly to the zero function i.e.  $\lim_m \|h_{m,\varepsilon}(\cdot)\| = 0$ . In this case we write

$$f_n - f = st_{A\text{-}equi} - o(a_n), \quad (\text{as } n \longrightarrow 0).$$

**Definition 5** Let  $A = (a_{mk})$  be a non-negative regular summability matrix and let  $(a_n)$  be a positive, non-increasing and real valued sequence. We say that the sequence  $(f_n)$  of functions is  $A$ -equistatistically bounded with the rate of  $O(a_n)$ , if there is a positive number  $S$  such that

$$h_{m,\varepsilon}(x) = \delta_A^m(\{n \in \mathbb{N} : |f_n(x)| \geq S a_n\})$$

converges uniformly to the zero function i.e.  $\lim_m \|h_{m,\varepsilon}(\cdot)\| = 0$ . In this case we write

$$f_n = st_{A\text{-}equi} - O(a_n), \quad (\text{as } n \longrightarrow \infty).$$

**Lemma 4.1** Let  $A = (a_{mk})$  be a non-negative regular summability matrix,  $(f_n)$  and  $(g_n)$  be sequences of continuous functions on  $X$ . Assume that  $f_n - f = st_{A\text{-}equi} - o(a_n)$  on  $X$  and  $g_n - g = st_{A\text{-}equi} - o(b_n)$  on  $X$  (as  $n \rightarrow \infty$ ) where  $a_n$  and  $b_n$  are both positive non-increasing sequences then the following hold:

- i)  $(f_n + g_n) \pm (f + g) = st_{A\text{-}equi} - o(c_n)$
  - ii)  $(f_n - f)(g_n - g) = st_{A\text{-}equi} - o(c_n)$
- $c_n = \max\{a_n, b_n\}$ .

**Proof.** (i) Given  $\varepsilon > 0$ , by the assumption, we have

$$\lim_{m \rightarrow \infty} \|h_{m,\varepsilon}(\cdot)\|_{C(X)} = 0 \text{ and } \lim_{m \rightarrow \infty} \|p_{m,\varepsilon}(\cdot)\|_{C(X)} = 0 \quad (4.1)$$

where

$$h_{m,\varepsilon}(x) = \delta_A^m \left( \left\{ n \in \mathbb{N} : |f_n(x) - f(x)| \geq \frac{\varepsilon}{2} a_n \right\} \right)$$

and

$$p_{m,\varepsilon}(x) = \delta_A^m \left( \left\{ n \in \mathbb{N} : |g_n(x) - g(x)| \geq \frac{\varepsilon}{2} b_n \right\} \right).$$

Let

$$r_{m,\varepsilon}(x) = \delta_A^m (\{n \in \mathbb{N} : |(f_n \pm g_n)(x) - (f \pm g)(x)| \geq \varepsilon c_n\})$$

where  $c_n = \max \{a_n, b_n\}$ . Then by the following inequality

$$r_{m,\varepsilon}(x) \leq h_{m,\varepsilon}(x) + p_{m,\varepsilon}(x), \quad x \in X$$

and (4.1), the result follows immediately.

(ii) Using a similar method and the fact that if  $ab \geq \varepsilon$ , then  $a \geq \frac{\sqrt{\varepsilon}}{2}$  or  $b \geq \frac{\sqrt{\varepsilon}}{2}$  for any  $a, b \geq 0$ , we get the result.

Note that similar results hold when the little "o" is replaced by the big "O".

Now we give the rate of  $A$ -equistatistical convergence of the operators  $L_n(f; x)$  to  $f(x)$  by using the modulus of continuity. It is known that the usual modulus of continuity is defined as

$$\omega(f; \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0, \infty)} |f(x+h) - f(x)|$$

and satisfies

$$\omega(f; |y-x|) \leq \left( 1 + \frac{|y-x|}{\delta} \right) \omega(f; \delta), \quad \delta > 0. \quad (4.2)$$

Then we have the following theorem:

**Theorem 4.2** *Let  $A = (a_{mk})$  be a non-negative regular summability matrix and let*

$$(i) \quad L_n(e_0(x)) - e_0(x) = st_{A-equi} o(a_n(x))$$

$$(ii) \quad w \left( f, \sqrt{L_n((y-x)^2; x)} \right) = st_{A-equi} o(b_n(x))$$

*where  $a_n(x)$  and  $b_n(x)$  are positive non-increasing sequences. Then*

$$L_n(f; x) - f(x) = st_{A-equi} o(c_n(x)), \text{ as } n \rightarrow \infty.$$

*where  $c_n(x) = \max \{a_n(x), b_n(x)\}$ . Similar result holds when little "o" replace by "O".*

**Proof.** Using linearity and positivity of the operators and then (4.2), we have

$$\begin{aligned}
|L_n(f; x) - f(x)| &\leq L_n(|f(y) - f(x)|; x) + |f(x)| |L_n(e_0; x) - e_0(x)| \\
&\leq L_n(\omega(f; |y - x|); x) + |f(x)| |L_n(e_0; x) - e_0(x)| \\
&\quad \omega(f; \delta) \left( e_0(x) + \frac{L_n(|y - x|; x)}{\delta} \right) \\
&\quad + |f(x)| |L_n(e_0; x) - e_0(x)|.
\end{aligned}$$

Applying the Cauchy- Schwarz inequality, we get

$$\begin{aligned}
|L_n(f; x) - f(x)| &\leq (1 + L_n(e_0(x))) w \left( f, \sqrt{L_n((y - x)^2; x)} \right) \\
&\quad + |f(x)| |L_n(e_0; x) - e_0(x)| \\
&\leq 2w \left( f, \sqrt{L_n((y - x)^2; x)} \right) + M |L_n(e_0; x) - e_0(x)| \\
&\quad + w \left( f, \sqrt{L_n((y - x)^2; x)} \right) |L_n(e_0; x) - e_0(x)|. \quad (4.3)
\end{aligned}$$

Using the inequality (4.3), conditions (i) and (ii), and Lemma 4.1, the proof is completed at once.

## 5 Concluding Remarks

In this section, we first construct an example to show that  $A$ -equistatistical convergence is stronger than equi-statistical convergence. Then, we modify the well known Bernstein polynomials and show that  $A$ -equistatistical convergence is valid, while the ordinary and equi-statistical convergence are not. Finally, we compute the rate of  $A$ -equistatistical convergence of these modified Bernstein polynomials.

We start with the following example:

**Example 4** Consider the sequence of functions

$$u_n(x) = \begin{cases} 0, & \text{if } n \text{ is even} \\ x, & \text{if } n \text{ is odd} \end{cases}, \quad x \in [0, 1]$$

and the non-negative regular summability matrix  $A = (a_{nk})$  where

$$a_{mk} = \begin{cases} \frac{1}{2}, & \text{if } k = 2n - 1 \text{ or } k = 2n + 1 \\ 0, & \text{otherwise.} \end{cases}$$

Since  $\delta_A^m(\{n \in \mathbb{N} : |u_n(x) - x| \geq \varepsilon\}) = \delta_A^m(2\mathbb{N}) = 0$ , for every  $\varepsilon > 0$  and  $x \in [0, 1]$ ,  $u_n(x)$  is  $A$ -equistatistically and  $A$ -statistically convergent to  $x$ . On the other hand, one can easily see that  $u_n(x)$  is not equi-statistically convergent to  $x$ .

Now let  $u_n(x)$  and  $A = (a_{nk})$  be the same as in the Example 6 and consider the following modified version of the Bernstein polynomials:

$$B_n^*(f; x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} u_n^k(x) (1 - u_n(x))^{n-k}, \quad (5.1)$$

$x \in [0, 1]$  and  $f \in C[0, 1]$ . We immediately see that

$$\begin{aligned} B_n^*(e_0; x) &= e_0(x), \\ B_n^*(e_1; x) &= u_n(x) \\ B_n^*(e_2; x) &= \left( u_n^2(x) + \frac{u_n(x)(1 - u_n(x))}{n} \right). \end{aligned} \quad (5.2)$$

Therefore we obtain that

$$B_n^*(f_i; x) \rightarrow_A f_i \text{ on } [0, 1] \text{ for } i = 0, 1, 2.$$

Hence by the Corollary 3.2 we have

$$B_n^*(f; x) \rightarrow_A f \text{ on } [0, 1] \text{ for all } f \in C[0, 1].$$

Also by (5.2) and the Theorem 4.2, we have

$$|B_n^*(f; x) - f(x)| = st_{A\text{-}equi}O(w(f, \delta_n(x))), \text{ as } n \rightarrow \infty.$$

$$\text{where } \delta_n^2(x) = 3 \cdot \max \left\{ 2x |u_n(x) - x|, |u_n^2(x) - x^2|, \left| \frac{u_n(x)(1 - u_n(x))}{n} \right| \right\}.$$

Under the light of the above interesting application, we see that the Corollary 2 of [4] and the Theorem 2.1 of [15] do not work for the operators defined by (5.1). This also shows that the Theorem 3.1 is a non-trivial extension of  $A$ -statistical and equi-statistical version of the Korovkin theorems considered in [4] and [15], respectively.

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# Approximating the solution of second order differential equation with retarded argument

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## Abstract

A new numerical method for second order nonlinear delay differential equations is constructed. The method combines the fixed point Banach-Picard technique with the trapezoidal quadrature rule and cubic spline interpolation. The effective error estimation of the method which gives a stopping criterion of the algorithm is obtained.

**Keywords and phrases :** second order delay differential equations, numerical method, fixed point technique, cubic splines.

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## 1 Introduction

Some of the existing numerical methods for second order delay differential equations are based on Runge-Kutta procedures (see [6]), on spline functions (see [2]), on Adomian decomposition (see [4]) and on Taylor expansion (see [7]). Most of them require high order smoothness conditions in order to obtain the convergence of the method (for instance, this is the situation of Runge-Kutta procedures and of the spline functions method). Here, we present a new numerical method which combines the Picard's sequence of successive approximations with a quadrature rule and use spline interpolation only on the points where the modified argument appears. The interpolation procedure is repeated at each step of iteration using the values computed at the previous step. All procedures included in the algorithm are recurrent and therefore, easy to programming. The method is developed, in the aim to be an alternative to the well-known spline functions method, in general cases for  $f$  (without smoothness conditions, for instance).

Consider the initial value problem:

$$\begin{cases} x''(t) = f(t, x(t), x(\varphi(t))), & t \in [0, a] \\ x(0) = x_0, \quad x'(0) = v_0 \end{cases} \quad (1)$$

where  $a > 0$ ,  $x_0 \in \mathbb{R}$  and  $\varphi : [0, a] \rightarrow [0, a]$  is such that  $\varphi(0) = 0$ ,  $\varphi(t) \leq t$  for all  $t \in [0, a]$ . The case  $\varphi(t) = \lambda t$  correspond to the second order pantograph equation used in electrodynamics (a numerical method for this equation can be viewed in [7]). We suppose that  $\varphi$  is Lipschitzian and  $f$  is Lipschitzian in each argument. These lead to the error estimation of the method which proves the convergence of the algorithm (the rate of convergence being  $O(h)$ ). The obtained error estimates lead to a practical stopping criterion of the algorithm. Finally, we present two numerical examples, of pantograph type, in order to illustrate the method.

## 2 Existence, uniqueness and approximation

Consider the following conditions:

(i)  $f \in C([0, a] \times \mathbb{R} \times \mathbb{R})$  and there is  $M_0 > 0$  such that

$$|f(s, u, v)| \leq M_0, \quad \text{for all } s \in [0, a], u, v \in \mathbb{R}$$

(ii)  $\varphi \in C[0, a]$ ,  $\varphi(0) = 0$  and  $\varphi(t) \leq t$  for all  $t \in [0, a]$ ,

(iii) exist  $L_1, L_2 > 0$  such that

$$|f(s, u, v) - f(s, u', v')| \leq L_1 |u - u'| + L_2 |v - v'|, \quad \text{for all } s \in [0, a], u, u', v, v' \in \mathbb{R}$$

(iv)  $a^2(L_1 + L_2) < 1$ ,

(v) exist  $\alpha, \gamma > 0$  such that

$$|f(s, u, v) - f(s', u, v)| \leq \gamma |s - s'|, \quad \text{for all } s, s' \in [0, a], u, v \in \mathbb{R}$$

and

$$|\varphi(s) - \varphi(s')| \leq \alpha |s - s'|, \quad \text{for all } s \in [0, a].$$

On  $C[0, a]$ , we apply the fixed point technique (based on the Picard-Banach principle) to the operator  $A : C[0, a] \rightarrow C[0, a]$ , given by

$$A(x(t)) = x_0 + v_0 t + \int_0^t (t-s) f(s, x(s), x(\varphi(s))) ds$$

and obtain sufficient conditions for the convergence of the sequence of successive approximations:  $x_0(t) = x_0$ ,  $t \in [0, a]$ ,

$$x_m(t) = x_0 + v_0 t + \int_0^t (t-s) f(s, x_{m-1}(s), x_{m-1}(\varphi(s))) ds, \quad t \in [0, a], m \in \mathbb{N}^* \quad (2)$$

to the unique solution  $x^*$ , of the initial value problem (1). So, we obtain:

**Theorem 1** *Under the conditions (i)-(iv) the initial value problem (1) has unique solution  $x^* \in C[0, a]$  and the sequence of successive approximations given by  $x_0(t) = x_0$ ,  $t \in [0, a]$ ,*

$$x_m(t) = x_0 + v_0 t + \int_0^t (t-s) f(s, x_{m-1}(s), x_{m-1}(\varphi(s))) ds, \quad t \in [0, a], m \in \mathbb{N}^*$$

*uniformly converges to  $x^*$ . Moreover, the following error estimation holds:*

$$|x_m(t) - x^*(t)| \leq \frac{(a)^{2m} (L_1 + L_2)^m}{1 - a^2 (L_1 + L_2)} \cdot (|v_0 a| + M_0 a^2), \quad \text{for all } t \in [0, a], m \in \mathbb{N}^*. \quad (3)$$

**Proof.** From condition (iv) the operator  $A$  is contraction and by the Picard-Banach fixed point principle follows the estimation

$$|x_m(t) - x^*(t)| \leq \frac{(a)^{2m} (L_1 + L_2)^m}{1 - a^2 (L_1 + L_2)} \cdot \max\{|x_1(t) - x_0(t)| : t \in [0, a]\}.$$

Since

$$|x_1(t) - x_0| \leq |v_0 t| + \int_0^t |t-s| \cdot |f(s, x_0, x_0)| ds \leq |v_0 a| + M_0 a^2, \quad \text{for all } t \in [0, a]$$

we obtain (3). Moreover, after elementary calculus, we infer that  $x^* \in C^2[0, a]$  and  $x^*$  is the unique solution of (1). ■

**Remark 1** Under the hypotheses of the above theorem follows that  $x_m \in C^2[0, a]$  for all  $m \in \mathbb{N}^*$ ,

$$x_m''(t) = f(t, x_{m-1}(t), x_{m-1}(\varphi(t)))$$

and

$$\begin{aligned} |x_m(t) - x_m(t')| &\leq |v_0| |t - t'| + \left| \int_t^{t'} (t-s) f(s, x_{m-1}(s), x_{m-1}(\varphi(s))) ds \right| + \\ &+ \int_0^{t'} |f(s, x_{m-1}(s), x_{m-1}(\varphi(s)))| \cdot |t - t'| ds \leq (|v_0| + 2aM_0) \cdot |t - t'| \end{aligned} \quad (4)$$

for all  $t, t' \in [0, a]$ ,  $m \in \mathbb{N}^*$ . Moreover, since  $x_m \in C^2[0, a]$  follows that  $x_m'$  is Lipschitzian with the Lipschitz constant,

$$\max\{|x_m''(t)| : t \in [0, a]\} = \max\{|f(t, x_{m-1}(t), x_{m-1}(\varphi(t)))| : t \in [0, a]\} \leq M_0.$$

Consider the functions  $F_m$  given by  $F_m : [0, a] \rightarrow \mathbb{R}$ ,

$$F_m(t) = f(t, x_m(t), x_m(\varphi(t))), \quad m \in \mathbb{N}.$$

It is easy to see that the functions  $x_m''$  and  $F_m$  have the same properties.

In the aim to compute the terms of the sequence of successive approximations consider the uniform partition of  $[0, a]$  given by the knots  $t_i = \frac{i \cdot a}{n}$ ,  $i = \overline{0, n}$ . On these knots, the relations (2) became

$$x_m(t_i) = x_0 + v_0 t_i + \int_0^{t_i} (t_i - s) f(s, x_{m-1}(s), x_{m-1}(\varphi(s))) ds, \quad i = \overline{0, n}, \quad m \in \mathbb{N}^*. \quad (5)$$

Let the functions  $G_{m,i} : [0, a] \rightarrow \mathbb{R}$ ,  $i = \overline{0, n}$ ,  $m \in \mathbb{N}$ , given by  $G_{m,i}(s) = (t_i - s) \cdot f(s, x_m(s), x_m(\varphi(s)))$ .

**Proposition 1** Under the conditions (i)-(v) the functions  $x_m''$  and  $F_m$ ,  $m \in \mathbb{N}^*$  are Lipschitzian with the same Lipschitz constant  $\bar{L} = \gamma + (|v_0| + 2aM_0)(L_1 + \alpha L_2)$ . Moreover, the functions  $G_{m,i}$ ,  $i = \overline{0, n}$ ,  $m \in \mathbb{N}$ , are Lipschitzian with the same constant  $L = a\bar{L} + M_0$ .

**Proof.** Let  $t, t' \in [0, a]$ . We have,  $|F_0(t) - F_0(t')| \leq \gamma |t - t'|$  and

$$\begin{aligned} |F_m(t) - F_m(t')| &\leq \gamma |t - t'| + L_1 |x_m(t) - x_m(t')| + L_2 |x_m(\varphi(t)) - x_m(\varphi(t'))| \leq \\ &\leq [\gamma + L_1(|v_0| + 2aM_0) \cdot |t - t'| + \alpha L_2(|v_0| + 2aM_0)] \cdot |t - t'| \end{aligned}$$

for all  $t, t' \in [0, a]$ ,  $m \in \mathbb{N}^*$ . On the other hand,

$$|x_1''(t) - x_1''(t')| = |f(t, x_0(t), x_0(\varphi(t))) - f(t', x_0(t'), x_0(\varphi(t')))| \leq \gamma |t - t'|$$

and

$$|x_m''(t) - x_m''(t')| \leq [\gamma + (|v_0| + 2aM_0)(L_1 + \alpha L_2)] |t - t'|,$$

for  $m \in \mathbb{N}^*$ . We see that

$$\begin{aligned} |G_{m,i}(t) - G_{m,i}(t')| &= |(t_i - t)F_m(t) - (t_i - t')F_m(t')| \leq \\ &\leq |t_i - t| \cdot |F_m(t) - F_m(t')| + |F_m(t')| \cdot |t - t'| \leq a\bar{L} |t - t'| + M_0 |t - t'| \end{aligned}$$

for all  $t, t' \in [0, a]$ ,  $i = \overline{0, n}$  and  $m \in \mathbb{N}^*$ . ■

To compute the integrals from (5) we apply the trapezoidal quadrature rule with recent remainder estimation obtained in [3] for Lipschitzian functions:

$$\int_a^b F(t) dt = \frac{(b-a)}{2n} \cdot \left[ \sum_{i=0}^{n-1} F\left(a + \frac{i(b-a)}{n}\right) + F\left(a + \frac{(i+1)(b-a)}{n}\right) \right] + R_n(F) \quad (6)$$

$$|R_n(F)| \leq \frac{L(b-a)^2}{4n} \quad (7)$$

where  $L > 0$  is the Lipschitz constant of  $F$ .

Applying the quadrature rule (6)-(7) to the integrals from (5) we obtain the following numerical method:  $x_m(t_0) = x_0$ , for all  $m \in \mathbb{N}^*$ ,

$$x_0(t_i) = x_0, \quad \text{for all } i = \overline{0, n} \quad (8)$$

$$\begin{aligned} x_m(t_i) &= x_0 + v_0 t_i + \int_0^{t_i} (t_i - s) f(s, x_{m-1}(s), x_{m-1}(\varphi(s))) ds = x_0 + v_0 t_i + \\ &+ \int_0^{t_i} G_{m-1,i}(s) ds = x_0 + v_0 t_i + \frac{a}{2n} \cdot \sum_{j=0}^{i-1} [(t_i - t_j) \cdot f(t_j, x_{m-1}(t_j), x_{m-1}(\varphi(t_j))) + \\ &+ (t_i - t_{j+1}) \cdot f(t_{j+1}, x_{m-1}(t_{j+1}), x_{m-1}(\varphi(t_{j+1})))] + R_{m,i}, \end{aligned} \quad (9)$$

for all  $i = \overline{1, n}$  and  $m \in \mathbb{N}^*$ .

Since the functions  $G_{m,i}$ ,  $i = \overline{0, n}$ ,  $m \in \mathbb{N}$ , are Lipschitzian with the same constant  $L = a\bar{L} + M_0$ , for the remainder estimation in (9) we have

$$|R_{m,i}| \leq \frac{La^2}{4n}, \quad \text{for all } i = \overline{1, n}, \quad m \in \mathbb{N}^*. \quad (10)$$

### 3 The algorithm

The relations (8)-(9) lead to the following algorithm:

$$x_0(t_i) = x_0, \quad \text{for all } i = \overline{0, n} \quad \text{and} \quad x_1(t_0) = x_0 \quad (11)$$

$$\begin{aligned} x_1(t_i) &= x_0 + v_0 t_i + \frac{a}{2n} \cdot \sum_{j=0}^{i-1} [(t_i - t_j) \cdot f(t_j, x_0, x_0) + (t_i - t_{j+1}) \cdot f(t_{j+1}, x_0, x_0)] + \\ &+ R_{1,i} = \overline{x_1(t_i)} + R_{1,i}, \quad \text{for all } i = \overline{1, n}, \end{aligned} \quad (12)$$

$x_2(t_0) = x_0$  and

$$\begin{aligned} x_2(t_i) &= x_0 + v_0 t_i + \frac{a}{2n} \cdot \sum_{j=0}^{i-1} [(t_i - t_j) \cdot f(t_j, \overline{x_1(t_j)} + R_{1,j}, x_1(\varphi(t_j))) + \\ &+ (t_i - t_{j+1}) \cdot f(t_{j+1}, \overline{x_1(t_{j+1})} + R_{1,j+1}, x_1(\varphi(t_{j+1})))] + R_{2,i} = x_0 + v_0 t_i + \\ &+ \frac{a}{2n} \cdot \sum_{j=0}^{i-1} [(t_i - t_j) \cdot f(t_j, \overline{x_1(t_j)}, s_1(\varphi(t_j))) + \\ &+ (t_i - t_{j+1}) \cdot f(t_{j+1}, \overline{x_1(t_{j+1})}, s_1(\varphi(t_{j+1})))] + \overline{R_{2,i}} = \overline{x_2(t_i)} + \overline{R_{2,i}} \end{aligned} \quad (13)$$

for all  $i = \overline{1, n}$ , where  $s_1 : [0, a] \rightarrow \mathbb{R}$ , is the cubic spline generated by initial conditions, inspired by the construction from [5], which interpolates the values  $x_0, x_1(t_i), i = \overline{1, n}$  and has the restrictions to the intervals  $[t_{i-1}, t_i], i = \overline{1, n}$ ,

$$s_1^{(1)}(t) = \frac{M_1^{(1)} - M_1^{(0)}}{6(t_1 - t_0)} \cdot (t - t_0)^3 + \frac{M_1^{(0)}}{2} \cdot (t - t_0)^2 + z_1^{(0)}(t - t_0) + x_0, \quad (14)$$

$t \in [t_0, t_1]$ ,

$$s_1^{(i)}(t) = \frac{M_1^{(i)} - M_1^{(i-1)}}{6(t_i - t_{i-1})} \cdot (t - t_{i-1})^3 + \frac{M_1^{(i-1)}}{2} \cdot (t - t_{i-1})^2 + z_1^{(i-1)}(t - t_{i-1}) + \overline{x_1(t_{i-1})}, \quad (15)$$

$t \in [t_{i-1}, t_i]$ ,  $i = \overline{2, n}$ . Here, the parameters  $M_1^{(i)}, z_1^{(i)}, i = \overline{1, n}$  are recurrent given in:

$$\begin{cases} z_1^{(1)} = \frac{3}{h_1} \cdot [\overline{x_1(t_1)} - x_0] - 2z_1^{(0)} - \frac{h_1}{2} \cdot M_1^{(0)} \\ M_1^{(1)} = \frac{6}{h_1^2} \cdot [\overline{x_1(t_1)} - x_0] - \frac{6}{h_1} \cdot z_1^{(0)} - 2M_1^{(0)} \end{cases} \quad (16)$$

$$\begin{cases} z_1^{(i)} = \frac{3}{h_i} \cdot [\overline{x_1(t_i)} - \overline{x_1(t_{i-1})}] - 2z_1^{(i-1)} - \frac{h_i}{2} \cdot M_1^{(i-1)} \\ M_1^{(i)} = \frac{6}{h_i^2} \cdot [\overline{x_1(t_i)} - \overline{x_1(t_{i-1})}] - \frac{6}{h_i} \cdot z_1^{(i-1)} - 2M_1^{(i-1)} \end{cases}, i = \overline{2, n} \quad (17)$$

starting from  $z_1^{(0)} = v_0$ ,  $M_1^{(0)} = f(0, x_0, x_0)$ . By induction, for  $m \geq 3$ , we obtain

$$\begin{aligned} x_m(t_i) &= x_0 + v_0 t_i + \frac{a}{2n} \cdot \sum_{j=0}^{i-1} [(t_i - t_j) \cdot f(t_j, \overline{x_{m-1}(t_j)} + \overline{R_{m-1,j}}, x_{m-1}(\varphi(t_j))) + \\ &+ (t_i - t_{j+1}) \cdot f(t_{j+1}, \overline{x_{m-1}(t_{j+1})} + \overline{R_{m-1,j+1}}, x_{m-1}(\varphi(t_{j+1}))) + R_{m,i} = x_0 + v_0 t_i + \\ &+ \frac{a}{2n} \cdot \sum_{j=0}^{i-1} [(t_i - t_j) \cdot f(t_j, \overline{x_{m-1}(t_j)}, s_{m-1}(\varphi(t_j))) + \\ &+ (t_i - t_{j+1}) \cdot f(t_{j+1}, \overline{x_{m-1}(t_{j+1})}, s_{m-1}(\varphi(t_{j+1}))) + \overline{R_{m,i}} = \overline{x_m(t_i)} + \overline{R_{m,i}} \end{aligned} \quad (18)$$

for all  $i = \overline{1, n}$ , where  $s_{m-1} : [0, a] \rightarrow \mathbb{R}$ , is the cubic spline generated by initial conditions, interpolating the values  $x_0, x_{m-1}(t_i), i = \overline{1, n}$  and having the restrictions to the intervals  $[t_{i-1}, t_i], i = \overline{1, n}$

$$s_{m-1}^{(1)}(t) = \frac{M_{m-1}^{(1)} - M_{m-1}^{(0)}}{6(t_1 - t_0)} \cdot (t - t_0)^3 + \frac{M_{m-1}^{(0)}}{2} \cdot (t - t_0)^2 + z_{m-1}^{(0)}(t - t_0) + x_0, \quad (19)$$

$t \in [t_0, t_1]$ ,

$$\begin{aligned} s_{m-1}^{(i)}(t) &= \frac{M_{m-1}^{(i)} - M_{m-1}^{(i-1)}}{6(t_i - t_{i-1})} \cdot (t - t_{i-1})^3 + \frac{M_{m-1}^{(i-1)}}{2} \cdot (t - t_{i-1})^2 + z_{m-1}^{(i-1)}(t - t_{i-1}) + \\ &+ \overline{x_{m-1}(t_{i-1})}, \quad t \in [t_{i-1}, t_i], \quad i = \overline{2, n}. \end{aligned} \quad (20)$$

Here, the parameters  $M_{m-1}^{(i)}, z_{m-1}^{(i)}, i = \overline{1, n}$  are recurrently given in:

$$\begin{cases} z_{m-1}^{(1)} = \frac{3}{h_1} \cdot [\overline{x_{m-1}(t_1)} - x_0] - 2z_{m-1}^{(0)} - \frac{h_1}{2} \cdot M_{m-1}^{(0)} \\ M_{m-1}^{(1)} = \frac{6}{h_1^2} \cdot [\overline{x_{m-1}(t_1)} - x_0] - \frac{6}{h_1} \cdot z_{m-1}^{(0)} - 2M_{m-1}^{(0)} \end{cases} \quad (21)$$

$$\begin{cases} z_{m-1}^{(i)} = \frac{3}{h_i} \cdot [\overline{x_{m-1}(t_i)} - \overline{x_{m-1}(t_{i-1})}] - 2z_{m-1}^{(i-1)} - \frac{h_i}{2} \cdot M_{m-1}^{(i-1)} \\ M_{m-1}^{(i)} = \frac{6}{h_i^2} \cdot [\overline{x_{m-1}(t_i)} - \overline{x_{m-1}(t_{i-1})}] - \frac{6}{h_i} \cdot z_{m-1}^{(i-1)} - 2M_{m-1}^{(i-1)} \end{cases} \quad (22)$$

$i = \overline{2, n}, m \in \mathbb{N}^*, m \geq 3$ , starting from

$$z_{m-1}^{(0)} = v_0, \quad M_{m-1}^{(0)} = f(0, x_0, x_0). \quad (23)$$

## 4 The main result

Firstly, we get a new result about the error estimation in spline approximation, which held in the interpolation of functions with Lipschitzian first derivative.

**Lemma 2** If  $f : [a, b] \rightarrow \mathbb{R}$  is a continuous function such that  $f(t_i) = y_i, i = \overline{0, n}$  and its restrictions to the intervals  $[t_{i-1}, t_i], i = \overline{1, n}$  are differentiable on  $(t_{i-1}, t_i)$  being Lipschitzian together with their first derivatives, then for any cubic spline  $s$  of interpolation of the values  $y_i, i = \overline{0, n}$  we have the following error estimations:

$$\|f - s\|_C \leq (L' + N(s)) \cdot h^2$$

and

$$|f'(t) - s'(t)| \leq (L' + N(s)) \cdot h$$

for all  $t \in [a, b] \setminus \{t_0, \dots, t_n\}$ , where  $N(s) = \max\{|M_i| : i = \overline{0, n}\}, M_i = s''(t_i), i = \overline{0, n}, L' = \max\{L_i : i = \overline{1, n}\}, h = \max\{h_i : i = \overline{1, n}\}$ . Here,  $L_i, i = \overline{1, n}$  are the Lipschitz constants of the derivatives of the restrictions of  $f$  to the intervals  $[t_{i-1}, t_i], i = \overline{1, n}$ .

**Proof.** If  $s \in C^2[a, b]$  is cubic spline then the Lipschitz constant of  $s'$  is  $N(s) = \|s''\|_C = \max\{|M_i| : i = \overline{0, n}\}$ . Let  $f_i, i = \overline{1, n}$  be the restrictions of  $f$  to the intervals  $[t_{i-1}, t_i], i = \overline{1, n}$ . Consider  $\varphi = f - s$  and let  $\varphi_i, i = \overline{1, n}$  be the restrictions of  $\varphi$  to the intervals  $[t_{i-1}, t_i], i = \overline{1, n}$ . We have  $\varphi \in C[a, b]$  and  $\varphi_i \in C^1(t_{i-1}, t_i), \forall i = \overline{1, n}$ . Moreover,  $\varphi(t_i) = f(t_i) - s(t_i) = 0$  for all  $i = \overline{0, n}$ , and according to the Rolle's theorem we infer that exists  $\xi_i \in (t_{i-1}, t_i)$  such that  $\varphi'_i(\xi_i) = 0, \forall i = \overline{1, n}$ . Therefore,  $f'_i(\xi_i) = s'_i(\xi_i)$  for all  $i = \overline{1, n}$ . Let arbitrary  $t \in [a, b]$ . Then there is  $j \in \{1, \dots, n\}$  such that  $t \in [t_{j-1}, t_j]$ . We get

$$\begin{aligned} |f'(t) - s'(t)| &\leq |f'_j(t) - f'_j(\xi_j)| + |s'_j(\xi_j) - s'_j(t)| \leq \\ &\leq (L_j + N(s)) |t - \xi_j| \leq (L' + N(s)) \cdot h \end{aligned}$$

and

$$\begin{aligned} |f(t) - s(t)| &= |f(t) - s(t) - (f(t_{j-1}) - s(t_{j-1}))| \leq \\ &\leq \int_{t_{j-1}}^t |f'_j(u) - s'_j(u)| du \leq (L_j + N(s)) \cdot h_j^2 \leq (L' + N(s)) \cdot h^2. \end{aligned}$$

■

#### 4.1 The error estimation

**Theorem 3** Under the conditions (i)-(v), if  $(L_1 + 2L_2)a^2 < 1$ , then the unique solution of the initial value problem (1),  $x^*$  is approximated on the knots  $t_i = \frac{i \cdot a}{n}, i = \overline{1, n}$  by the sequence  $(\overline{x_m(t_i)})_m$  given in (4)-(6), (13) and the apriori error estimation is:

$$\begin{aligned} |x^*(t_i) - \overline{x_m(t_i)}| &\leq \frac{a^{2m} (L_1 + L_2)^m}{1 - a^2 (L_1 + L_2)} \cdot (|v_0 a| + M_0 a^2) + \frac{a^2 L}{4n[1 - a^2 (L_1 + 2L_2)]} + \\ &+ \frac{a^2 L_2}{1 - a^2 (L_1 + 2L_2)} \cdot (M_0 + \max\{|M_k^{(i)}| : k = \overline{1, m-1}, i = \overline{0, n}\}) \cdot \frac{a^2}{n^2} \end{aligned} \quad (24)$$

for all  $m \in \mathbb{N}^*$  and  $i = \overline{1, n}$ .

**Proof.** For  $m \in \mathbb{N}^*$  and  $i = \overline{1, n}$ , we have  $|x^*(t_i) - \overline{x_m(t_i)}| \leq |x^*(t_i) - x_m(t_i)| + |x_m(t_i) - \overline{x_m(t_i)}|$ ,  $|x_m(t_i) - \overline{x_m(t_i)}| = |\overline{R_{m,i}}|$  and

$$|x^*(t_i) - x_m(t_i)| \leq \frac{a^{2m} (L_1 + L_2)^m}{1 - a^2 (L_1 + L_2)} \cdot (|v_0 a| + M_0 a^2), \quad (25)$$

according to Theorem 1. From (12) and (10), follows

$$|x_1(t_i) - \overline{x_1(t_i)}| = |R_{1,i}| \leq \frac{La^2}{4n}, \quad \forall i = \overline{1, n}.$$

We define the functions  $V_m$ ,  $m \in \mathbb{N}^*$ ,  $V_m : [0, a] \rightarrow \mathbb{R}$  given by its restrictions to the subintervals  $[t_{i-1}, t_i]$ ,  $i = \overline{1, n}$ , by

$$\begin{aligned} V_m(t) &= x_m(t) + [\overline{x_m(t_1)} - x_m(t_1)] \cdot \frac{t - t_0}{t_1 - t_0}, \quad t \in [t_0, t_1] \\ V_m(t) &= x_m(t) + [\overline{x_m(t_i)} - x_m(t_i)] \cdot \frac{t - t_{i-1}}{t_i - t_{i-1}} + [\overline{x_m(t_{i-1})} - x_m(t_{i-1})] \cdot \\ &\quad \cdot \frac{t_i - t}{t_i - t_{i-1}}, \quad t \in [t_{i-1}, t_i], i = \overline{2, n}. \end{aligned} \quad (26)$$

We see that  $V_m(t_0) = x_m(t_0) = x_0$  and  $V_m(t_i) = \overline{x_m(t_i)}$ ,  $V_m(t_{i-1}) = \overline{x_m(t_{i-1})}$  for all  $i = \overline{2, n}$  and  $m \in \mathbb{N}^*$ . So, the spline function  $s_{m-1}$  interpolates  $V_{m-1}$  on the knots  $t_i$ ,  $i = \overline{0, n}$ . Moreover, the restrictions of  $V_m$  given in (26) have the same properties as the terms of the sequence of successive approximations,  $x_m$  (that is, with Lipschitzian first derivative having the same Lipschitz constant  $M_0$ ). On the other hand, for the splines given in (14)-(17) and (19)-(23) we obtain,

$$|x_k(t) - s_k(t)| \leq |x_k(t) - V_k(t)| + |V_k(t) - s_k(t)|, \quad \forall t \in [0, a],$$

with

$$\begin{aligned} |V_k(t) - x_k(t)| &\leq \left| \overline{x_k(t_i)} - x_k(t_i) \right| \cdot \left| \frac{t - t_{i-1}}{t_i - t_{i-1}} \right| + \\ &+ \left| \overline{x_k(t_{i-1})} - x_k(t_{i-1}) \right| \cdot \left| \frac{t_i - t}{t_i - t_{i-1}} \right| \leq 2 \cdot \max(|\overline{R_{k,i}}|, |\overline{R_{k,i-1}}|) \end{aligned}$$

for all  $i = \overline{0, n-1}$  and  $k = \overline{1, m-1}$ . From Lemma 2 follows

$$\begin{aligned} |V_k(t) - s_k(t)| &\leq \left( M_0 + \max\{|M_k^{(i)}| : i = \overline{0, n}\} \right) \cdot \frac{a^2}{n^2} \leq \\ &\leq \left( M_0 + \max\{|M_k^{(i)}| : i = \overline{0, n}, k = \overline{1, m-1}\} \right) \cdot \frac{a^2}{n^2}, \quad \forall t \in [0, a], \forall m \in \mathbb{N}^*. \end{aligned} \quad (27)$$

Now, we proceed to estimate the remainders  $\overline{R_{m,i}}$ ,  $m \in \mathbb{N}^*$ ,  $m \geq 2$ ,  $i = \overline{1, n}$ . In this aim, firstly we get

$$\begin{aligned} |\overline{R_{2,i}}| &= \left| x_2(t_i) - \overline{x_2(t_i)} \right| \leq |R_{2,i}| + \frac{a}{2n} \cdot \sum_{j=0}^{i-1} [(t_i - t_j) \cdot (L_1 |R_{1,j}| + L_2(\omega_1 \cdot \frac{a^2}{n^2} + \\ &+ \max(2 |R_{1,j}|, 2 |R_{1,j+1}|)) + (t_i - t_{j+1}) \cdot (L_1 |R_{1,j+1}| + L_2(\omega_1 \cdot \frac{a^2}{n^2} + \max(2 |R_{1,j}|, 2 |R_{1,j+1}|)))] \leq \\ &\leq \frac{La^2}{4n} (1 + a^2(L_1 + 2L_2)) + a^2\omega_1 L_2 \cdot \frac{a^2}{n^2}, \quad \forall i = \overline{1, n} \end{aligned}$$

where  $\omega_1 = M_0 + \max\{|M_1^{(i)}| : i = \overline{0, n}\}$ .

Analogously,

$$\begin{aligned} |\overline{R_{3,i}}| &= \left| x_3(t_i) - \overline{x_3(t_i)} \right| \leq |R_{3,i}| + \frac{a}{2n} \cdot \sum_{j=0}^{i-1} [(t_i - t_j) \cdot (L_1 |\overline{R_{2,j}}| + L_2(\max(2 |\overline{R_{2,j}}|, \\ &, 2 |\overline{R_{2,j+1}}|) + \omega_2 \cdot \frac{a^2}{n^2})) + (t_i - t_{j+1}) \cdot (L_1 |\overline{R_{2,j+1}}| + L_2(\max(2 |\overline{R_{2,j}}|, 2 |\overline{R_{2,j+1}}|) + \omega_2 \cdot \frac{a^2}{n^2})))] \leq \\ &\leq \frac{La^2}{4n} \cdot [1 + a^2(L_1 + 2L_2) + (a^2(L_1 + 2L_2))^2] + a^2\omega_1 L_2 a^2 (L_1 + 2L_2) \cdot \frac{a^2}{n^2} + a^2 L_2 \omega_2 \cdot \frac{a^2}{n^2} \end{aligned}$$

where  $\omega_2 = M_0 + \max\{|M_2^{(i)}| : i = \overline{0, n}\}$ . Let  $\omega = \max(\omega_1, \omega_2)$ . Then

$$|\overline{R_{3,i}}| \leq [1 + a^2(L_1 + 2L_2) + a^4(L_1 + 2L_2)^2] \cdot \frac{La^2}{4n} +$$

$$+a^2L_2[1+a^2(L_1+2L_2)] \cdot \omega \frac{a^2}{n^2}, \quad \forall i = \overline{1, n}.$$

By induction for  $m \in \mathbb{N}^*, m \geq 3$  let  $\omega_k = M_0 + \max\{|M_k^{(i)}| : i = \overline{0, n}\}$ ,  $k = \overline{3, m-1}$  and  $\omega = \max\{\omega_1, \omega_2, \dots, \omega_{m-1}\}$ .

We get

$$\begin{aligned} |x_m(t_i) - \overline{x_m(t_i)}| &= |\overline{R_{m,i}}| \leq |R_{m,i}| + \frac{a}{2n} \left[ \sum_{j=0}^{i-1} [(t_i - t_j) \cdot (L_1 |\overline{R_{m-1,j}}| + \right. \\ &\quad \left. + L_2(2 \max(|\overline{R_{m-1,j}}|, |\overline{R_{m-1,j+1}}|) + \omega \cdot \frac{a^2}{n^2})) + \right. \\ &\quad \left. + (t_i - t_{j+1}) \cdot (L_1 |\overline{R_{m-1,j+1}}| + L_2(2 \max(|\overline{R_{m-1,j}}|, |\overline{R_{m-1,j+1}}|) + \omega \cdot \frac{a^2}{n^2})) \right] \leq \\ &\leq [1 + a^2(L_1 + 2L_2) + a^4(L_1 + 2L_2)^2 + \dots + (a^2)^{m-1}(L_1 + 2L_2)^{m-1}] \cdot \frac{La^2}{4n} + \\ &\quad + a^2L_2[1 + a^2(L_1 + 2L_2) + \dots + (a^2)^{m-2}(L_1 + 2L_2)^{m-2}] \cdot \frac{\omega a^2}{n^2}, \quad \forall i = \overline{1, n}. \end{aligned}$$

We can see that  $\omega = M_0 + \max\{|M_k^{(i)}| : i = \overline{0, n}, k = \overline{1, m-1}\}$ . Consequently,

$$\begin{aligned} |x_m(t_i) - \overline{x_m(t_i)}| &= |\overline{R_{m,i}}| \leq \frac{1 - a^{2m}(L_1 + 2L_2)^m}{1 - a^2(L_1 + 2L_2)} \cdot \frac{La^2}{4n} + \\ &\quad + a^2L_2 \frac{1 - a^{2(m-1)}(L_1 + 2L_2)^{m-1}}{1 - a^2(L_1 + 2L_2)} \cdot \frac{\omega a^2}{n^2}. \end{aligned}$$

Since  $a^2(L_1 + 2L_2) < 1$  we conclude

$$\begin{aligned} |x_m(t_i) - \overline{x_m(t_i)}| &= |\overline{R_{m,i}}| \leq \frac{La^2}{4n[1 - a^2(L_1 + 2L_2)]} + \\ &\quad + \frac{\omega a^4 L_2}{n^2[1 - a^2(L_1 + 2L_2)]} \stackrel{\text{notation}}{=} \Omega, \quad \forall i = \overline{1, n}, \forall m \in \mathbb{N}^*, m \geq 2 \end{aligned}$$

and

$$|x_1(t_i) - \overline{x_1(t_i)}| = |R_{1,i}| \leq \frac{La^2}{4n}, \quad \forall i = \overline{1, n}.$$

These complete the proof. ■

**Remark 2** From estimation (24) follows the convergence of the method and of its algorithm. Now, we can see that the same estimation can give a practical stopping criterion of the algorithm. This can be stated as follows: For given  $\varepsilon' > 0$  and given  $n \in \mathbb{N}^*$  (previously chosen) it determines the first natural number  $m \in \mathbb{N}^*$  for which,

$$|\overline{x_m(t_i)} - \overline{x_{m-1}(t_i)}| < \varepsilon' \quad \text{for all } i = \overline{1, n}$$

and we stop to this  $m$ , retaining the approximations  $\overline{x_m(t_i)}$ ,  $i = \overline{1, n}$ , of the solution. A demonstration of this criterion is the following:

For each  $i = \overline{1, n}$  we have

$$\begin{aligned} |x^*(t_i) - \overline{x_m(t_i)}| &\leq |x^*(t_i) - x_m(t_i)| + |x_m(t_i) - \overline{x_m(t_i)}| \leq \\ &\leq \frac{a^2(L_1 + L_2)}{1 - a^2(L_1 + L_2)} \cdot |x_m(t_i) - x_{m-1}(t_i)| + |\overline{R_{m,i}}| \end{aligned}$$



and

$$\begin{aligned} |x_m(t_i) - x_{m-1}(t_i)| &\leq |x_m(t_i) - \overline{x_m(t_i)}| + |\overline{x_m(t_i)} - \overline{x_{m-1}(t_i)}| + \\ &+ |\overline{x_{m-1}(t_i)} - x_{m-1}(t_i)| = |\overline{R_{m,i}}| + |\overline{R_{m-1,i}}| + |\overline{x_m(t_i)} - \overline{x_{m-1}(t_i)}|. \end{aligned}$$

So,

$$\begin{aligned} |x^*(t_i) - \overline{x_m(t_i)}| &\leq |\overline{R_{m,i}}| + \frac{a^2(L_1 + L_2)}{1 - a^2(L_1 + L_2)} \cdot |\overline{x_m(t_i)} - \overline{x_{m-1}(t_i)}| + \\ &+ \frac{a^2(L_1 + L_2)}{1 - a^2(L_1 + L_2)} \cdot (|\overline{R_{m,i}}| + |\overline{R_{m-1,i}}|). \end{aligned}$$

Then

$$|x^*(t_i) - \overline{x_m(t_i)}| \leq \Omega \cdot \frac{1 + a^2(L_1 + L_2)}{1 - a^2(L_1 + L_2)} + \frac{a^2(L_1 + L_2)}{1 - a^2(L_1 + L_2)} \cdot |\overline{x_m(t_i)} - \overline{x_{m-1}(t_i)}|.$$

For given  $\varepsilon > 0$  we require

$$\Omega \cdot \frac{1 + a^2(L_1 + L_2)}{1 - a^2(L_1 + L_2)} < \frac{\varepsilon}{2} \quad (28)$$

and

$$\frac{a^2(L_1 + L_2)}{1 - a^2(L_1 + L_2)} \cdot |\overline{x_m(t_i)} - \overline{x_{m-1}(t_i)}| < \frac{\varepsilon}{2}.$$

Since

$$\Omega < \frac{La^2}{4n[1 - a^2(L_1 + 2L_2)]} + \frac{\omega a^4 L_2}{4n[1 - a^2(L_1 + 2L_2)]}$$

for  $n \geq 5$ , we can chose the least natural number  $n$  for which the inequality (28) holds. Afterward we find the least natural number  $m$  (this is the last iterative step to be made) for which

$$|\overline{x_m(t_i)} - \overline{x_{m-1}(t_i)}| < \frac{\varepsilon}{2} \cdot \frac{1 - a^2(L_1 + L_2)}{a^2(L_1 + L_2)}$$

for all  $i = \overline{1, n}$ . With these, we obtain  $|x^*(t_i) - \overline{x_m(t_i)}| < \varepsilon$ , for all  $i = \overline{1, n}$ .

**Remark 3** It is easy to prove that if  $f \in C^1([0, a] \times \mathbb{R} \times \mathbb{R})$  has bounded partial derivatives  $\frac{\partial f}{\partial x}$ ,  $\frac{\partial f}{\partial y}$  such that all its first order partial derivatives are Lipschitzian in each argument, and if  $\varphi \in C^1[0, a]$  with Lipschitzian first derivative, then  $F_m \in C^1[0, a]$  and the functions  $F'_m$  are Lipschitzian with the same Lipschitz constant  $L'$ .

**Theorem 4** Under the conditions (i)-(v) and in the conditions mentioned in Remark 3, if  $a^2(L_1 + 2L_2) < 1$  then the unique solution  $x^*$ , of the initial value problem (1), is approximated on the knots  $t_i = \frac{i \cdot a}{n}$ ,  $i = \overline{1, n}$  by the sequence  $(\overline{x_m(t_i)})_m$  given in (4)-(6), (13) and the apriori error estimation is:

$$\begin{aligned} |x^*(t_i) - \overline{x_m(t_i)}| &\leq \frac{a^{2m}(L_1 + L_2)^m}{1 - a^2(L_1 + L_2)} \cdot (|v_0 a| + M_0 a^2) + \frac{a^3 L'}{12n^2[1 - a^2(L_1 + 2L_2)]} + \\ &+ \frac{a^2 L_2}{1 - a^2(L_1 + 2L_2)} \cdot (M_0 + \max\{|M_k^{(i)}| : k = \overline{1, m-1}, i = \overline{0, n}\}) \cdot \frac{a^2}{n^2} \end{aligned} \quad (29)$$

for all  $m \in \mathbb{N}^*$  and  $i = \overline{1, n}$ .

**Proof.** Is similar to the proof of Theorem 3, with the difference that instead of inequality (10) it is used the estimation

$$|R_{m,i}| \leq \frac{L' a^3}{12n^2}, \quad \text{for all } i = \overline{1, n}, \quad m \in \mathbb{N}^*. \quad (30)$$

The inequality (30) is obtained using the remainder estimation in (6)

$$|R_n(F)| \leq \frac{L'(b-a)^3}{12n^2}$$

which holds for functions with Lipschitzian first derivative (see [1]). Here,  $L'$  is the Lipschitz constant of  $F'$ . ■

## 4.2 Numerical examples

**Example 5** Firstly, we consider the initial value problem (1) with  $\varphi(t) = \lambda t$ ,  $a = \frac{1}{2}$ ,  $\lambda = \frac{1}{2}$ ,  $x_0 = 1$  and  $f(s, u, v) = \frac{2u}{3} + \frac{v}{3} \cdot e^{s/2}$  which has the exact solution  $x^*(t) = e^t$ . The stopping condition is  $\left| \overline{x_m(t_i)} - \overline{x_{m-1}(t_i)} \right| < \varepsilon'$ , for all  $i = \overline{1, n}$ . On this example, the condition of convergence  $(L_1 + 2L_2)a^2 = \frac{1+\sqrt[4]{e}}{6} < 1$  is fulfilled. Using the algorithm with  $\varepsilon' = 10^{-5}$ ,  $n = 10$ , we get  $m = 4$ , the number of iterations, and the approximations  $\overline{x_m(t_i)}$  of the exact solution on the corresponding knots (which can be compared with the exact values of  $e^{t_i}$ ) are in Table 1. We observe a very good accuracy and the order of convergence stated in Theorem 2 is confirmed to be  $O(h^2)$ , where  $h = \frac{a}{n}$ . Moreover, on the interval  $[0, 0.3]$  the accuracy is better, having a convergence order  $O(h^4)$  on this interval. If we take  $\varepsilon' = 10^{-9}$  and  $n = 10$ , we get  $m = 7$ , and the values are represented in Table 2 (we can observe a similar accuracy as in Table 1).

$t_i$	$\overline{x_m(t_i)}$	$x^*(t_i) = e^{t_i}$	$t_i$	$\overline{x_m(t_i)}$	$x^*(t_i) = e^{t_i}$
0.05	1.0512500	1.05127109637	0.3	1.34946602	1.3498588075
0.1	1.1051273	1.10517091807	0.35	1.41811653	1.4190675485
0.15	1.1617571	1.16183424272	0.4	1.49342171	1.4918246976
0.2	1.2213094	1.22114027581	0.45	1.56962813	1.5683121854
0.25	1.2839754	1.28402541668	0.5	1.63131058	1.6487212707

Table 1

$t_i$	$\overline{x_m(t_i)}$	$x^*(t_i) = e^{t_i}$	$t_i$	$\overline{x_m(t_i)}$	$x^*(t_i) = e^{t_i}$
0.05	1.0512500000	1.05127109637	0.3	1.34946602403	1.3498588075
0.1	1.1051273393	1.10517091807	0.35	1.41811653455	1.4190675485
0.15	1.1617571992	1.16183424272	0.4	1.49342174410	1.4918246976
0.2	1.2213094121	1.22114027581	0.45	1.56962803549	1.5683121854
0.25	1.2839754824	1.28402541668	0.5	1.63131062924	1.6487212707

Table 2

**Example 6** Consider the initial value problem

$$\begin{cases} x''(t) = \left| \frac{1}{3} \cdot \sin t \cdot \cos(x(t)) + \frac{1}{5} \cdot \cos t \cdot \sin(x(0.5t)) \right|, & t \in [0, 1] \\ x(0) = 1, & x'(0) = 0.5 \end{cases}$$

which is nonlinear and the kernel function cannot be differentiable. On the other hand, this function is continuous and Lipschitzian in each argument having the Lipschitz constants  $\gamma = \frac{8}{15}$ ,  $L_1 = \frac{1}{3}$ ,  $L_2 = \frac{1}{5}$  and the convergence condition is  $(L_1 + 2L_2)a^2 = \frac{11}{15} < 1$ . For this example we cannot use the well-known methods (Runge-Kutta, collocation, splines) because these methods require, in the proof of convergence, high order of smoothness properties for the kernel function. But here, we can use the above presented method, which according to the previous example, has good accuracy. The stopping criterion is similar to the previous example. For  $\varepsilon' = 10^{-5}$ ,  $n = 10$  we get  $m = 2$  and the results are in Table 3.

$t_i$	$\overline{x_m(t_i)}$	$t_i$	$\overline{x_m(t_i)}$
0.1	1.05	0.6	1.30
0.2	1.10	0.7	1.35
0.3	1.15	0.8	1.40
0.4	1.20	0.9	1.45
0.5	1.25	1	1.50

Table 3

## 5 Conclusions

A new numerical method for second order delay differential equations is presented. This method combines the Picard sequence of successive approximations, the trapezoidal quadrature rule and spline interpolation. The interpolation is used only on the modified argument. The method's algorithm has recurrent form easy to programming and a practical stopping criterion of the algorithm is obtained using the error estimate. The main result of the paper is Theorem 3.

The method is created in the aim to be an alternative to the well-known methods (Runge-Kutta, collocation, spline functions method) in the cases in which these methods are not applicable (when the kernel function are not smooth). The above presented method is convergent even in the case of Lipschitzian kernel function and such example is presented (in Example 2). Consequently, the presented method covers the situations unsolved by other methods. The accuracy of the method is illustrated in Example 1.

The principle of the method (the use, in numerical integration, of an interpolation procedure only on the points where the argument is modified) gives its generality, being extensible to other types of functional equations with modified argument.

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# Rate of Convergence of the Integrated MKZ-Bézier Operators

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**Abstract** In this paper the pointwise approximation of the integrated MKZ-Bézier operators for bounded functions is studied for the case  $0 < \alpha < 1$ . By using the method of [*J.Approx.Theory* 104(2000) 330-344] and some analysis techniques, we obtain an estimate formula on this bézier type operators. Our results extend the work of Zeng [*Comput.Math.Appl.* 39(2000) 1-13].

**Keywords:** MKZ-Bézier operators, Rate of convergence, Approximation, Bounded functions, Lebesgue-Stieltjes integral

**Classification(MSC 2000):** 41A30, 41A35, 41A36, 41A60

## 1. Introduction

For a function  $f \in L_1[0, 1]$  and  $\alpha > 0$ , the integrated MKZ-Bézier operators are defined as

$$M_{n,\alpha}(f, x) = \sum_{k=0}^{\infty} Q_{nk}^{(\alpha)}(x) \frac{(n+k)(n+k+1)}{n} \int_{I_k} f(t) dt, \quad (1)$$

where  $Q_{nk}^{(\alpha)}(x) = (J_{n,k}(x))^{\alpha} - (J_{n,k+1}(x))^{\alpha}$ ,  $J_{n,k}(x) = \sum_{j=k}^{\infty} m_{nj}(x)$ , and

$$I_k = [\frac{k}{n+k}, \frac{k+1}{n+k+1}], m_{nj}(x) = \binom{n+j-1}{j} x^j (1-x)^n, j = 0, 1, 2, \dots$$

The approximation of bounded variation functions with the operators  $M_{n,\alpha}$  for the case  $\alpha \geq 1$  was studied by Zeng [1]. Some related investigation about this bézier type operators, we can see Refs. [2-4]. In this paper, we discuss the pointwise approximation of  $M_{n,\alpha}$  for the case  $0 < \alpha < 1$ . Our results extend the work of Zeng.

The main theorem of this paper is as follows:

**Theorem 1** Let  $f$  be a bounded function on  $[0, 1]$ ,  $0 < \alpha < 1$ , and  $f(x+), f(x-)$  exist at a fixed point  $x \in (0, 1)$ , then for  $n$  sufficiently large, we have

$$\begin{aligned} & \left| M_{n,\alpha}(f, x) - \frac{1}{2^{\alpha}} f(x+) - (1 - \frac{1}{2^{\alpha}}) f(x-) \right| \\ & \leq \frac{6}{\sqrt{nx} + 1} |f(x+) - f(x-)| + \frac{5 + 4C}{nx(1-x)} \sum_{k=1}^n \Omega_x(g_x, 1/\sqrt{k}) \end{aligned} \quad (2)$$

where  $C$  is a positive constant,  $\Omega_x(f, \lambda) = \sup_{t \in [x-\lambda, x+\lambda]} |f(t) - f(x)|$  and

$$g_x(t) = \begin{cases} f(t) - f(x+), & x < t \leq 1, \\ 0, & t = x, \\ f(t) - f(x-), & 0 \leq t < x. \end{cases} \quad (3)$$

For further properties of  $\Omega_x(f, \lambda)$ , we refer the readers to [10].

Let

$$K_{n,\alpha}(x, t) = \sum_{k=0}^{\infty} Q_{nk}^{(\alpha)}(x) \chi_k(t) |I_k|^{-1}, \quad (4)$$

and

$$\lambda_{n,\alpha}(x, t) = \int_0^t K_{n,\alpha}(x, u) du, \quad (5)$$

where  $\chi_k$  is the characteristic function of the interval  $I_k = [k/(n+k), (k+1)/(n+k+1)]$  with respect to  $I = [0, 1]$ , and  $|I_k|$  denotes the length of  $I_k$ .

Then by Lebesgue-Stieltjes integral representations, there holds

$$M_{n,\alpha}(f, x) = \int_0^1 f(t) K_{n,\alpha}(x, t) dt = \int_0^1 f(t) d_t \lambda_{n,\alpha}(x, t). \quad (6)$$

## 2. Auxiliary results

The proof of our result is based on the following lemmas.

**Lemma 1.** [11, Theorem 2] *For every  $k \in N, x \in (0, 1]$ , we have*

$$m_{nk}(x) < \frac{1}{\sqrt{2enx}}. \quad (7)$$

**Lemma 2.** *Let  $0 < \alpha \leq 1$  and  $x \in (0, 1)$ , as  $n > 144/x$  and  $k' = [nx/(1-x)]$ , we have*

$$(i) \quad \left| \left( \sum_{k > nx/(1-x)} m_{nk}(x) \right)^\alpha - \frac{1}{2^\alpha} \right| \leq \frac{4}{\sqrt{nx} + 1}, \quad (8)$$

$$(ii) \quad Q_{nk'}^{(\alpha)}(x) < \frac{2}{\sqrt{nx} + 1}. \quad (9)$$

**Proof.** (i) By mean value theorem, we have

$$\left| \left( \sum_{k > nx/(1-x)} m_{nk}(x) \right)^\alpha - \frac{1}{2^\alpha} \right| = \alpha (\xi_{nk}(x))^{\alpha-1} \left| \sum_{k > nx/(1-x)} m_{nk}(x) - \frac{1}{2} \right|, \quad (10)$$

where  $\xi_{nk}(x)$  lies between  $\frac{1}{2}$  and  $\sum_{k > nx/(1-x)} m_{nk}(x)$ .

From [1, Lemma 4], there holds

$$\left| \sum_{k > nx/(1-x)} m_{nk}(x) - \frac{1}{2} \right| \leq \frac{3}{\sqrt{nx} + 1}. \quad (11)$$

Thus

$$\xi_{nk}(x) > \sum_{k > nx/(1-x)} m_{nk}(x) > \frac{1}{4} \quad (12)$$

for all  $n > 144/x$ .

From (10),(11),(12) and noting  $3\alpha < 4^\alpha$ , we get

$$\left| \left( \sum_{k > nx/(1-x)} m_{nk}(x) \right)^\alpha - \frac{1}{2^\alpha} \right| \leq \frac{\alpha}{4^{\alpha-1}} \frac{3}{\sqrt{nx} + 1} < \frac{4}{\sqrt{nx} + 1}.$$

(ii) Noting the expression of  $Q_{nk'}^{(\alpha)}(x)$ , and along the same method of (i), we can get

$$Q_{nk'}^{(\alpha)}(x) \leq \alpha 4^{1-\alpha} m_{n,k'}(x) \leq \frac{\alpha 4^{1-\alpha}}{\sqrt{2enx}} < \frac{1}{\sqrt{nx}}. \quad (13)$$

The second inequality of (13) is obtained from Lemma 1.

Since  $Q_{nk'}^{(\alpha)}(x) \leq 1$ , (9) is proved.

**Lemma 3.** For  $0 < \alpha \leq 1$  and  $0 \leq t < x < 1$ , there holds

$$\lambda_{n,\alpha}(x, t) = \int_0^t K_{n,\alpha}(x, u) du \leq \int_0^t K_{n,1}(x, u) du \leq \frac{2x(1-x)^2}{n(x-t)^2}. \quad (14)$$

**Proof.** Let  $t \in [k^*/(n+k^*), (k^*+1)/(n+k^*+1))$ . Then we can write  $t = [(n+k^*+1)k^* + \delta]/(n+k^*)(n+k^*+1)$  ( $0 \leq \delta < n$ ). So

$$\begin{aligned} \int_0^t K_{n,\alpha}(x, u) du &= \int_0^t \sum_{k=0}^{\infty} Q_{nk}^{(\alpha)}(x) \chi_k(u) |I_k|^{-1} du = \sum_{k=0}^{\infty} Q_{nk}^{(\alpha)}(x) |I_k|^{-1} \int_0^t \chi_k(u) du \\ &= \sum_{k=0}^{k^*-1} Q_{nk}^{(\alpha)}(x) + |I_{k^*}|^{-1} Q_{nk^*}^{(\alpha)}(x) \int_{k^*/(n+k^*)}^{[(n+k^*+1)k^*+\delta]/(n+k^*)(n+k^*+1)} 1 du \\ &= \sum_{k=0}^{k^*-1} Q_{nk}^{(\alpha)}(x) + \delta Q_{nk^*}^{(\alpha)}(x) = 1 - (1-\delta)(J_{n,k^*}(x))^\alpha - \delta(J_{n,k^*+1}(x))^\alpha \\ &\leq 1 - (1-\delta)J_{n,k^*}(x) - \delta J_{n,k^*+1}(x) = \int_0^t K_{n,1}(x, u) du. \end{aligned}$$

The right hand inequality of (14) follows from [1, Lemma 6]. The proof is complete.

**Lemma 4.** For  $0 < \alpha \leq 1$  and  $0 \leq x < t < 1$ , there holds

$$1 - \lambda_{n,\alpha}(x, t) = 1 - \int_0^t K_{n,\alpha}(x, u) du \leq \frac{Cx^{1-\alpha}(1-x)^{2-\alpha}}{n(t-x)^2}, \quad (15)$$

where  $C$  is a positive constant.

**Proof.**

$$\begin{aligned} 1 - \int_0^t K_{n,\alpha}(x, u) du &= \int_t^1 K_{n,\alpha}(x, u) du = \sum_{k=0}^{\infty} Q_{nk}^{(\alpha)}(x) |I_k|^{-1} \int_t^1 \chi_k(u) du \\ &\leq \sum_{t \leq \frac{k+1}{n+k+1}} Q_{nk}^{(\alpha)}(x) |I_k|^{-1} \int_{\frac{k}{n+k}}^{\frac{k+1}{n+k+1}} \chi_k(u) du \\ &= \sum_{t \leq \frac{k+1}{n+k+1}} Q_{nk}^{(\alpha)}(x) = \sum_{k \geq \frac{nt}{1-t}-1} (J_{n,k}^\alpha(x) - J_{n,k+1}^\alpha(x)) \\ &= \left( \sum_{k \geq \frac{nt}{1-t}-1} m_{nk}(x) \right)^\alpha = \left( \sum_{k \geq \frac{nt}{1-t}} m_{n,k-1}(x) \right)^\alpha \end{aligned}$$

$$\begin{aligned}
 &= \left( \sum_{k \geq \frac{nt}{1-t}} \frac{nk}{(n+k)(n+k-1)x(1-x)} \binom{n+k}{k} x^k (1-x)^{n+1} \right)^\alpha \\
 &\leq \frac{1}{(x(1-x))^\alpha} \left( \sum_{k \geq \frac{nt}{1-t}} M_{nk(x)} \right)^\alpha,
 \end{aligned}$$

where  $M_{nk(x)} = \binom{n+k}{k} x^k (1-x)^{n+1}$ .

Since  $0 \leq x < t < 1$ , so  $|\frac{k}{n+k} - x| \geq |t - x|$  for all  $k \geq \frac{nt}{1-t}$ . Thus we have

$$\begin{aligned}
 1 - \int_0^t K_{n,\alpha}(x, u) du &\leq \frac{1}{(x(1-x))^\alpha} \left( \sum_{k \geq \frac{nt}{1-t}} \frac{|\frac{k}{n+k} - x|^{2/\alpha}}{|t-x|^{2/\alpha}} M_{nk(x)} \right)^\alpha \\
 &\leq \frac{1}{(x(1-x))^\alpha (t-x)^2} \left( \sum_{k=0}^{\infty} \left| \frac{k}{n+k} - x \right|^{2/\alpha} M_{nk(x)} \right)^\alpha.
 \end{aligned}$$

we can choose  $p, q > 1$ , i.e.,  $\frac{1}{p} + \frac{1}{q} = 1$ . Then by the Hölder inequality, we have

$$\begin{aligned}
 \left( \sum_{k=0}^{\infty} \left| \frac{k}{n+k} - x \right|^{2/\alpha} M_{nk(x)} \right)^\alpha &= \left( \sum_{k=0}^{\infty} \left| \frac{k}{n+k} - x \right|^{2/\alpha} M_{nk}^{1/p}(x) M_{nk}^{1/q}(x) \right)^\alpha \\
 &\leq \left( \sum_{k=0}^{\infty} \left| \frac{k}{n+k} - x \right|^{2p/\alpha} M_{nk}(x) \right)^{\alpha/p}
 \end{aligned}$$

since  $(\sum_{k=0}^{\infty} M_{nk}(x))^\alpha = 1$ .

Choosing  $p = \alpha[1/\alpha + 1]$ , then  $2p/\alpha = 2[1/\alpha + 1]$  is a positive even integer. By [9, Theorem 2.1] and with a simple calculation, we obtain

$$\left( \sum_{k=0}^{\infty} \left| \frac{k}{n+k} - x \right|^{2p/\alpha} M_{nk}(x) \right)^{\alpha/p} \leq \frac{Cx(1-x)^2}{n}.$$

This completes the proof of Lemma 4.

**Lemma 5.** Let  $f$  be a bounded function on  $(0, 1)$ , when  $n$  sufficiently large, we have

$$|M_{n,\alpha}(g_x, x)| \leq \frac{5+4C}{nx(1-x)} \sum_{k=1}^n \Omega_x(g_x, 1/\sqrt{k}). \quad (16)$$

**Proof.** By (6), we have

$$M_{n,\alpha}(g_x, x) = \int_0^1 g_x(t) d_t \lambda_{n,\alpha}(x, t) = \Sigma_1 + \Sigma_2 + \Sigma_3, \quad (17)$$

where

$$\begin{aligned}
 \Sigma_1 &= \int_0^{x-\frac{x}{\sqrt{n}}} g_x(t) d_t \lambda_{n,\alpha}(x, t), \quad \Sigma_2 = \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{1-x}{\sqrt{n}}} g_x(t) d_t \lambda_{n,\alpha}(x, t), \\
 \Sigma_3 &= \int_{x+\frac{1-x}{\sqrt{n}}}^1 g_x(t) d_t \lambda_{n,\alpha}(x, t).
 \end{aligned}$$

Observing that  $g_x(x) = 0$ , we first have

$$|\Sigma_2| \leq \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{1-x}{\sqrt{n}}} |g_x(t) - g_x(x)| d_t \lambda_{n,\alpha}(x, t)$$

$$\leq \Omega_x(g_x, 1/\sqrt{n}) \leq \frac{1}{n} \sum_{k=1}^n \Omega_x(g_x, 1/\sqrt{k}). \quad (18)$$

To estimate  $\Sigma_1$ , let  $y = x - x/\sqrt{n}$  and using Lebesgue-Stieltjes integration by parts, we have

$$\begin{aligned} |\Sigma_1| &\leq \int_0^y \Omega_x(g_x, x-t) d_t \lambda_{n,\alpha}(x, t) \\ &= \Omega_x(g_x, x-y) \lambda_{n,\alpha}(x, y+) + \int_0^y \widehat{\lambda}_{n,\alpha}(x, t) d_t (-\Omega_x(g_x, x-t)), \end{aligned}$$

where  $\widehat{\lambda}_{n,\alpha}(x, t)$  is the normalized form of  $\lambda_{n,\alpha}(x, t)$ .

Also  $\widehat{\lambda}_{n,\alpha}(x, t) \leq \lambda_{n,\alpha}(x, t)$  on  $[0, 1]$ , by (14), it follows that

$$|\Sigma_1| \leq \Omega_x(g_x, x-y) \frac{2x(1-x)^2}{n(x-y)^2} + \frac{2x(1-x)^2}{n} \int_0^y \frac{1}{(x-t)^2} d_t (-\Omega_x(g_x, x-t)). \quad (19)$$

Since

$$\int_0^y \frac{1}{(x-t)^2} d_t (-\Omega_x(g_x, x-t)) = -\frac{\Omega_x(g_x, x-t)}{(x-t)^2} \Big|_0^{y+} + \int_0^y \frac{2\Omega_x(g_x, x-t)}{(x-t)^3} dt,$$

from (19) it follows that

$$|\Sigma_1| \leq \frac{2(1-x)^2}{nx} \Omega_x(g_x, x) + \frac{2x(1-x)^2}{n} \int_0^{x-\frac{x}{\sqrt{n}}} \frac{2\Omega_x(g_x, x-t)}{(x-t)^3} dt.$$

Putting  $t = x - x/\sqrt{u}$  for the last integral, we get

$$\begin{aligned} |\Sigma_1| &\leq \frac{2(1-x)^2}{nx} \Omega_x(g_x, x) + \frac{2(1-x)^2}{nx} \int_1^n \Omega_x(g_x, x/\sqrt{u}) du \\ &\leq \frac{4(1-x)^2}{nx} \sum_{k=1}^n \Omega_x(g_x, 1/\sqrt{k}). \end{aligned} \quad (20)$$

Using the similar method to estimate  $|\Sigma_3|$ , we obtain

$$|\Sigma_3| \leq \frac{2C(1-x)^{2-\alpha}}{nx^{1+\alpha}} \sum_{k=1}^n \Omega_x(g_x, 1/\sqrt{k}). \quad (21)$$

Combining the estimates of (17), (18), (20) and (21), also noting the properties of  $\Omega_x(f, \lambda)$ , we get the required result.

### 3. Proof of Theorem 1

Let  $f$  satisfy the conditions of Theorem 1, we can decompose  $f(t)$  into four parts as

$$\begin{aligned} f(t) &= \frac{1}{2^\alpha} f(x+) + (1 - \frac{1}{2^\alpha}) f(x-) + g_x(t) + \frac{f(x+) - f(x-)}{2^\alpha} \widehat{sign}(t-x) \\ &\quad + \delta_x(t) \left[ f(x) - \frac{1}{2^\alpha} f(x+) - (1 - \frac{1}{2^\alpha}) f(x-) \right], \end{aligned} \quad (22)$$

where  $g_x(t)$  is as defined in (3),

$$\widehat{sign}(t-x) = \begin{cases} 2^\alpha - 1, & t > x, \\ 0, & t = x, \\ -1, & t < x, \end{cases}$$



and

$$\delta_x(t) = \begin{cases} 1, & t = x, \\ 0, & t \neq x. \end{cases}$$

Therefore,

$$\begin{aligned} \left| M_{n,\alpha}(f, x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right| &\leq |M_{n,\alpha}(g_x, x)| \\ &+ \left| \frac{f(x+) - f(x-)}{2^\alpha} M_{n,\alpha}(\widehat{\text{sign}}(t - x), x) \right| \\ &+ \left| \left[ f(x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right] M_{n,\alpha}(\delta_x, x) \right|. \end{aligned} \quad (23)$$

It is obvious that

$$M_{n,\alpha}(\delta_x, x) = 0. \quad (24)$$

Next we estimate  $M_{n,\alpha}(\widehat{\text{sign}}(t - x), x)$ .

Let  $x \in [\frac{k'}{n+k'}, \frac{k'+1}{n+k'+1}]$ , we have

$$\begin{aligned} M_{n,\alpha}(\widehat{\text{sign}}(t - x), x) &= \sum_{k=0}^{k'-1} (-1) Q_{nk}^{(\alpha)}(x) + \frac{Q_{nk'}^{(\alpha)}(x)}{|I_{k'}|} \int_{\frac{k'}{n+k'}}^x (-1) dt \\ &+ \frac{Q_{nk'}^{(\alpha)}(x)}{|I_{k'}|} \int_x^{\frac{k'+1}{n+k'+1}} (2^\alpha - 1) dt + \sum_{k=k'+1}^{\infty} (2^\alpha - 1) Q_{nk}^{(\alpha)}(x) \\ &= \sum_{k=k'+1}^{\infty} 2^\alpha Q_{nk}^{(\alpha)}(x) - 1 + \frac{Q_{nk'}^{(\alpha)}(x)}{|I_{k'}|} \int_x^{\frac{k'+1}{n+k'+1}} 2^\alpha dt \\ &\leq \sum_{k=k'+1}^{\infty} 2^\alpha Q_{nk}^{(\alpha)}(x) - 1 + \frac{Q_{nk'}^{(\alpha)}(x)}{|I_{k'}|} \int_{I_{k'}} 2^\alpha dt. \end{aligned}$$

Hence, we get

$$\begin{aligned} \left| M_{n,\alpha}(\widehat{\text{sign}}(t - x), x) \right| &\leq \left| \sum_{k=k'+1}^{\infty} 2^\alpha Q_{nk}^{(\alpha)}(x) - 1 \right| + 2^\alpha Q_{nk'}^{(\alpha)}(x) \\ &= 2^\alpha \left| \sum_{k > nx/(1-x)} Q_{nk}^{(\alpha)}(x) - \frac{1}{2^\alpha} \right| + 2^\alpha Q_{nk'}^{(\alpha)}(x) \\ &= 2^\alpha \left| \left( \sum_{k > nx/(1-x)} m_{nk}(x) \right)^\alpha - \frac{1}{2^\alpha} \right| + 2^\alpha Q_{nk'}^{(\alpha)}(x). \end{aligned} \quad (25)$$

Theorem 1 now follows from (23),(24),(25), Lemma 2 and Lemma 5.

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# Stability of moving on $(2+\alpha)$ chain <sup>1</sup>

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## Abstract

There is considered the system of differential equations on graph which is the mathematical model of traffic flows on network. The qualitative properties of solutions, stationary points and system behavior in the neighborhood of critical regimes are studied.

The paper continues the investigation of flow stability on graphs [1]-[3].

1. We consider an one direction moving on the graph as shown in Fig. 1.

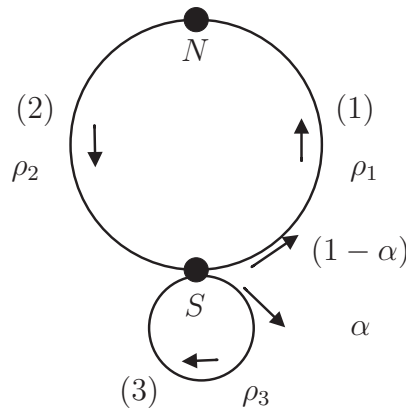


Figure 1:  $(2+\alpha)$  chain

The graph has two nodes  $N$  and  $S$ , three edges of the same length 1 and the state function, which means the dependence of the flow intensity from density. In this case the state function is defined by  $x(1-x)$ ,  $0 \leq x \leq 1$ . Next, an  $\alpha$ -part of flow from edge (2) ( $\vec{N}S$ ) goes to the edge (3) ( $\vec{S}S$ ), and  $(1-\alpha)$  - part of flow goes to edge (1) ( $\vec{S}N$ ).

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We assume that  $\theta(t)$  is the step function with two values. The value of the function is equal to 0 if the movement is forbidden over node  $S$  'on' and 'out' the edge (3) and the value is equal to 1 if the movement is permitted.

Thus if  $\rho_i$ ,  $1 \leq i \leq 3$  denote the densities on edges of the graph, then

$$\begin{cases} \frac{d\rho_1}{dt} = -f(\rho_1) + (1 - \alpha)f(\rho_2) + \theta f(\rho_3), \\ \frac{d\rho_2}{dt} = f(\rho_1) + ((1 - \alpha) + \theta\alpha) f(\rho_2), \\ \frac{d\rho_3}{dt} = \theta\alpha f(\rho_2) - \theta f(\rho_3), \end{cases} \quad (1)$$

i.e.

$$\begin{pmatrix} \frac{d\rho_1}{dt} \\ \frac{d\rho_2}{dt} \\ \frac{d\rho_3}{dt} \end{pmatrix} = \begin{pmatrix} -1 & (1 - \alpha) & 0 \\ 1 & -(1 - \alpha) - \theta\alpha & 0 \\ 0 & \theta\alpha & -\theta \end{pmatrix} \begin{pmatrix} f(\rho_1) \\ f(\rho_2) \\ f(\rho_3) \end{pmatrix} \quad (2)$$

It is clear that

$$\left( \frac{d\rho_1}{dt} + \frac{d\rho_2}{dt} + \frac{d\rho_3}{dt} \right) = (\rho_1 + \rho_2 + \rho_3)' \equiv 0,$$

i.e.

$$\rho_1(t) + \rho_2(t) + \rho_3(t) \equiv C, \quad (3)$$

where  $C$  is the flow mass,  $0 \leq C \leq 3$ . Since

$$0 \leq \rho_i(t) \leq 1, \quad i = 1, 2, 3 \quad (4)$$

and if any function  $\rho_{i_0}(T) = 1$ , then

$$\rho_{i_0}(t) \equiv 1 \quad \forall t > T. \quad (5)$$

The solution (5) is called *critical*.

## 2. The maximal network ( $\theta \equiv 1$ )

We consider in this section that the edge with number (3) is always open. The system (1)-(2) will be transformed as below:

$$\begin{cases} \frac{d\rho_1}{dt} = -f(\rho_1) + (1 - \alpha)f(\rho_2) + f(\rho_3), \\ \frac{d\rho_2}{dt} = f(\rho_1) - f(\rho_2), \\ \frac{d\rho_3}{dt} = \alpha f(\rho_2) - f(\rho_3) \end{cases} \quad (6)$$

By virtue of (4)-(5), if at some time  $T$  the density on any edges becomes maximum, then the input and output flows become zero for all  $t \geq T$ .

Therefore, the behavior of solutions (6) in neighbourhood of admisable value cube's (4) boundary is a very important characteristic.

When  $\alpha = 1$  the system (6) becomes a full 3-vertex graph and is equivalent to the graph as shown in Fig. 2 [1].

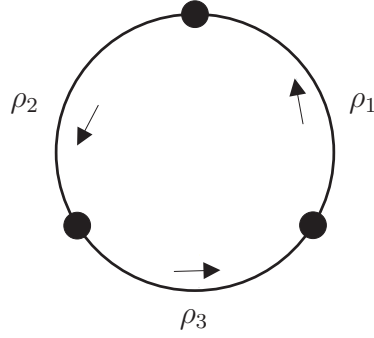


Figure 2: 3 vertices full graph (with 3 edges)

When  $\alpha = 0$ , we get system

$$\begin{cases} \frac{d\rho_1}{dt} = -f(\rho_1) + f(\rho_2) + f(\rho_3), \\ \frac{d\rho_2}{dt} = f(\rho_1) - f(\rho_2), \\ \frac{d\rho_3}{dt} = -f(\rho_3) \end{cases} \quad (7)$$

and the initial conditions are

$$\rho_1(0) = \rho_{10}, \rho_2(0) = \rho_{20}, \rho_3(0) = \rho_{30}. \quad (8)$$

From (7) we have

$$\int_{\rho_{30}}^{\rho_3} \frac{d\rho_3}{f(\rho_3)} = - \int_0^t dt = -t,$$

and according to  $f(\rho_3) = \rho_3(1 - \rho_3)$ , we get

$$\ln \frac{\rho_3}{(1 - \rho_3)} \Big|_{\rho_{30}}^{\rho_3} = -t,$$

i.e.

$$\begin{aligned} \frac{\rho_3(t)}{1 - \rho_3(t)} &= \frac{1 - \rho_{30}}{\rho_{30}} e^{-t}, \\ \rho_3(t) &= \left( 1 + \frac{\rho_{30}}{1 - \rho_{30}} e^t \right)^{-1}. \end{aligned} \quad (9)$$

It is clear that the function  $\rho_3(t)$  monotonously decreases and from (3)

$$\rho_1(t) + \rho_2(t) \nearrow C. \quad (10)$$

If  $C < 1$ , then

$$\rho_1(t) + \rho_2(t) = C - \rho_3(t) \nearrow C < 1,$$

and from (2) it follows that the flow on two edges will be stable.

If  $C > 1$ , then for finite time  $T_1$  the condition will be achieved

$$\rho_1(t) + \rho_2(t) > 1$$

for all  $t > T_1$ .

Thence, from [2] it follows that the flow on subgraph with edges (1)-(2) will certainly be in critical regime.

Hence, for  $\alpha = 0$ , the  $(2 + 0)-$  flow is equivalent to the two-section graph position [2].

### 3. Qualitative properties of solutions (6) in boundary neighbourhood

We consider the set  $D$  of permissible values  $\bar{\rho} = (\rho_1, \rho_2, \rho_3)$  meet Eqs. (3), (4) in dependence on parameter  $C$ .

When  $0 < C < 1$ , we have the set  $D$  as shown in Fig. 3.

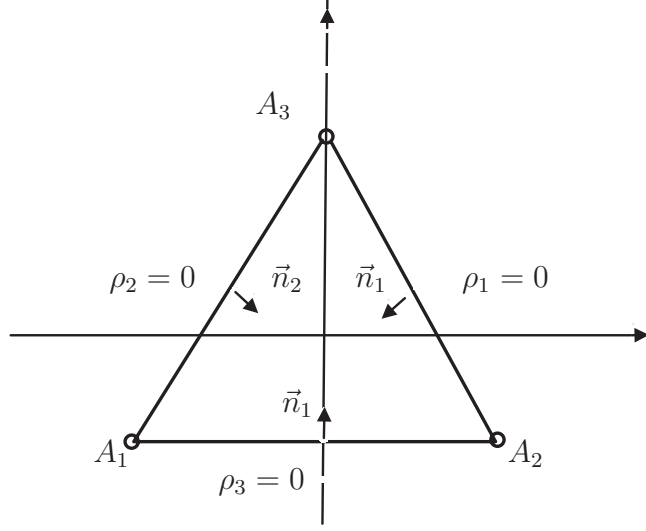


Figure 3: The set  $D$  when  $\rho_1 + \rho_2 + \rho_3 = C < 1$

If  $1 < C < 2$ , the set  $D$  is shown in Fig. 4.

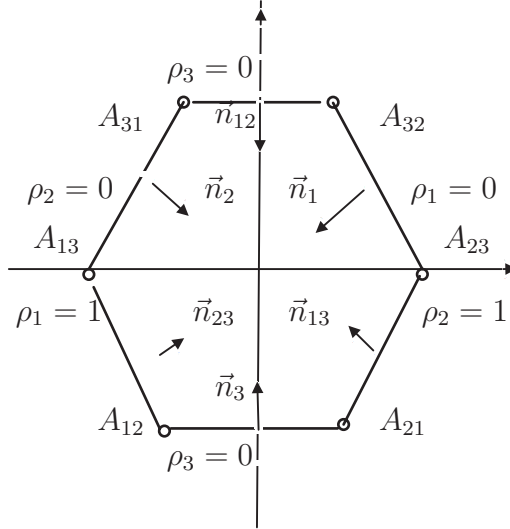


Figure 4: The set  $D$  when  $1 < \rho_1 + \rho_2 + \rho_3 < 2$

If  $C = \frac{3}{2}$ , we have the regular hexagon as shown in Fig. 4. The cross-sections of the

hexagon are symmetric with certain accuracy that surely depends on rotation on angle  $\pi/3$  with regard to the above mention value of parameter  $C$ .

The similar property is trust also for the set of parameters  $(0 < C < 1)$  and  $(2 < C < 3)$ .

Hence, if  $2 < C < 3$ , we have the set  $D$  as shown in Fig. 5.

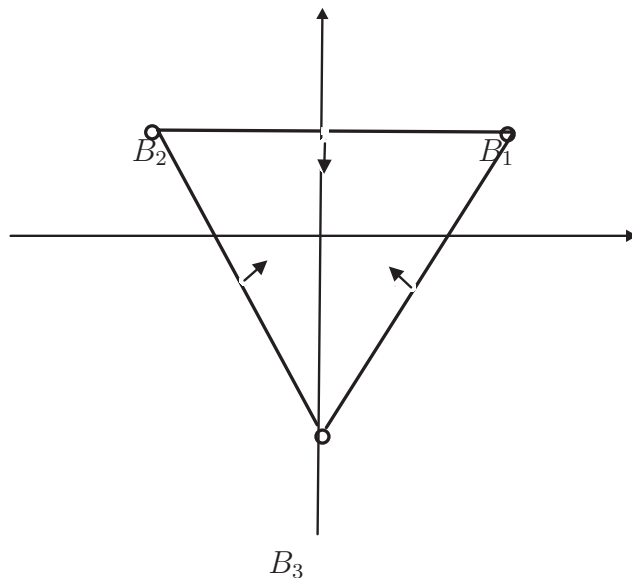


Figure 5: The set  $D$  when  $\rho_1 + \rho_2 + \rho_3 = C > 2$

Assume that  $C < 1$ .

The normal vector  $\vec{n}_1$  to  $A_2A_3$  in Fig. 3 is equal to

$$\left(2/\sqrt{6}, -1/\sqrt{6}, -1/\sqrt{6}\right).$$

We consider the function

$$F = \dot{\vec{\rho}} \cdot \vec{n}_1,$$

which represents the velocity of system moving away from bounding (critical) regime  $\vec{\rho}$ .

If  $0 < C < 1$ , we have

$$F = \frac{d\rho_1}{dt} - \frac{1}{2} \frac{d\rho_2}{dt} - \frac{1}{2} \frac{d\rho_3}{dt} =$$



$$\begin{aligned}
 & \left( -f(\rho_1) + (1 - \alpha) f(\rho_2) + f(\rho_3) - \frac{1}{2}f(\rho_1) + \right. \\
 & \quad \left. + \frac{1}{2}f(\rho_2) - \frac{1}{2}\alpha f(\rho_2) + \frac{1}{2}f(\rho_3) \right) \frac{\sqrt{2}}{6} = \\
 & = \left( -\sqrt{\frac{3}{2}}f(\rho_1) + \sqrt{\frac{3}{2}}f(\rho_2)(1 - \alpha) + \sqrt{\frac{3}{2}}f(\rho_3) \right).
 \end{aligned}$$

If  $\vec{\rho} \in A_2A_3$ , then  $\rho_1 = 0$ ,  $\rho_2 + \rho_3 = C$  and

$$\vec{F}|_{\rho_2+\rho_3=C, \rho_1=0} = \sqrt{\frac{3}{2}}((1 - \alpha) f(\rho_2) + f(\rho_3)). \quad (11)$$

Analogously,  $\vec{n}_2 = (-1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6})$ , and

$$\vec{F}|_{\rho_1+\rho_3=C, \rho_2=0} = \frac{3}{\sqrt{6}}f(\rho_1) = \sqrt{\frac{3}{2}}f(\rho_1). \quad (12)$$

Thus  $\vec{n}_3 = (-1/\sqrt{6}, -1/\sqrt{6}, 2/\sqrt{6})$ ,

$$\vec{F}|_{\rho_1+\rho_2=C, \rho_3=0} = \frac{3\alpha}{\sqrt{6}}f(\rho_2) = \sqrt{\frac{3}{2}}\alpha f(\rho_2) \quad (13)$$

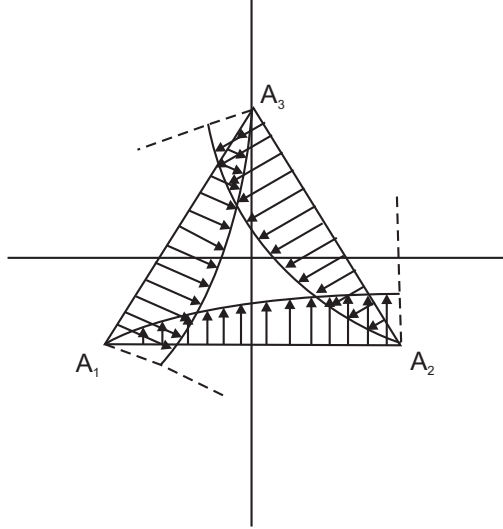
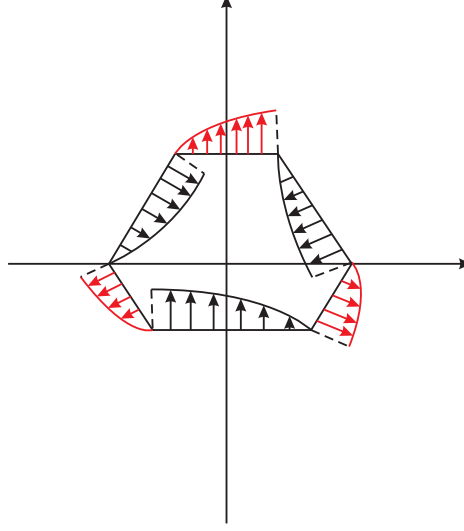


Figure 6: Tria-gram of velocities,  $C < 1$

Therefore, the following is existed.

**Proposition 1.** *If  $C < 1$ , the system (6) moves away from bounding regime with velocity close to zero (11)-(13).*

Figure 7: Diagram of velocities,  $1 < C < 2$ 

Then we have  $\vec{n}_{12} = -\vec{n}_3$ ,  $\vec{n}_{13} = -\vec{n}_2$ , and  $\vec{n}_{23} = -\vec{n}_1$ .

Hence, from (11)-(13), if  $C > 1$ ,  $C < 2$ ,

$$\vec{F}|_{\rho_2+\rho_3=C-1, \rho_1 \equiv 1} = -\sqrt{\frac{3}{2}}((1-\alpha)f(\rho_2) + f(\rho_3)),$$

$$\vec{F}|_{\rho_1+\rho_3=C-1, \rho_2 \equiv 1} = -\sqrt{\frac{3}{2}}f(\rho_1),$$

$$\vec{F}|_{\rho_1+\rho_2=C-1, \rho_3 \equiv 1} = -\sqrt{\frac{3}{2}}\alpha f(\rho_1).$$

Summarizing the above statements, we have the diagram of velocities on the boundary as shown in Fig. 7, if  $1 < C < 2$ .

When  $C > 2$ , we obtain the case as shown in Fig.8.

#### 4. On quantity of stationary points (6)

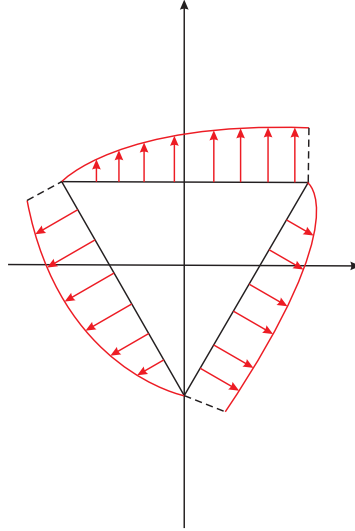
Let  $\rho_\alpha$  be a minimum solution of the equation

$$f(\rho) = \frac{1}{\alpha}f(\rho_\alpha), \quad (14)$$

where  $\rho_\alpha = \rho_\alpha(\rho)$ ,  $0 < \alpha < 1$ ,  $0 < \rho < 1$ .

It is clear that

$$\rho_\alpha = (1 - \rho)_\alpha. \quad (15)$$

Figure 8: Case  $C > 2$ 

Moreover,  $0_\alpha = 0$  and

$$\rho_\alpha \nearrow (0.5)_\alpha \quad \text{if} \quad \rho \in (0, 1). \quad (16)$$

Analogously, if  $\rho \in (0, 0.5)$ , then it holds

$$\rho_\alpha \nearrow \rho_1 \equiv \rho \quad \text{if} \quad \alpha \in (0, 1]. \quad (17)$$

**Proposition 2.** The inequality is true

$$\rho_\alpha + \rho \leq 1 \quad (18)$$

for all  $\rho \in [0, 1]$  and  $\alpha \in [0, 1]$ .

**Proof:** If  $\rho \in (0, 1/2)$ , from  $\rho_\alpha < \rho$ , we get

$$\rho_\alpha + \rho < 2\rho < 1.$$

If  $\rho \in (1/2, 1)$ , then  $\rho_\alpha = (1 - \rho)_\alpha < 1 - \rho$ , consequently,  $\rho + \rho_\alpha < 1$ .

The stationary points (6) are the solutions of the system

$$\begin{cases} f(\rho_1) = f(\rho_2), \\ f(\rho_3) = \alpha f(\rho_2). \end{cases} \quad (19)$$

From symmetry of the function  $f(\rho)$  relatively to  $\rho = 0.5$ , we get the following stationary points

- (1)  $(\rho, \rho, \rho_\alpha), \quad 0 \leq 2\rho + \rho_\alpha \leq 2;$
- (2)  $(1 - \rho, 1 - \rho, \rho_\alpha), \quad 0 \leq 2 - 2\rho + \rho_\alpha \leq 2;$
- (3)  $(\rho, 1 - \rho, \rho_\alpha), \quad 1 \leq 1 + \rho_\alpha \leq 1 + (0.5)_\alpha;$
- (4)  $(1 - \rho, \rho, \rho_\alpha), \quad 1 \leq 1 + \rho_\alpha \leq 1 + (0.5)_\alpha;$
- (5)  $(\rho, \rho, 1 - \rho_\alpha), \quad 1 \leq 2\rho + 1 - \rho_\alpha \leq 3;$
- (6)  $(1 - \rho, 1 - \rho, 1 - \rho_\alpha), \quad 1 \leq 2 - \rho \leq 3 - 2\rho - \rho_\alpha \leq 3;$
- (7)  $(\rho, 1 - \rho, 1 - \rho_\alpha), \quad 2 - (0.5)_\alpha \leq 2 - 2\rho_\alpha \leq 2;$
- (8)  $(1 - \rho, \rho, 1 - \rho_\alpha), \quad 2 - (0.5)_\alpha \leq 2 - 2\rho_\alpha \leq 2;$

From  $\rho_\alpha = (1 - \rho)_\alpha$ , the family (1) and (2) are equivalent.

Hence, the following families are equivalent:

$$(3) \iff (4), \quad (5) \iff (6), \quad (7) \iff (8).$$

Thus it is true

**Proposition 3.**

When  $\alpha \in (0, 1)$  is fixed, there exists four families of stable states of system (6)  $0 \leq \rho \leq 1$ :

- (a)  $(\rho, \rho, \rho_\alpha), \quad 0 \leq C \leq 2;$
- (b)  $(\rho, 1 - \rho, \rho_\alpha), \quad 1 \leq C \leq 1 + (0.5)_\alpha;$
- (c)  $(\rho, \rho, 1 - \rho_\alpha), \quad 1 \leq C \leq 3;$
- (d)  $(\rho, 1 - \rho, 1 - \rho_\alpha), \quad 2 - (0.5)_\alpha \leq C \leq 2.$

**Corollary 1.** When  $\alpha = 1$

$$\begin{aligned} \rho_\alpha &= \{\rho, 0 \leq \rho \leq 0.5; 1 - \rho, 0.5 \leq \rho \leq 1\} = \\ &= \min \{\rho, 1 - \rho\}, \quad (0.5)_1 = 0.5. \end{aligned}$$

Combining the families (a) and (c), we have

$$\begin{aligned} (a') \quad &(\rho, \rho, \rho), \quad 0 \leq \rho \leq 1; \\ (c') \quad &(\rho, \rho, 1 - \rho), \quad 1 \leq \rho \leq 1; \end{aligned}$$

Analogically, from (b) and combining the families (d), we get the following

$$\begin{aligned} (b') \quad & (\rho, 1 - \rho, \rho), \quad 0 \leq \rho \leq 1; \\ (d') \quad & (\rho, 1 - \rho, 1 - \rho), \quad 1 \leq \rho \leq 1, \end{aligned}$$

where sets  $(a') - (d')$  are derived in [1].

**Definition.** Let  $m(C)$  be a quantity of stable point (6),  $m(C) = m(C, \alpha)$ .

**Theorem 1.** For all  $\alpha \in (0, 1)$

$$m(C) = m(C, \alpha) = \begin{cases} 1, & 0 < C < 1; \\ 4, & 1 < C < 1 + (0.5)_\alpha; \\ 2, & 1 + (0.5)_\alpha < C < 2 - (0.5)_\alpha; \\ 4, & 2 - (0.5)_\alpha < C < 2; \\ 1, & 2 < C < 3. \end{cases}$$

**Proof.**

As  $C - 2\rho + \rho_\alpha(\rho)$  monotonically increases, then for  $C \in (0, 1)$ , the family (a) generates the unique stable point.

Analogically, when  $C \in (2, 3)$ , the family (c) is uniquely admissible.

And the equation

$$2\rho + 1 - \rho_\alpha = \rho + 1 + (\rho - \rho_\alpha) = C \quad (20)$$

has the unique solution as the function  $C - 2\rho + \rho_\alpha(\rho)$  is monotonic.

Analogically, if the parameter  $C$  satisfies the condition  $1 + (0.5)_\alpha < C < 2 - (0.5)_\alpha$ , then the equation (20) has the only solution from the family (c) and one solution from the family (a) with the same reason.

If  $1 < C < 1 + (0.5)_\alpha$ , then the following families give one solution for each: (a) and (c) and the family (b) gives exactly two solutions.

The function

$$\rho + (1 - \rho) + \rho_\alpha(\rho) = 1 + \rho_\alpha(\rho)$$

grows monotonously for  $\rho \in (0, 0.5)$  and is symmetrically relative to the point  $\rho = 0.5$ .

From similar consideration, the statement is true in the case  $2 - (0.5)_\alpha < C < 2$ .

The Theorem 2 is proved.

**Remark.** For  $\alpha = 1$ , we get the following values [1]

$$m(C) = m(C, 1) = \begin{cases} 1, & 0 < C < 1; \\ 4, & 1 < C < 1.5; \\ 1, & C = 1.5; \\ 4, & 1.5 < C < 2; \\ 1, & 2 \leq C < 3. \end{cases}$$

## 5. On qualitative properties of the solutions in the neighbourhood of the stationary point

We consider the system of differential equations (6) in general form

$$\dot{\vec{\rho}} = \vec{F}(\vec{\rho}) = \vec{F}(\vec{\rho}, \alpha), \quad (21)$$

where  $\vec{\rho} = (\rho_1, \dots, \rho_n)$ ,

$$0 < \rho_i < 1, \quad i = 1, \dots, n, \quad (22)$$

$$\|\vec{\rho}\|_{\ell_1^n} = \rho_1 + \dots + \rho_n \equiv C. \quad (23)$$

Let  $\vec{\rho}^*$  be a stationary point, i.e. it means that

$$\vec{F}(\vec{\rho}^*) = \vec{0}, \quad 0 < \rho_1^* < 1,$$

$$i = 1, \dots, n, \quad \|\vec{\rho}^*\|_{\ell_1^n} = C.$$

Let  $\vec{\rho}$  be an arbitrary admissible point satisfying (21), (22) and (23).

Then

$$\begin{aligned} & (\dot{\vec{\rho}} - \dot{\vec{\rho}}^*, \vec{\rho} - \vec{\rho}^*) \\ &= (\vec{F}(\vec{\rho}, \alpha), \vec{\rho} - \vec{\rho}^*). \end{aligned}$$

Suppose  $\vec{\Delta} = \vec{\rho} - \vec{\rho}^*$  and let  $T$  denotes the ortonormal matrix of size  $n \times n$  satisfying

$$\vec{\Delta} \equiv T\vec{x} \quad (24)$$

where  $\vec{x} = (x_1, \dots, x_n)$ ,

$$x_n = \Delta_1 + \dots + \Delta_n = 0. \quad (25)$$

Thus the map

$$\vec{\rho} = \vec{\rho}^* + T\vec{x}$$

fulfills the orthogonal tranformation. It is a rotation and transferring of co-ordinates so that the stationary point becomes point of origin, and the hyperplane (23) has an equation  $x_n = 0$  in new co-ordinates.

Hence, we have the follows

$$\begin{aligned} \frac{1}{2} \left( \|\vec{\rho} - \vec{\rho}^*\|_{\ell_1^n} \right)'_t &= \left( \vec{F}(\vec{\rho}, \alpha), \vec{\rho} - \vec{\rho}^* \right) = \left( \vec{F}(\vec{\rho}^* + \vec{\Delta}, \alpha), \vec{\Delta} \right) = \\ &= \left( \vec{F}(\vec{\rho}^* + T\vec{x}, \alpha), T\vec{x} \right) = G(\vec{x}). \end{aligned}$$

**Proposition 3.**

*The function  $G(\vec{x}) = G(\vec{x}, \vec{\rho}^*)$  satisfies the following properties*

$$\begin{aligned} (a) \quad & G(\vec{0}) = 0; \\ (b) \quad & \text{grad } G|_{\vec{x}=\vec{0}} = \vec{0}. \end{aligned}$$

**Proof.** It is proved by substitution in right part.

Let  $H = H(\vec{\rho}^*, \alpha)$  denotes the matrix  $\vec{F}'|_{\vec{x}=\vec{0}}$ , and by  $T^* = T^{-1}$  the adjoint of the matrix  $T$ .

**Theorem 2.**

*If matrix  $T^*HT$  is positively definite on hyperplane  $x_n = 0$ , then the stationary point  $\vec{\rho}^*$  is unstable.*

*If the matrix  $T^*HT$  is negatively definite on the same set, then the point  $\vec{\rho}^*$  is stable.*

**Proof.**

As

$$\left( \|\vec{\rho} - \vec{\rho}^*\|_{\ell_1^n} \right)'_t = 2G(\vec{x})$$

,

for the function  $G(\vec{x})$  when  $\vec{x} = \vec{0}$  and the sufficient conditions of extremum is fulfilled. Then the function  $\|\vec{\rho} - \vec{\rho}^*\|_{\ell_1^n}$  is monotonous, where from the property of strict unstability of the point follows.



## 6. Transform to $(x_1, x_2, x_3) = \vec{x}$ co-ordinates

In this part, the following system is introduced for reference

$$\begin{cases} \vec{e}_1 = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0), \\ \vec{e}_2 = (-\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}), \\ \vec{e}_3 = (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}). \end{cases}$$

Let

$$\begin{cases} x_1 = (\rho_1, \rho_2, \rho_3) \cdot \vec{e}_1 \\ x_2 = (\rho_1, \rho_2, \rho_3) \cdot \vec{e}_2 \\ x_3 = (\rho_1, \rho_2, \rho_3) \cdot \vec{e}_3 \end{cases}$$

i.e.

$$\rho_1 \vec{e}_1 + \rho_2 \vec{e}_2 + \rho_3 \vec{e}_3 \equiv x_1 \vec{e}_1 + x_2 \vec{e}_2 + x_3 \vec{e}_3, \quad \vec{x} = T^* \vec{\rho},$$

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} \\ \iff \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} &= \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \end{aligned} \quad (26)$$

$$\vec{\rho} = T^* \vec{x} = (T^*)^{-1} \vec{x} = T \vec{x}.$$

$$\text{As } \dot{\vec{x}} = T^* \dot{\vec{\rho}}, \text{ and } \dot{\vec{x}} = T^* \vec{F}(\vec{\rho}, \alpha) \iff$$

$$\begin{cases} \dot{x}_1 = \sqrt{2}f(\rho_1) - \frac{\sqrt{2}}{2}(2 - \alpha)f(\rho_2) - \frac{\sqrt{2}}{2}f(\rho_3) \\ \dot{x}_2 = 0 \cdot f(\rho_1) + \frac{\sqrt{6}}{2}\alpha f(\rho_2) - \frac{\sqrt{6}}{2}f(\rho_3) \\ \dot{x}_3 = 0 \end{cases}$$

Consider that  $(\rho_1 + \rho_2 + \rho_3) \equiv C$ , we get

$$\begin{cases} \dot{x}_1 = \sqrt{2}f(\rho_1) - \frac{\sqrt{2}}{2}(2 - \alpha)f(\rho_2) - \frac{\sqrt{2}}{2}f(c - \rho_1 - \rho_2) \\ \dot{x}_2 = 0 + \frac{\sqrt{6}}{2}\alpha f(\rho_2) - \frac{\sqrt{6}}{2}f(c - \rho_1 - \rho_2) \end{cases} \quad (27)$$

$\iff$

$$\dot{\vec{x}} = W(\vec{\rho}, c), \vec{x} = (x_1, x_2), \vec{\rho} = (\rho_1, \rho_2)$$

The properties of solutions in the neighbourhood of stationary point is defined by eigenvalues of the matrix

$$W' = - \begin{pmatrix} \sqrt{2}f'(\rho_1) + \frac{\sqrt{2}}{2}f'(C - \rho_1 - \rho_2) & \frac{\sqrt{2}}{2}(\alpha - 2)f'(\rho_2) + \frac{\sqrt{2}}{2}f'(C - \rho_1 - \rho_2) \\ \frac{\sqrt{6}}{2}f'(C - \rho_1 - \rho_2) & \frac{\sqrt{6}}{2}\alpha f'(\rho_2) + \frac{\sqrt{6}}{2}f'(C - \rho_1 - \rho_2) \end{pmatrix} \quad (28)$$

We consider all branches of stationary points one by one (a) – (d), found in Proposition 2.

$$\begin{aligned} (a) \quad W'(\rho, \rho, \rho_\alpha) &= - \begin{pmatrix} \sqrt{2}f'(\rho) + \frac{\sqrt{2}}{2}f'(\rho_\alpha) & \frac{\sqrt{2}}{2}(\alpha - 2)f'(\rho) + \frac{\sqrt{2}}{2}f'(\rho_\alpha) \\ \frac{\sqrt{6}}{2}f'(\rho_\alpha) & \frac{\sqrt{6}}{2}\alpha f'(\rho) + \frac{\sqrt{6}}{2}f'(\rho_\alpha) \end{pmatrix} = \\ &= - \begin{pmatrix} 2f'(\rho) + f'(\rho_\alpha) & (\alpha - 2)f'(\rho) + f'(\rho_\alpha) \\ f'(\rho_\alpha) & \alpha f'(\rho) + f'(\rho_\alpha) \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{6}}{2} \end{pmatrix} \end{aligned}$$

As multiplying at diagonal matrix is equivalent to the stretching of system of reference, so the picture of qualitative behavior of the matrix  $W'$  in stationary points neighbourhood is equivalent to the matrix given below

$$\tilde{W}' = - \begin{pmatrix} 2f'(\rho) + f'(\rho_\alpha) & (\alpha - 2)f'(\rho) + f'(\rho_\alpha) \\ f'(\rho_\alpha) & \alpha f'(\rho) + f'(\rho_\alpha) \end{pmatrix}, \quad (29)$$

When  $\alpha = 1$ , we obtain

$$\tilde{W}'(\rho, \rho, \rho) = - \begin{pmatrix} 3f'(\rho) & 0 \\ f'(\rho) & 2f'(\rho) \end{pmatrix},$$

wherefrom

$$\begin{pmatrix} \lambda_1 < 0, \lambda_2 < 0 & \iff & \rho < 0.5 \\ \lambda_1 > 0, \lambda_2 > 0 & \iff & \rho > 0.5 \end{pmatrix}$$

This case is investigated in [1].

The characteristical polynomial (29) has the following form

$$P(\lambda) = \lambda^2 + ((2 + \alpha)f'(\rho) + 2f'(\rho_\alpha))\lambda + (2\alpha(f'(\rho))^2 + 4f'(\rho)f'(\rho_\alpha)). \quad (30)$$

Obviously that when  $\rho < 0.5$ , the both zeroes of  $P(\lambda)$  are negative, and when  $\rho > 0.5$

$$2f'(\rho)(\alpha f'(\rho) + 2f'(\rho_\alpha)) < 0,$$

since

$$\alpha f'(\rho) + f'(\rho_\alpha) \quad \forall \quad \rho \in (0, 1)$$

by virtue of the convex up of the function  $f$ .

$$\begin{aligned} (c) \quad W'(\rho, \rho, 1 - \rho_\alpha) = \\ = - \begin{pmatrix} \sqrt{2}f'(\rho) + \frac{\sqrt{2}}{2}f'(1 - \rho_\alpha) & \frac{\sqrt{2}}{2}(\alpha - 2)f'(\rho) + \frac{\sqrt{2}}{2}f'(1 - \rho_\alpha) \\ \frac{\sqrt{6}}{2}f'(1 - \rho_\alpha) & \frac{\sqrt{6}}{2}\alpha f'(\rho) + \frac{\sqrt{6}}{2}f'(1 - \rho_\alpha) \end{pmatrix} \end{aligned}$$

and analogically

$$\tilde{W}' = - \begin{pmatrix} 2f'(\rho) + f'(1 - \rho_\alpha) & (\alpha - 2)f'(\rho) + f'(1 - \rho_\alpha) \\ f'(1 - \rho_\alpha) & \alpha f'(\rho) + f'(1 - \rho_\alpha) \end{pmatrix},$$

$$\begin{aligned} P(\lambda) &= \lambda^2 + ((2 + \alpha)f'(\rho) + 2f'(1 - \rho_\alpha))\lambda + (2\alpha(f'(\rho))^2 + 4f'(\rho)f'(1 - \rho_\alpha)) = \\ &= \lambda^2 + ((2 + \alpha)f'(\rho) - 2f'(\rho_\alpha))\lambda + (2\alpha f'(\rho) - 4f'(\rho_\alpha))f'(\rho) \end{aligned} \quad (31)$$

the polynomial  $P(\lambda)$  has  $\forall \rho \in [0, 1)$  at least one positive null.

Consequently, the branch (c) is absolutely unstable.

## 7. Application: vector fields maps ( $\alpha = 0.5$ )

### 7.1. $0 < C < 1$

$(\rho, \rho, \rho_\alpha)$

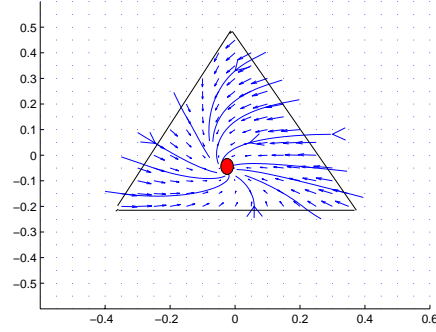


Figure 9: Velocity field and integral cuirves in the case  $C = 0.6$

### 7.2. $1 < C < 1 + (0.5)_\alpha$

(a)  $(\rho, \rho, \rho_\alpha)$

(b)  $(\rho, 1 - \rho, \rho_\alpha); (1 - \rho, \rho, \rho_\alpha);$

(c)  $(\rho, \rho, 1 - \rho_\alpha)$

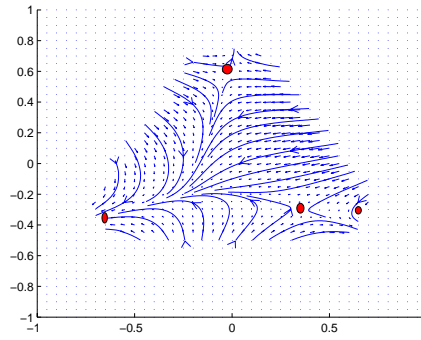
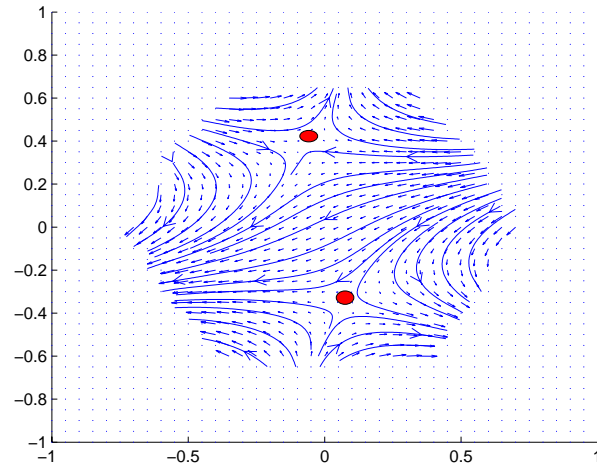


Figure 10: Velocity field and integral cuirves in the case  $C = 1.1$

### 7.3. $1 + (0.5)_\alpha < C < 2 - (0.5)_\alpha$

(a)  $(\rho, \rho, \rho_\alpha)$

(c)  $(\rho, \rho, 1 - \rho_\alpha)$

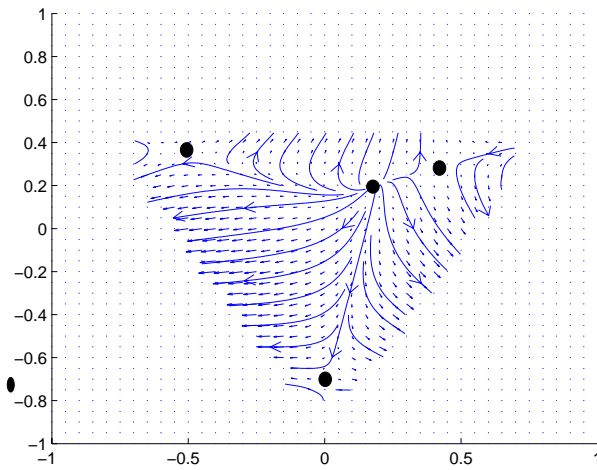
Figure 11: Velocity field and integral cuirves in the case  $C = 1.5$ 

7.4.  $2 - (0.5)_\alpha < C < 2$

(a)  $(\rho, \rho, \rho_\alpha)$

(c)  $(\rho, \rho, 1 - \rho_\alpha)$

(d)  $(\rho, 1 - \rho, 1 - \rho_\alpha); (1 - \rho, \rho, 1 - \rho_\alpha)$

Figure 12: Velocity field and integral cuirves in the case  $C = 1.8$ 

7.5.  $C > 2$

(c)  $(\rho, \rho, 1 - \rho_\alpha)$

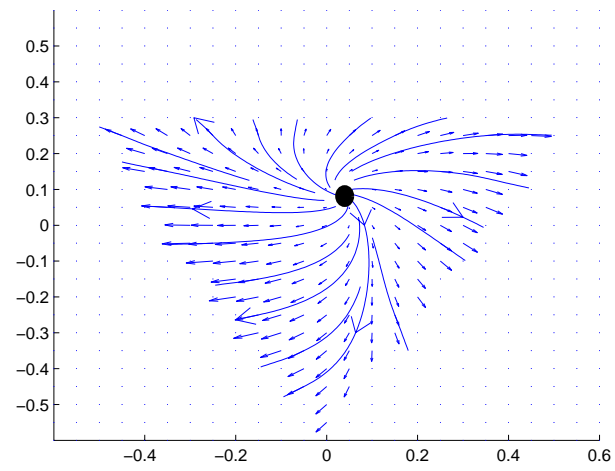


Figure 13: Case  $C = 2.3$

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## ON CONVERGENCE OF ISHIKAWA'S TYPE ITERATION TO A COMMON FIXED POINT IN $\mathcal{L}$ -FUZZY METRIC SPACES

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**Abstract.** In this paper, it is shown that, if Ishikawa's type iterative sequence associated with the nonlinear mappings  $G$  and  $H$  converges to a point, then the limit point is a common fixed point of the mappings  $G$  and  $H$  in  $\mathcal{L}$ -fuzzy metric spaces.

**2000 AMS Subject Classification:** 54E40; 54E35; 54H25.

**Key Words and Phrases:** Ishikawa's type iterative sequence;  $\mathcal{L}$ -fuzzy metric space; common fixed point.

### 1. INTRODUCTION

In the recent years, several authors [3, 8, 9] have studied the convergence of the sequence of the Mann iterative sequence [5] of a mapping  $H$  to a fixed point of  $H$  under various contractive conditions. The Ishikawa iteration scheme [3] was first used to establish the strong convergence for a pseudo contractive self-mapping of a convex compact subset of a Hilbert space. Very soon, both

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iterative sequences were used to establish the strong convergence of these iterative sequences for some contractive type mappings in Hilbert spaces and then in more general normed linear spaces.

In this paper, we used Ishikawa's type iterative sequence for finding a common fixed point for two mappings in  $\mathcal{L}$ -fuzzy metric spaces.

In the sequel, we shall adopt the usual terminology, notation and conventions of  $\mathcal{L}$ -fuzzy metric spaces introduced by Saadati et al. [10].

**Definition 1.1.** ([2]) Let  $\mathcal{L} = (L; \leq_L)$  be a complete lattice (i.e., a partially ordered set in which every nonempty subset admits supremum and infimum) and  $U$  a non-empty set called the universe. An  $\mathcal{L}$ -fuzzy set in  $U$  is defined as a mapping  $U \rightarrow L$ . For each  $u$  in  $U$ ,  $\mathcal{A}(u)$  represents the degree (in  $L$ ) to which  $u$  satisfies  $\mathcal{A}$ .

Classically, a *triangular norm*  $T$  on  $([0, 1], \leq)$  is defined as an increasing, commutative, associative mapping  $T : [0, 1]^2 \rightarrow [0, 1]$  satisfying  $T(1, x) = x$  for all  $x \in [0, 1]$ . These definitions can be straightforwardly extended to any lattice  $\mathcal{L} = (L, \leq_L)$ .

**Definition 1.2.** ([1, 4]) A *triangular norm* ( $t$ -norm) on  $\mathcal{L}$  is a mapping  $\mathcal{T} : L^2 \rightarrow L$  satisfying the following conditions:

- (i)  $(\forall x \in L)(\mathcal{T}(x, 1_{\mathcal{L}}) = x)$  ( $\therefore$  boundary condition);
- (ii)  $(\forall (x, y) \in L^2)(\mathcal{T}(x, y) = \mathcal{T}(y, x))$  ( $\therefore$  commutativity);
- (iii)  $(\forall (x, y, z) \in L^3)(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$  ( $\therefore$  associativity);
- (iv)  $(\forall (x, x', y, y') \in L^4)(x \leq_L x' \text{ and } y \leq_L y' \Rightarrow \mathcal{T}(x, y) \leq_L \mathcal{T}(x', y'))$  ( $\therefore$  monotonicity).

A  $t$ -norm  $\mathcal{T}$  on  $\mathcal{L}$  is said to be *continuous* if, for any  $x, y \in L$ , any sequences  $\{x_n\}$  and  $\{y_n\}$  in  $\mathcal{L}$  which converge to  $x$  and  $y$ , respectively,

$$\lim_{n \rightarrow \infty} \mathcal{T}(x_n, y_n) = \mathcal{T}(x, y).$$

For example,  $\mathcal{T}(x, y) = \min(x, y)$  and  $\mathcal{T}(x, y) = xy$  are two continuous  $t$ -norms on  $[0, 1]$ .

A  $t$ -norm  $\mathcal{T}$  is said to be of *Hadžić type* if  $\mathcal{T}(x, y) \geq_L \wedge(x, y)$  for all  $x, y \in L$ , where

$$\wedge(x, y) = \begin{cases} x & \text{if } x \leq_L y, \\ y & \text{if } y \leq_L x. \end{cases}$$

The  $t$ -norm  $\mathcal{T}$  can be defined recursively as an  $(n+1)$ -ary operation ( $n \in \mathbf{N}$ ) by  $\mathcal{T}^1 = \mathcal{T}$  and

$$\mathcal{T}^n(x_1, \dots, x_{n+1}) = \mathcal{T}(\mathcal{T}^{n-1}(x_1, \dots, x_n), x_{n+1})$$

for all  $n \geq 2$  and  $x_i \in L$ .

**Definition 1.3.** A *negator* on  $\mathcal{L}$  is any decreasing mapping  $\mathcal{N} : L \rightarrow L$  satisfying  $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$  and  $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$ . If  $\mathcal{N}(\mathcal{N}(x)) = x$  for all  $x \in L$ , then  $\mathcal{N}$  is called an *involution negator*.

Next, we use a fixed involutive negator. The negator  $N_s$  on  $([0, 1], \leq)$  defined as  $N_s(x) = 1 - x$  for all  $x \in [0, 1]$  is called the *standard negator* on  $([0, 1], \leq)$ .

**Definition 1.4.** The triple  $(X, \mathcal{M}, \mathcal{T})$  is said to be an  $\mathcal{L}$ -fuzzy metric space if  $X$  is an arbitrary (non-empty) set,  $\mathcal{T}$  is a continuous  $t$ -norm on  $\mathcal{L}$  and  $\mathcal{M}$  is an  $\mathcal{L}$ -fuzzy set on  $X^2 \times ]0, +\infty[$  satisfying the following conditions: for every  $x, y, z \in X$  and  $t, s \in ]0, +\infty[$ ,

- (a)  $\mathcal{M}(x, y, t) >_L 0_{\mathcal{L}}$ ;
- (b)  $\mathcal{M}(x, y, t) = 1_{\mathcal{L}}$  for all  $t > 0$  if and only if  $x = y$ ;
- (c)  $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$ ;
- (d)  $\mathcal{T}(\mathcal{M}(x, y, t), \mathcal{M}(y, z, s)) \leq_L \mathcal{M}(x, z, t + s)$ ;
- (e)  $\mathcal{M}(x, y, \cdot) : ]0, \infty[ \rightarrow L$  is continuous;
- (f)  $\lim_{t \rightarrow \infty} \mathcal{M}(x, y, t) = 1_{\mathcal{L}}$ .

In this case,  $\mathcal{M}$  is called an  $\mathcal{L}$ -fuzzy metric.

If, in the above definition, the triangular inequality (d) is replaced by

$$(NA) \quad \mathcal{T}(\mathcal{M}(x, y, t), \mathcal{M}(y, z, s)) \leq_L \mathcal{M}(x, z, \max\{t, s\})$$

for all  $x, y, z \in X$  and  $t, s > 0$  or, equivalently,

$$\mathcal{T}(\mathcal{M}(x, y, t), \mathcal{M}(y, z, t)) \leq_L \mathcal{M}(x, z, t)$$

for all  $x, y, z \in X$  and  $t > 0$ , then the triple  $(X, \mathcal{M}, \mathcal{T})$  is called a *non-Archimedean  $\mathcal{L}$ -fuzzy metric space* [6, 7].

**Definition 1.5.** (1) A sequence  $\{x_n\}$  in an  $\mathcal{L}$ -fuzzy metric space  $(X, \mathcal{M}, \mathcal{T})$  is called a *Cauchy sequence* if, for each  $\varepsilon \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$  and  $t > 0$ , there exists  $n_0 \in \mathbf{N}$  such that, for all  $n, m \geq n_0$ ,

$$\mathcal{M}(x_n, x_m, t) >_L \mathcal{N}(\varepsilon).$$

(2) A sequence  $\{x_n\}$  is said to be *converges* to a point  $x \in X$ , which is denoted by  $x_n \xrightarrow{\mathcal{M}} x$  if  $\mathcal{M}(x_n, x, t) \rightarrow 1_{\mathcal{L}}$  whenever  $n \rightarrow +\infty$  for all  $t > 0$ .

(3) A  $\mathcal{L}$ -fuzzy metric space  $(X, \mathcal{M}, \mathcal{T})$  is said to be *complete* if every Cauchy sequence in  $X$  is convergent.

**Lemma 1.6.** ([10]) Let  $(X, \mathcal{M}, \mathcal{T})$  be an  $\mathcal{L}$ -fuzzy metric space. If we define a mapping  $E_{\lambda, \mathcal{M}} : X^2 \longrightarrow \mathbf{R}^+ \cup \{0\}$  by

$$E_{\lambda, \mathcal{M}}(x, y) = \inf\{t > 0 : \mathcal{M}(x, y, t) >_L \mathcal{N}(\lambda)\}$$

for all  $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$  and  $x, y \in X$ . Then we have the following:

(1) For any  $\mu \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ , there exists  $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$  such that

$$E_{\mu, \mathcal{M}}(x, z) \leq E_{\lambda, \mathcal{M}}(x, y) + E_{\lambda, \mathcal{M}}(y, z)$$

for all  $x, y, z \in X$ ;

(2) A sequence  $\{x_n\}$  is convergent with the  $\mathcal{L}$ -fuzzy metric  $\mathcal{M}$  if and only if

$$E_{\lambda, \mathcal{M}}(x_n, x) \rightarrow 0.$$

Also, a sequence  $\{x_n\}$  is a Cauchy sequence with the  $\mathcal{L}$ -fuzzy metric  $\mathcal{M}$  if and only if it is a Cauchy sequence with the  $E_{\lambda, \mathcal{M}}$ .

## 2. MAIN RESULT

**Definition 2.1.** Let  $(X, \mathcal{M}, \mathcal{T})$  be an  $\mathcal{L}$ -fuzzy metric space and  $I = [0, 1]$  the closed unit interval.

(1) A continuous mapping  $W : X^2 \times I \longrightarrow X$  is said to be a *convex structure* on  $X$  if, for all  $x, y \in X$ ,  $k \in I$  and  $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ ,

$$(2.1) \quad E_{\lambda, \mathcal{M}}[u, W(x, y, k)] \leq kE_{\lambda, \mathcal{M}}(u, x) + (1 - k)E_{\lambda, \mathcal{M}}(u, y)$$

for all  $u \in X$ .

(2) A  $\mathcal{L}$ -fuzzy metric space  $(X, \mathcal{M}, \mathcal{T})$  together with a convex structure is called a *convex  $\mathcal{L}$ -fuzzy metric space*.

**Theorem 2.2.** Let  $C$  be a nonempty closed convex subset of a non-Archimedean convex  $\mathcal{L}$ -fuzzy metric space  $(X, \mathcal{M}, \mathcal{T})$  in which  $\mathcal{T}$  is of Hadžić type. Let  $G, H : X \longrightarrow X$  be two self-mappings satisfying the following condition:

$$(2.2) \quad \mathcal{M}(Gx, Hy, t) \geq_L \mathcal{T}^2\left(\mathcal{M}\left(x, y, \frac{t}{h}\right), \mathcal{M}\left(x, Hy, \frac{t}{h}\right), \mathcal{M}\left(Gx, y, \frac{t}{h}\right)\right)$$

for all  $x, y \in C$  and  $t > 0$  in which  $h \in (0, 1)$ . Suppose that  $\{x_n\}$  is Ishikawa's type iterative sequence with  $G$  and  $T$  defined by

$$\begin{cases} x_0 \in C, \\ y_n = W(Gx_n, x_n, \beta_n), \\ x_{n+1} = W(Hy_n, x_n, \alpha_n) \end{cases} \quad (*)$$

for all  $n \geq 0$ , where the sequences  $\{\alpha_n\}$  and  $\{\beta_n\}$  satisfy  $0 \leq \alpha_n, \beta_n \leq 1$  and  $\{\alpha_n\}$  is away from zero. If  $\{x_n\}$  converges to a point  $p \in C$ , then  $p$  is a common fixed point of  $G$  and  $H$ .

*Proof.* From  $(*)$ , it follows that

$$E_{\lambda, \mathcal{M}}(x_n, x_{n+1}) = E_{\lambda, \mathcal{M}}[x_n, W(Hy_n, x_n, \alpha_n)] = \alpha_n E_{\lambda, \mathcal{M}}(x_n, Hx_n).$$

Since  $x_n \longrightarrow p$ ,  $\mathcal{M}(x_n, x_{n+1}, t) \longrightarrow 1_{\mathcal{L}}$  and, by Lemma 1.6 (2), it follows that

$$E_{\lambda, \mathcal{M}}(x_n, x_{n+1}) \longrightarrow 0.$$

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Since  $\{\alpha_n\}$  is away from zero, it follows that

$$(2.3) \quad \lim_{n \rightarrow \infty} E_{\lambda, \mathcal{M}}(x_n, Hy_n) = 0.$$

Using (2.7), we get

$$(2.4) \quad \begin{aligned} & \mathcal{M}(Gx_n, Hy_n, t) \\ & \geq_L \mathcal{T}^2(\mathcal{M}(x_n, y_n, \frac{t}{h}), \mathcal{M}(x_n, Hy_n, \frac{t}{h}), \mathcal{M}(Gx_n, y_n, \frac{t}{h})). \end{aligned}$$

By the property of  $E$  and (2.4), we have

$$\begin{aligned} & E_{\lambda, \mathcal{M}}(Gx_n, Hy_n) \\ & = \inf\{t > 0 : \mathcal{M}(Gx_n, Hy_n, t) >_L \mathcal{N}(\lambda)\} \\ & \leq \inf\{t > 0 : \mathcal{T}^2(\mathcal{M}(x_n, y_n, \frac{t}{h}), \mathcal{M}(x_n, Hy_n, \frac{t}{h}), \mathcal{M}(Gx_n, y_n, \frac{t}{h})) >_L \mathcal{N}(\lambda)\} \\ & \leq h \inf\{t > 0 : \mathcal{T}^2(\mathcal{M}(x_n, y_n, t), \mathcal{M}(x_n, Hy_n, t), \mathcal{M}(Gx_n, y_n, t)) >_L \mathcal{N}(\lambda)\} \\ & \leq h \inf\{t > 0 : \wedge^2(\mathcal{M}(x_n, y_n, t), \mathcal{M}(x_n, Hy_n, t), \mathcal{M}(Gx_n, y_n, t)) >_L \mathcal{N}(\lambda)\} \\ & \leq h[E_{\lambda, \mathcal{M}}(x_n, y_n) + E_{\lambda, \mathcal{M}}(x_n, Hy_n) + E_{\lambda, \mathcal{M}}(Gx_n, y_n)]. \end{aligned}$$

It follows from (\*) that

$$\begin{aligned} E_{\lambda, \mathcal{M}}(x_n, y_n) & = E_{\lambda, \mathcal{M}}[x_n, W(Gx_n, x_n, \beta_n)] = \beta_n E_{\lambda, \mathcal{M}}(x_n, Gx_n), \\ E_{\lambda, \mathcal{M}}(Gx_n, y_n) & = E_{\lambda, \mathcal{M}}[Gx_n, W(Gx_n, x_n, \beta_n)] = (1 - \beta_n) E_{\lambda, \mathcal{M}}(x_n, Gx_n). \end{aligned}$$

Thus we have

$$(2.5) \quad E_{\lambda, \mathcal{M}}(Gx_n, Hy_n) \leq h[E_{\lambda, \mathcal{M}}(x_n, Gx_n) + E_{\lambda, \mathcal{M}}(x_n, Hy_n)].$$

By the triangular inequality (NA), we have

$$(2.6) \quad \mathcal{T}(\mathcal{M}(Hy_n, Gx_n, t), \mathcal{M}(x_n, Hy_n, t)) \leq_L \mathcal{M}(x_n, Gx_n, t).$$

By the property of  $E$ , since the  $t$ -norm is of Hadžić type, we have

$$\begin{aligned} & E_{\lambda, \mathcal{M}}(x_n, Gx_n) \\ & = \inf\{t > 0 : \mathcal{M}(x_n, Gx_n, t) >_L \mathcal{N}(\lambda)\} \\ & \leq \inf\{t > 0 : \mathcal{T}(\mathcal{M}(Hy_n, Gx_n, t), \mathcal{M}(x_n, Hy_n, t)) >_L \mathcal{N}(\lambda)\} \\ & \leq \inf\{t > 0 : \wedge(\mathcal{M}(Hy_n, Gx_n, t), \mathcal{M}(x_n, Hy_n, t)) >_L \mathcal{N}(\lambda)\} \\ & \leq E_{\lambda, \mathcal{M}}(Hy_n, Gx_n) + E_{\lambda, \mathcal{M}}(x_n, Hy_n). \end{aligned}$$

Hence, from (2.5) and the last inequality, we have

$$E_{\lambda, \mathcal{M}}(Hy_n, Gx_n) \leq \frac{2h}{1-h} E_{\lambda, \mathcal{M}}(Hy_n, x_n).$$

Taking the limit as  $n \rightarrow \infty$ , by (2.3), we obtain

$$\lim_{n \rightarrow \infty} E_{\lambda, \mathcal{M}}(Hy_n, Gx_n) = 0.$$

Since  $Hy_n \rightarrow p$ , we have  $Gx_n \rightarrow p$ . From  $E_{\lambda, \mathcal{M}}(x_n, y_n) = \beta_n E_{\lambda, \mathcal{M}}(x_n, Gx_n)$ , it follows that  $y_n \rightarrow p$ . Further, from (2.2) and the property of  $E$ , we have

$$E_{\lambda, \mathcal{M}}(Gx_n, Hp) \leq h[E_{\lambda, \mathcal{M}}(x_n, p) + E_{\lambda, \mathcal{M}}(x_n, Hp) + E_{\lambda, \mathcal{M}}(p, Gx_n)].$$

Taking the limit as  $n \rightarrow \infty$ , we obtain

$$E_{\lambda, \mathcal{M}}(p, Hp) \leq hE_{\lambda, \mathcal{M}}(p, Hp).$$

Since  $h \in (0, 1)$ , we have  $E_{\lambda, \mathcal{M}}(p, Hp) = 0$  and hence  $Hp = p$ .

Similarly, from (2.2) and the property of  $E$ , it follows that

$$E_{\lambda, \mathcal{M}}(Gp, Hx_n) \leq h[E_{\lambda, \mathcal{M}}(x_n, p) + E_{\lambda, \mathcal{M}}(p, Hx_n) + E_{\lambda, \mathcal{M}}(x_n, Gp)].$$

Taking the limit as  $n \rightarrow \infty$ , we obtain

$$E_{\lambda, \mathcal{M}}(p, Gp) \leq hE_{\lambda, \mathcal{M}}(p, Gp).$$

Since  $h \in (0, 1)$ , we have  $E_{\lambda, \mathcal{M}}(p, Gp) = 0$  and hence  $Gp = p$ . Therefore,  $Gp = Hp = p$  and so the point  $p$  is a common fixed point of the mappings  $H$  and  $G$ . This completes the proof.  $\square$

If  $G = H$  in Theorem 2.2, then we have the following:

**Corollary 2.3.** *Let  $C$  be a nonempty closed convex subset of a non-Archimedean convex  $\mathcal{L}$ -fuzzy metric space  $(X, \mathcal{M}, \mathcal{T})$  in which  $\mathcal{T}$  is of Hadžić type. Let  $G : X \rightarrow X$  be a self-mapping satisfying the following condition:*

$$(2.7) \quad \mathcal{M}(Gx, Gy, t) \geq_L \mathcal{T}^2(\mathcal{M}(x, y, \frac{t}{h}), \mathcal{M}(x, Gy, \frac{t}{h}), \mathcal{M}(Gx, y, \frac{t}{h}))$$

for all  $x, y \in C$  and  $t > 0$  in which  $h \in (0, 1)$ . Suppose that  $\{x_n\}$  is Ishikawa's type iterative sequence with  $G$  defined by

$$\begin{cases} x_0 \in C, \\ y_n = W(Gx_n, x_n, \beta_n), \\ x_{n+1} = W(Gy_n, x_n, \alpha_n) \end{cases}$$

for all  $n \geq 0$ , where the sequences  $\{\alpha_n\}$  and  $\{\beta_n\}$  satisfy  $0 \leq \alpha_n, \beta_n \leq 1$  and  $\{\alpha_n\}$  is away from zero. If  $\{x_n\}$  converges to a point  $p \in C$ , then  $p$  is a fixed point of  $G$ .

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IDENTITIES OF ROGERS-RAMANUJAN TYPE VIA “ $\pm 1$ ”WENCHANG CHU<sup>†</sup> AND CHENYING WANG

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ABSTRACT. By modifying the initial terms with “ $\pm 1$ ”, we review few known identities of Rogers-Ramanujan type and establish several new ones. The approach is fundamentally based on known theta function identities.

For two indeterminate  $q$  and  $x$ , the shifted-factorial of  $x$  with base  $q$  is defined by

$$(x; q)_0 = 1 \quad \text{and} \quad (x; q)_n = (1-x)(1-xq) \cdots (1-xq^{n-1}) \quad \text{for } n \in \mathbb{N}.$$

When  $|q| < 1$ , the  $q$ -shifted factorial of infinite order and the modified Jacobi theta function read respectively as

$$(x; q)_\infty = \prod_{k=0}^{\infty} (1 - q^k x) \quad \text{and} \quad \langle x; q \rangle_\infty = (x; q)_\infty (q/x; q)_\infty.$$

For the sake of brevity, their multiparameter forms are abbreviated to

$$\begin{aligned} [\alpha, \beta, \dots, \gamma; q]_\infty &= (\alpha; q)_\infty (\beta; q)_\infty \cdots (\gamma; q)_\infty, \\ \langle \alpha, \beta, \dots, \gamma; q \rangle_\infty &= \langle \alpha; q \rangle_\infty \langle \beta; q \rangle_\infty \cdots \langle \gamma; q \rangle_\infty. \end{aligned}$$

Following Gasper-Rahman [12], the basic hypergeometric series is defined by

$${}_{1+r}\phi_s \left[ \begin{matrix} a_0, a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; z \right] = \sum_{n=0}^{\infty} \left\{ (-1)^n q^{\binom{n}{2}} \right\}^{s-r} \left[ \begin{matrix} a_0, a_1, \dots, a_r \\ q, b_1, \dots, b_s \end{matrix} \middle| q \right]_n z^n$$

where the base  $q$  will be restricted to  $|q| < 1$  for nonterminating  $q$ -series.

In the  $q$ -series theory, there are numerous identities expressing infinite sums as infinite products. Two typical ones are the celebrated Rogers-Ramanujan identities (see Andrews [1, §1] and Chu [8, §4] for example):

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{[q, q^4; q^5]_\infty} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \frac{1}{[q^2, q^3; q^5]_\infty}.$$

For a comprehensive investigation of Rogers-Ramanujan type identities, refer to Slater [19] in her collection of 130 identities and Sills [17] with an excellent up-to-date annotation.

The purpose of this paper is to present a new approach to a class of Rogers-Ramanujan type identities. By employing the theta function identities appeared mainly in [9], we shall review few known Rogers-Ramanujan identities and establish several new ones simply by modifying the initial terms with “ $\pm 1$ ”.

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§1. In this section, we shall establish three pairs of new identities of Rogers-Ramanujan type through the following two theta function equations (cf. [4, Entry 25(i-ii), P40] and [9, Eqs 3.1a and 3.1b]):

$$(-q; q^2)_\infty + (q; q^2)_\infty = \frac{2}{(q^4; q^4)_\infty} [q^{16}, -q^6, -q^{10}; q^{16}]_\infty, \quad (1a)$$

$$(-q; q^2)_\infty - (q; q^2)_\infty = \frac{2q}{(q^4; q^4)_\infty} [q^{16}, -q^2, -q^{14}; q^{16}]_\infty. \quad (1b)$$

Firstly, recalling the  $q$ -Kummer theorem (cf. Gasper-Rahman [12, II-9])

$${}_2\phi_1 \left[ \begin{matrix} a, c \\ qa/c \end{matrix} \middle| q; -q/c \right] = (-q; q)_\infty \frac{[qa, q^2a/c^2; q^2]_\infty}{[qa/c, -q/c; q]_\infty} \quad \text{where } |q/c| < 1$$

we may derive, by letting  $a \rightarrow -q^2$  and  $c \rightarrow \infty$ , the identity [20, Eq 3.14]

$$\sum_{m=0}^{\infty} \frac{(-q; q)_{m+1}}{(q; q)_m} q^{\binom{m+1}{2}} = \frac{(-q; q)_\infty}{(q; q)_\infty} [q^4, q^2, q^2; q^4]_\infty = \frac{(-q; q^2)_\infty}{(q; q^2)_\infty}.$$

According to (1a) and (1b), modifying the initial term of the last sum by “ $\pm 1$ ” leads us to the following two identities.

**Theorem 1** (New identities of Rogers-Ramanujan type).

$$1 + \sum_{m=0}^{\infty} \frac{(-q; q)_{m+1}}{(q; q)_m} q^{\binom{m+1}{2}} = 2 \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} [q^{16}, -q^6, -q^{10}; q^{16}]_\infty, \quad (2a)$$

$$1 + \sum_{m=0}^{\infty} \frac{(-q; q)_{m+2}}{(q; q)_{m+1}} q^{\binom{m+2}{2}} = 2 \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} [q^{16}, -q^2, -q^{14}; q^{16}]_\infty. \quad (2b)$$

Secondly, recall the  $q$ -Gauss summation formula [12, II-8]

$${}_2\phi_1 \left[ \begin{matrix} a, b \\ c \end{matrix} \middle| q; c/ab \right] = \frac{[c/a, c/b; q]_\infty}{[c, c/ab; q]_\infty} \quad \text{where } |c/ab| < 1. \quad (3)$$

When  $q \rightarrow q^2$ ,  $a \rightarrow -q^2$ ,  $c \rightarrow q^3$  and  $b \rightarrow \infty$ , it reduces to [7, Eq 2.4]

$${}_1\phi_1 \left[ \begin{matrix} -q^2 \\ q^3 \end{matrix} \middle| q^2; -q \right] \frac{1}{1-q} = \sum_{m=0}^{\infty} \frac{(-q^2; q^2)_m}{(q; q)_{2m+1}} q^{m^2} = \frac{(-q; q^2)_\infty}{(q; q^2)_\infty}.$$

Combining this equation with “ $\pm 1$ ”, we derive the two identities.

**Theorem 2** (New identities of Rogers-Ramanujan type).

$$1 + \sum_{m=0}^{\infty} \frac{(-q^2; q^2)_m}{(q; q)_{2m+1}} q^{m^2} = 2 \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} [q^{16}, -q^6, -q^{10}; q^{16}]_\infty, \quad (4a)$$

$$1 + \sum_{m=0}^{\infty} \frac{(-q^2; q^2)_{m+1}}{(q^2; q)_{2m+2}} q^{m^2+2m} = 2 \frac{(-q^3; q^2)_\infty}{(q^4; q^2)_\infty} [q^{16}, -q^2, -q^{14}; q^{16}]_\infty. \quad (4b)$$

Finally, the  $q$ -Gauss summation formula (3) may be specified, by  $q \rightarrow q^2$ ,  $a \rightarrow -1$ ,  $c \rightarrow q^3$  and  $b \rightarrow \infty$ , to another identity

$$(1+q) \sum_{m=0}^{\infty} \frac{(-1; q^2)_m}{(q; q)_{2m+1}} q^{m^2+2m} = \frac{(-q; q^2)_\infty}{(q; q^2)_\infty}.$$

Its combinations with “ $\pm 1$ ” yields the following two identities.



**Theorem 3** (New identities of Rogers-Ramanujan type).

$$1 + (1 + q) \sum_{m=1}^{\infty} \frac{(-q^2; q^2)_{m-1}}{(q^2; q)_{2m}} q^{m^2+2m} = \frac{(-q^3; q^2)_{\infty}}{(q^4; q^2)_{\infty}} [q^{16}, -q^6, -q^{10}; q^{16}]_{\infty}, \quad (5a)$$

$$1 + (1 + q) \sum_{m=0}^{\infty} \frac{(-q^2; q^2)_m}{(q^2; q)_{2m+2}} q^{m^2+4m+2} = \frac{(-q^3; q^2)_{\infty}}{(q^4; q^2)_{\infty}} [q^{16}, -q^2, -q^{14}; q^{16}]_{\infty}. \quad (5b)$$

Furthermore, it is not difficult to verify that (1.7a-1.7b) and (1.9a-1.9b) in Chu-Wang [10] are two equivalent pairs under the “ $\pm 1$ ” rule in view of (1a) and (1b).

§2. Next, we reproduce the two theta function identities from [4, Entry 25(vi-v), P40] (see [9, Eqs 3.1c and 3.1d] also)

$$(-q; q^2)_{\infty}^2 + (q; q^2)_{\infty}^2 = \frac{2}{(q^2; q^2)_{\infty}} [q^8, -q^4, -q^4; q^8]_{\infty}, \quad (6a)$$

$$(-q; q^2)_{\infty}^2 - (q; q^2)_{\infty}^2 = \frac{4q}{(q^2; q^2)_{\infty}} [q^8, -q^8, -q^8; q^8]_{\infty}. \quad (6b)$$

They will be used to prove two pairs of Rogers-Ramanujan type identities.

Recall the  $q$ -analogue of the second Gauss  ${}_2F_1(-1)$  sum [12, II-11]

$${}_2\phi_2 \left[ \begin{matrix} a^2, & b^2 \\ abq^{1/2}, & -abq^{1/2} \end{matrix} \middle| q; -q \right] = \frac{[qa^2, qb^2; q^2]_{\infty}}{[q, qa^2b^2; q^2]_{\infty}}. \quad (7)$$

For  $a = b = \sqrt{-1}$ , it reduces to the following identity

$$\sum_{m=0}^{\infty} \frac{(-1; q)_m^2}{(q; q)_m (q; q^2)_m} q^{\binom{m+1}{2}} = \frac{(-q; q^2)_{\infty}^2}{(q; q^2)_{\infty}^2}.$$

In view of (6a) and (6b), we may modify the initial term of the last sum by “ $\pm 1$ ” and therefore establish the following two identities.

**Theorem 4** (New identities of Rogers-Ramanujan type).

$$1 + \sum_{m=1}^{\infty} \frac{(-1; q)_m (-q; q)_{m-1}}{(q; q)_m (q; q^2)_m} q^{\binom{m+1}{2}} = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} [q^8, -q^4, -q^4; q^8]_{\infty}, \quad (8a)$$

$$\sum_{m=0}^{\infty} \frac{(-q; q)_m^2 q^{\binom{m}{2}+2m}}{(q; q)_{m+1} (q; q^2)_{m+1}} = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} [q^8, -q^8, -q^8; q^8]_{\infty}. \quad (8b)$$

Similarly, for  $a = \sqrt{-1}$  and  $b = q\sqrt{-1}$  in (7), the corresponding identity reads as

$$\sum_{m=0}^{\infty} \frac{(-1; q)_m (-q; q)_{m+1}}{(q; q)_m (q; q^2)_{m+1}} q^{\binom{m+1}{2}} = \frac{(-q; q^2)_{\infty}^2}{(q; q^2)_{\infty}^2}.$$

Its combinations with “ $\pm 1$ ” lead us to the further two identities.

**Theorem 5** (New identities of Rogers-Ramanujan type).

$$1 + \sum_{m=1}^{\infty} \frac{(-q; q)_{m-1} (-q; q)_{m+1}}{(q; q)_m (q^3; q^2)_m} q^{\binom{m+1}{2}} = \frac{(-q; q)_{\infty}}{(q^2; q)_{\infty}} [q^8, -q^4, -q^4; q^8]_{\infty}, \quad (9a)$$

$$1 + \sum_{m=0}^{\infty} \frac{(-q; q)_m (-q; q)_{m+2}}{(q; q)_{m+1} (q^3; q^2)_{m+1}} q^{\binom{m}{2}+2m} = 2 \frac{(-q; q)_{\infty}}{(q^2; q)_{\infty}} [q^8, -q^8, -q^8; q^8]_{\infty}. \quad (9b)$$

§3. Recall the two cubic theta function identities (cf. [5, Theorem 2 and Corollary 2] and [9, Eqs 3.16a and 3.16b]):

$$\frac{(-q; q)_\infty}{(-q^3; q^3)_\infty} + \frac{(q; q)_\infty}{(q^3; q^3)_\infty} = \frac{2}{(q^3; q^3)_\infty} [q^{12}, -q^5, -q^7; q^{12}]_\infty, \quad (10a)$$

$$\frac{(-q; q)_\infty}{(-q^3; q^3)_\infty} - \frac{(q; q)_\infty}{(q^3; q^3)_\infty} = \frac{2q}{(q^3; q^3)_\infty} [q^{12}, -q, -q^{11}; q^{12}]_\infty. \quad (10b)$$

They will be employed to show three pairs of Rogers-Ramanujan type identities.

Firstly, combining “ $\pm 1$ ” with the identity [19, Eqs 24 and 30]

$$\sum_{m=0}^{\infty} \frac{(-1; q)_{2m}}{(q^2; q^2)_m} q^m = \frac{(-q; q)_\infty}{(q; q)_\infty} [q^6, q^3, q^3; q^6]_\infty = \frac{(-q; q)_\infty}{(q; q)_\infty} \frac{(q^3; q^3)_\infty}{(-q^3; q^3)_\infty}$$

and then using (10a) and (10b), we derive the following two identities.

**Theorem 6** (New identities of Rogers-Ramanujan type).

$$1 + \sum_{m=1}^{\infty} \frac{(-q; q)_{2m-1}}{(q^2; q^2)_m} q^m = \frac{[q^{12}, -q^5, -q^7; q^{12}]_\infty}{(q; q)_\infty}, \quad (11a)$$

$$\sum_{m=0}^{\infty} \frac{(-q; q)_{2m+1}}{(q^2; q^2)_{m+1}} q^m = \frac{[q^{12}, -q, -q^{11}; q^{12}]_\infty}{(q; q)_\infty}. \quad (11b)$$

Similarly, combining “ $\pm 1$ ” with the identity [14, Proposition 6C]

$$\sum_{m=0}^{\infty} \frac{(-1; q)_m (-q; q)_{m+1}}{(q; q)_{2m+1}} q^{m^2+m} = \frac{[q^3, -q, -q^2; q^3]_\infty}{(q; q)_\infty} = \frac{(-q; q)_\infty}{(q; q)_\infty} \frac{(q^3; q^3)_\infty}{(-q^3; q^3)_\infty}$$

leads us to the following two Rogers-Ramanujan type identities.

**Theorem 7** (New identities of Rogers-Ramanujan type).

$$1 + \sum_{m=1}^{\infty} \frac{(-q; q)_{m-1} (-q; q)_{m+1}}{(q^2; q)_{2m}} q^{m^2+m} = \frac{[q^{12}, -q^5, -q^7; q^{12}]_\infty}{(q^2; q)_\infty}, \quad (12a)$$

$$1 + \sum_{m=0}^{\infty} \frac{(-q; q)_m (-q; q)_{m+2}}{(q^2; q)_{2m+2}} q^{m^2+3m+1} = \frac{[q^{12}, -q, -q^{11}; q^{12}]_\infty}{(q^2; q)_\infty}. \quad (12b)$$

Finally, combining “ $\pm 1$ ” with the following identity [19, Eq 26]

$$\sum_{m=0}^{\infty} \frac{(-q; q)_m}{(q; q)_m (q; q^2)_{m+1}} q^{m^2} = \frac{(-q; q)_\infty}{(q; q)_\infty} [q^6, q^3, q^3; q^6]_\infty = \frac{(-q; q)_\infty}{(q; q)_\infty} \frac{(q^3; q^3)_\infty}{(-q^3; q^3)_\infty}$$

we establish the two identities displayed in the following theorem.

**Theorem 8** (New identities of Rogers-Ramanujan type).

$$1 + \sum_{m=0}^{\infty} \frac{(-q; q)_m}{(q; q)_m (q; q^2)_{m+1}} q^{m^2} = 2 \frac{[q^{12}, -q^5, -q^7; q^{12}]_\infty}{(q; q)_\infty}, \quad (13a)$$

$$1 + \sum_{m=0}^{\infty} \frac{(-q; q)_{m+1}}{(q; q)_{m+1} (q^3; q^2)_{m+1}} q^{m^2+2m} = 2 \frac{[q^{12}, -q, -q^{11}; q^{12}]_\infty}{(q^2; q)_\infty}. \quad (13b)$$

In addition, there is also a fourth equivalent pair based on relations (10a) and (10b), which can be found in Chu-Wang [10, Eqs 1.18a and 1.18b].

§4. Rewrite the theta function equation [9, Example 4] as

$$\frac{(-q; q)_\infty}{(q^4; -q^4)_\infty} - \frac{(q; q)_\infty}{(-q^4; -q^4)_\infty} = 2q \frac{(-q; q)_\infty}{(-q^4; -q^4)_\infty} [q^8, -q^8, -q^8; q^8]_\infty. \quad (14)$$

Taking into account of

$$\frac{(-q; q)_\infty}{(q; q)_\infty} [q^8, -q^4, -q^4; q^8]_\infty = \frac{(-q; q)_\infty}{(q; q)_\infty} \frac{(-q^4; -q^4)_\infty}{(q^4; -q^4)_\infty}$$

and then applying the “ $-1$ ” rule to Theorems 4 and 5, we find that the identities displayed there constitute two equivalent pairs of Rogers-Ramanujan type identities. Similarly, one can show that the following identities due to Chu-Wang [10, Eqs 1.12a and 1.12b] result also in an equivalent pair

$$1 + 2 \sum_{m=1}^{\infty} \frac{(-q^2; q^2)_{m-1}^2}{(q; q)_{2m}} q^m = \frac{(-q; q)_\infty}{(q; q)_\infty} [q^8, -q^4, -q^4; q^8]_\infty, \quad (15a)$$

$$\sum_{m=0}^{\infty} \frac{(-q^2; q^2)_m^2}{(q; q)_{2m+2}} q^m = \frac{(-q; q)_\infty}{(q; q)_\infty} [q^8, -q^8, -q^8; q^8]_\infty. \quad (15b)$$

§5. According to the two quintuple theta function equations (cf. Chu [9, Eqs 3.17a and 3.17b])

$$\frac{(-q; q)_\infty}{(-q^5; q^5)_\infty} + \frac{(q; q)_\infty}{(q^5; q^5)_\infty} = 2(-q^5; q^5)_\infty \langle -q^3, -q^4; q^{10} \rangle_\infty, \quad (16a)$$

$$\frac{(-q; q)_\infty}{(-q^5; q^5)_\infty} - \frac{(q; q)_\infty}{(q^5; q^5)_\infty} = 2q(-q^5; q^5)_\infty \langle -q, -q^2; q^{10} \rangle_\infty; \quad (16b)$$

we may apply the “ $\pm 1$ ” rule to the identity due to Sills [18, Eq 5]

$$\sum_{m=0}^{\infty} \frac{(-1; q)_m}{(q; q)_m (q; q^2)_m} q^{\binom{m+1}{2}} = \frac{(-q; q)_\infty}{(q; q)_\infty} [q^{10}, q^5, q^5; q^{10}]_\infty = \frac{(-q; q)_\infty}{(q; q)_\infty} \frac{(q^5; q^5)_\infty}{(-q^5; q^5)_\infty}$$

and derive consequently the following two strange identities, where the second one is due to Bowman-Laughlin-Sills [7, Eq 2.28].

**Theorem 9** (Identities of Rogers-Ramanujan type).

$$1 + \sum_{m=1}^{\infty} \frac{(-q; q)_{m-1}}{(q; q)_m (q; q^2)_m} q^{\binom{m+1}{2}} = \frac{(-q; q)_\infty}{(q; q)_\infty} \frac{(q^{10}; q^{10})_\infty}{\langle -q, -q^2; q^{10} \rangle_\infty}, \quad (17a)$$

$$\sum_{m=0}^{\infty} \frac{(-q; q)_m}{(q; q)_{m+1} (q; q^2)_{m+1}} q^{\binom{m}{2} + 2m} = \frac{(-q; q)_\infty}{(q; q)_\infty} \frac{(q^{10}; q^{10})_\infty}{\langle -q^3, -q^4; q^{10} \rangle_\infty}. \quad (17b)$$

§6. Recall the theta function equation (cf. [6, Eq 2.16] and [9, Example 16])

$$\frac{(-q; q)_\infty}{(-q^9; q^9)_\infty} - \frac{(q; q)_\infty}{(q^9; q^9)_\infty} = 2q \frac{(-q; q)_\infty}{(q^9; q^9)_\infty} [q^{18}, q^3, q^{15}; q^{18}]_\infty. \quad (18)$$

Taking into account of the equality

$$\frac{(-q; q)_\infty}{(q; q)_\infty} [q^{18}, q^9, q^9; q^{18}]_\infty = \frac{(-q; q)_\infty (q^9; q^9)_\infty}{(q; q)_\infty (-q^9; q^9)_\infty}$$

and then applying the “−1” rule to the two identities appeared in [2, Eq D3], [3, Eq 7.2] and [19, Eq 78]

$$1 + \sum_{m=1}^{\infty} \frac{(q^3; q^3)_{m-1}(-1; q)_m}{(q; q)_{2m-1}(q; q)_m} q^{\binom{m+1}{2}} = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} [q^{18}, q^9, q^9; q^{18}]_{\infty}, \quad (19a)$$

$$1 + 2 \sum_{m=1}^{\infty} \frac{(q^6; q^6)_{m-1} q^m}{(q^2; q^2)_{m-1}(q; q)_{2m}} = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} [q^{18}, q^9, q^9; q^{18}]_{\infty} \quad (19b)$$

we recover, respectively, the two identities of Rogers-Ramanujan type.

**Theorem 10** (Bailey [2, Eq D1] and [3, Eq 7.1]).

$$\sum_{m=0}^{\infty} \frac{(q^3; q^3)_m (-q; q)_m}{(q; q)_{2m+1}(q; q)_{m+1}} q^{\binom{m}{2}+2m} = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} [q^{18}, q^3, q^{15}; q^{18}]_{\infty}, \quad (20a)$$

$$\sum_{m=0}^{\infty} \frac{(q^6; q^6)_m q^m}{(q^2; q^2)_m (q; q)_{2m+2}} = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} [q^{18}, q^3, q^{15}; q^{18}]_{\infty}. \quad (20b)$$

§7. In view of the theta function equation (cf. [9, Example 18] and [16, Eq 4.3])

$$3 \frac{(-q; q)_{\infty}}{(-q^9; q^9)_{\infty}} - \frac{(q; q)_{\infty}}{(q^9; q^9)_{\infty}} = 2 \frac{(-q; q)_{\infty}}{(q^9; q^9)_{\infty}} [q^6, -q, -q^5; q^6]_{\infty} \times [q^4, q^8; q^{12}]_{\infty} \quad (21)$$

applying the “−1” rule to the three times of (19a) and (19b), we derive the following two Rogers-Ramanujan type identities, respectively.

**Theorem 11** (New identities of Rogers-Ramanujan type).

$$1 + 3 \sum_{m=1}^{\infty} \frac{(q^3; q^3)_{m-1}(-q; q)_{m-1}}{(q; q)_{2m-1}(q; q)_m} q^{\binom{m+1}{2}} \quad (22a)$$

$$= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} [q^6, -q, -q^5; q^6]_{\infty} \times [q^4, q^8; q^{12}]_{\infty}; \quad (22b)$$

$$1 + 3 \sum_{m=1}^{\infty} \frac{(q^6; q^6)_{m-1} q^m}{(q^2; q^2)_{m-1}(q; q)_{2m}} \quad (23a)$$

$$= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} [q^6, -q, -q^5; q^6]_{\infty} \times [q^4, q^8; q^{12}]_{\infty}. \quad (23b)$$

§8. For  $A, b, c, d, e$  subject to  $A^2 = bcde$ , there holds Weierstrass’ classical three term relation (cf. Chu [9, Theorem 1]):

$$\langle A/b, A/c, A/d, A/e; q \rangle_{\infty} - \langle b, c, d, e; q \rangle_{\infty} = b \langle A, A/bc, A/bd, A/be; q \rangle_{\infty}$$

which can be used to prove the following theta function equations

$$[q^2, qw, q^3w; q^3w]_{\infty} - (q; q)_{\infty} = q(1-w)[q^{27}, q^6, q^{21}; q^{27}]_{\infty}, \quad (24a)$$

$$[q, q^2w, q^3w; q^3w]_{\infty} - (q; q)_{\infty} = q^2(1-w)[q^{27}, q^3, q^{24}; q^{27}]_{\infty}; \quad (24b)$$

where  $\omega \neq 1$  denotes the cubic root of unity. In fact, manipulating the difference displayed in equation (24a)

$$\begin{aligned} & [q^2, qw, q^3w; q^3w]_{\infty} - (q; q)_{\infty} \\ &= [q^9, q^2, q^7; q^9]_{\infty} \{ \langle qw, q^3w, q^5w; q^9 \rangle_{\infty} - \langle q, q^3, q^5; q^9 \rangle_{\infty} \} \\ &= \frac{[q^9, q^2, q^7; q^9]_{\infty}}{[q^3w^2, q^6w; q^9]_{\infty}} \{ \langle qw, q^3w, q^5w, q^3w^2; q^9 \rangle_{\infty} - \langle q, q^3, q^5, q^3w^2; q^9 \rangle_{\infty} \} \\ &= q \frac{[q^9, q^2, q^7; q^9]_{\infty}}{[q^3w^2, q^6w; q^9]_{\infty}} \langle q^6w, w, q^2w, q^2w^2; q^9 \rangle_{\infty} \end{aligned}$$

and then simplifying the products in the last line, we get the right member stated in (24a). The equation (24b) follows analogously from the reformulation:

$$\begin{aligned}
& [q, q^2w, q^3w; q^3w]_\infty - (q; q)_\infty \\
&= [q^9, q, q^8; q^9]_\infty \{ \langle q^2w, q^3w, q^4w; q^9 \rangle_\infty - \langle q^2, q^3, q^4; q^9 \rangle_\infty \} \\
&= \frac{[q^9, q, q^8; q^9]_\infty}{[q^3w^2, q^6w; q^9]_\infty} \{ \langle q^2w, q^3w, q^4w, q^3w^2; q^9 \rangle_\infty - \langle q^2, q^3, q^4, q^3w^2; q^9 \rangle_\infty \} \\
&= q^2 \frac{[q^9, q, q^8; q^9]_\infty}{[q^3w^2, q^6w; q^9]_\infty} \langle q^6w, w, qw, qw^2; q^9 \rangle_\infty.
\end{aligned}$$

Recall the following two Rogers-Ramanujan type identities due to Bailey [2, Eqs B2 and 10.8] (see [19, Eqs 91 and 90] also)

$$\sum_{m=0}^{\infty} \frac{(q^3; q^3)_m q^{m^2+2m}}{(q; q)_m (q; q)_{2+2m}} = \frac{[q^{27}, q^6, q^{21}; q^{27}]_\infty}{(q; q)_\infty}, \quad (25a)$$

$$\sum_{m=0}^{\infty} \frac{(q^3; q^3)_m q^{m^2+3m}}{(q; q)_m (q; q)_{2+2m}} = \frac{[q^{27}, q^3, q^{24}; q^{27}]_\infty}{(q; q)_\infty}. \quad (25b)$$

Applying the “+1” rule to these two identities multiplied, respectively, by  $q(1-w)$  and  $q^2(1-w)$ , we recover the two strange identities of Rogers-Ramanujan type.

**Theorem 12** (Ismail-Stanton [14, Theorem 9]).

$$1 + (1-w) \sum_{m=1}^{\infty} \frac{(q^3; q^3)_{m-1} q^{m^2}}{(q; q)_{m-1} (q; q)_{2m}} = \frac{[q^3w, q^2, qw; q^3w]_\infty}{(q; q)_\infty}, \quad (26a)$$

$$1 + (1-w) \sum_{m=1}^{\infty} \frac{(q^3; q^3)_{m-1} q^{m^2+m}}{(q; q)_{m-1} (q; q)_{2m}} = \frac{[q^3w, q, q^2w; q^3w]_\infty}{(q; q)_\infty}. \quad (26b)$$

§9. According to the two theta function equations [9, Eqs 3.3a and 3.3b]

$$\frac{(-q; q^2)_\infty}{(q; q^2)_\infty} + \frac{(q^2; q^4)_\infty^2}{(-q^2; q^4)_\infty^2} = 2 \frac{(-q; q)_\infty}{(-q^2; q^4)_\infty^2} [-q^3, -q^4, -q^5, -q^8; q^8]_\infty^2, \quad (27a)$$

$$\frac{(-q; q^2)_\infty}{(q; q^2)_\infty} - \frac{(q^2; q^4)_\infty^2}{(-q^2; q^4)_\infty^2} = 2q \frac{(-q; q)_\infty}{(-q^2; q^4)_\infty^2} [-q, -q^4, -q^7, -q^8; q^8]_\infty^2; \quad (27b)$$

modifying with “ $\pm 1$ ” the initial term of the following identity [10, Eq 1.11a]

$$\sum_{m=0}^{\infty} \frac{(-1; q)_{2m}}{(q; q)_{2m}} q^m = \frac{(-q; q)_\infty}{(q; q)_\infty} [q^4, -q^2, -q^2; q^4]_\infty = \frac{(-q; q^2)_\infty}{(q; q^2)_\infty} \frac{(-q^2; q^4)_\infty^2}{(q^2; q^4)_\infty^2}$$

we derive further the two identities with strange product expressions.

**Theorem 13** (New identities of Rogers-Ramanujan type).

$$1 + \sum_{m=1}^{\infty} \frac{(-q; q)_{2m-1}}{(q; q)_{2m}} q^m = \frac{(-q; q)_\infty}{(q^2; q^4)_\infty^2} [-q^3, -q^4, -q^5, -q^8; q^8]_\infty^2, \quad (28a)$$

$$\sum_{m=0}^{\infty} \frac{(-q; q)_{2m+1}}{(q; q)_{2m+2}} q^m = \frac{(-q; q)_\infty}{(q^2; q^4)_\infty^2} [-q, -q^4, -q^7, -q^8; q^8]_\infty^2. \quad (28b)$$

§10. **Further bisections on triple products.** In a recent paper [10] by the authors, several identities of Rogers-Ramanujan type have been reviewed by applying the multisection method to Jacobi’s triple product identity [15]

$$[q, x, q/x; q]_\infty = \sum_{n=-\infty}^{+\infty} (-1)^n q^{\binom{n}{2}} x^n. \quad (29)$$

Splitting the sum into two parts according to the parity of summation index  $n$ , we get the following expression

$$[q, x, q/x; q]_{\infty} = \sum_{n=-\infty}^{+\infty} q^{2n^2-n} x^{2n} - x \sum_{n=-\infty}^{+\infty} q^{2n^2+n} x^{2n}.$$

Factorizing each infinite series by means of (29) leads to the equivalent identity

$$[q, x, q/x; q]_{\infty} = [q^4, -qx^2, -q^3/x^2; q^4]_{\infty} - x[q^4, -q/x^2, -q^3x^2; q^4]_{\infty} \quad (30)$$

which can also be derived from the sextuple product identity appeared in a recent paper by Chu-Yan [11, Corollary 2].

Letting  $x \rightarrow q$  and  $q \rightarrow q^3$  in (30), then writing alternatively

$$(q; q)_{\infty} = [q^3, q, q^2; q^3]_{\infty}$$

we find the following theta function equation:

$$(q; q)_{\infty} = [q^{12}, -q^5, -q^7; q^{12}]_{\infty} - q[q^{12}, -q, -q^{11}; q^{12}]_{\infty}. \quad (31)$$

According to the “-1” rule, it is not hard to check that the following identities [19, Eqs 58 and 56] (see Chu-Wang [10, Eqs 1.18a-1.18b] for correction) are equivalent:

$$1 + \sum_{m=1}^{\infty} \frac{(-q; q)_{m-1} q^{m^2}}{(q; q)_m (q; q^2)_m} = \frac{[q^{12}, -q^5, -q^7; q^{12}]_{\infty}}{(q; q)_{\infty}}, \quad (32a)$$

$$\sum_{m=0}^{\infty} \frac{(-q; q)_m q^{m^2+2m}}{(q; q)_{m+1} (q; q^2)_{m+1}} = \frac{[q^{12}, -q, -q^{11}; q^{12}]_{\infty}}{(q; q)_{\infty}}. \quad (32b)$$

Similarly by expressing the factorial fraction in terms of triple product

$$\frac{(q^2; q^2)_{\infty}}{(-q; q^2)_{\infty}} = [q^4, q, q^3; q^4]_{\infty}$$

we may specify (30), under  $x \rightarrow q$  and  $q \rightarrow q^4$ , to another theta function equation

$$\frac{(q^2; q^2)_{\infty}}{(-q; q^2)_{\infty}} = [q^{16}, -q^6, -q^{10}; q^{16}]_{\infty} - q[q^{16}, -q^2, -q^{14}; q^{16}]_{\infty}. \quad (33)$$

In accordance with the “-1” rule, we have consequently the following three equivalent pairs of Rogers-Ramanujan type identities.

- Chu-Wang [10, Eqs 1.4a and 1.4b]:

$$1 + \sum_{m=1}^{\infty} \frac{(-q^2; q^2)_{m-1}}{(q^2; q^2)_m} q^m = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} [q^{16}, -q^6, -q^{10}; q^{16}]_{\infty}, \quad (34a)$$

$$\sum_{m=0}^{\infty} \frac{(-q^2; q^2)_m}{(q^2; q^2)_{m+1}} q^m = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} [q^{16}, -q^2, -q^{14}; q^{16}]_{\infty}. \quad (34b)$$

- Gessel-Stanton [13, Eqs 7.13 and 7.15]:

$$1 + \sum_{m=1}^{\infty} \frac{(-q; q)_{m-1}}{(q; q)_m} q^{\binom{m+1}{2}} = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} [q^{16}, -q^6, -q^{10}; q^{16}]_{\infty}, \quad (35a)$$

$$\sum_{m=0}^{\infty} \frac{(-q; q)_m}{(q; q)_{m+1}} q^{\binom{m}{2}+2m} = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} [q^{16}, -q^2, -q^{14}; q^{16}]_{\infty}. \quad (35b)$$

- Slater [19, Eqs 69 and 72]:

$$\sum_{m=0}^{\infty} \frac{(-q^2; q^2)_m}{(q; q)_{2m+2}} q^{m^2+2m} = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} [q^{16}, -q^2, -q^{14}; q^{16}]_{\infty}, \quad (36a)$$

$$1 + \sum_{m=1}^{\infty} \frac{(-q^2; q^2)_{m-1}}{(q; q)_{2m}} q^{m^2} = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} [q^{16}, -q^6, -q^{10}; q^{16}]_{\infty}. \quad (36b)$$

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# VOLTERRA COMPOSITION OPERATORS FROM MIXED NORM SPACES TO BLOCH-TYPE SPACES

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**Abstract:** Let  $\varphi$  be a holomorphic self-map and  $g$  a fixed holomorphic function on the unit ball  $B$  in  $\mathbb{C}^n$ . This paper studies the boundedness and compactness of the following Volterra composition operator

$$T_{g,\varphi}f(z) = \int_0^1 f(\varphi(tz)) \Re g(tz) \frac{dt}{t},$$

from the mixed-norm space  $H(p, q, \varphi)$  to the Bloch-type space of holomorphic functions on  $B$ .

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## 1 Introduction

Let  $B$  be the unit ball of  $\mathbb{C}^n$  and  $H(B)$  the space of all holomorphic functions on  $B$ . Let  $dv$  be the normalized Lebesgue measure of  $B$ , i.e.  $v(B) = 1$ . For  $f \in H(B)$ , let

$$\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$$

represent the radial derivative of  $f \in H(B)$ .

A positive continuous function  $\mu$  on the interval  $[0, 1)$  is called normal if there is  $\delta \in [0, 1)$  and  $s$  and  $t$ ,  $0 < s < t$  such that

$$\begin{aligned} \frac{\mu(r)}{(1-r)^s} &\text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r \rightarrow 1} \frac{\mu(r)}{(1-r)^s} = 0; \\ \frac{\mu(r)}{(1-r)^t} &\text{ is increasing on } [\delta, 1) \text{ and } \lim_{r \rightarrow 1} \frac{\mu(r)}{(1-r)^t} = \infty. \end{aligned} \quad (1)$$



If we say that  $\mu : B \rightarrow [0, \infty)$  is normal we will assume that  $\mu(z) = \mu(|z|)$ ,  $z \in B$ .

Let  $\mu : B \rightarrow [0, \infty)$  be normal. For  $0 < p, q < \infty$ , the mixed norm space  $H(p, q, \nu) = H(p, q, \nu)(B)$  consists of all  $f \in H(B)$  such that

$$\|f\|_{H(p,q,\nu)} = \left( \int_0^1 M_q^p(f, r) \frac{\varphi^p(r)}{1-r} dr \right)^{1/p} < \infty,$$

where

$$M_q(f, r) = \left( \int_S |f(r\zeta)|^q d\sigma(\zeta) \right)^{1/q}.$$

For  $p = q$  and  $\varphi(r) = (1 - r^2)^{\alpha+1}$ , the mixed norm space is equivalent to the weighted Bergman space  $A_\alpha^p = A_\alpha^p(B)$ , which consisting of all  $f \in H(B)$  such that

$$\|f\|_{A_\alpha^p}^p = \int_B |f(z)|^p (1 - |z|^2)^\alpha dv(z) < \infty.$$

Let  $\mu : B \rightarrow [0, \infty)$  be normal. The Bloch-type space  $\mathcal{B}_\mu = \mathcal{B}_\mu(B)$  is the space of all functions  $f \in H(B)$  such that

$$b_\mu(f) = \sup_{z \in B} \mu(z) |\Re f(z)| < \infty.$$

$\mathcal{B}_\mu$  is Banach space with the norm  $\|f\|_{\mathcal{B}_\mu} = |f(0)| + b_\mu(f)$ . The little Bloch-type space  $\mathcal{B}_{\mu,0} = \mathcal{B}_{\mu,0}(B)$  consists of all  $f \in H(B)$  such that

$$\lim_{|z| \rightarrow 1} \mu(z) |\Re f(z)| = 0.$$

It is easy to see that  $\mathcal{B}_{\mu,0}$  is a closed subspace of  $\mathcal{B}_\mu$ . When  $\mu(r) = (1 - r^2)^\alpha$ ,  $\alpha \in (0, \infty)$ , we obtain  $\alpha$ -Bloch spaces and little  $\alpha$ -Bloch spaces (see, e.g., [21]).

Let  $\varphi$  be a holomorphic self-map of  $B$ . The composition operator  $C_\varphi$  is defined by

$$(C_\varphi f)(z) = (f \circ \varphi)(z), \quad f \in H(B).$$

The book [2] contains plenty of information on composition operators on various function spaces.

Suppose that  $g : B \rightarrow \mathbb{C}^1$  is a holomorphic map, define

$$T_g f(z) = \int_0^1 f(tz) \Re g(tz) \frac{dt}{t}, \quad f \in H(B), \quad z \in B.$$

This operator is called the extended Cesàro operator (or the Riemann-Stieltjes operator), which was introduced in [3], and studied in [1, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 17, 19, 20].

Motivated by the definition of operators  $C_\varphi$  and  $T_g$ , in [22] Zhu defined the so-called Volterra composition operator as follows.

$$T_{g,\varphi} f(z) = \int_0^1 f(\varphi(tz)) \Re g(tz) \frac{dt}{t}, \quad f \in H(B), \quad z \in B. \quad (2)$$

It is easy to see that  $T_{g,z} = T_g$ . In the setting of the unit disk  $D$ , the Volterra composition operator has the following form

$$T_{g,\varphi}f(z) = \int_0^z (f \circ \varphi)(\xi)g'(\xi)d\xi, \quad f \in H(D), \quad z \in D,$$

which was first introduced and studied in [6].

In [22], Zhu studied the boundedness and compactness of the Volterra composition operator from generalized Bergman spaces to Bloch type spaces. In [23], Zhu studied the boundedness and compactness of the Volterra composition operator on logarithmic Bloch spaces. Recall that a linear operator is said to be bounded if the image of a bounded set is a bounded set, while a linear operator is compact if it takes bounded sets to sets with compact closure.

In this paper, we study the operator  $T_{g,\varphi}$  from mixed norm spaces to Bloch type spaces in the unit ball. The sufficient and necessary conditions for the operator  $T_{g,\varphi}$  to be bounded or compact are given. As some corollaries, we can obtain the characterization of the boundedness and compactness of the extended Cesàro operator from mixed norm spaces to Bloch type spaces, which generalize the corresponding results in [13]. We omit the details.

Throughout the paper, constants are denoted by  $C$ , they are positive and may not be the same in every occurrence.

## 2 Main Results and Proofs

In this section we state and prove the main results of this paper. First we quote several lemmas which are used in the proofs of the main results. The following characterization of compactness can be proved in a standard way (see, e.g., the proofs of Proposition 3.11 of [2]), hence we omit its proof.

**Lemma 1.** *Assume that  $0 < p, q < \infty$ ,  $g \in H(B)$ ,  $\varphi$  is a holomorphic self-map of  $B$ ,  $\nu : B \rightarrow [0, \infty)$  and  $\mu : B \rightarrow [0, \infty)$  are normal. Then the operator  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_\mu$  is compact if and only if  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_\mu$  is bounded and for every bounded sequence  $(f_k)_{k \in \mathbb{N}} \subset H(p, q, \nu)$  converging to 0 uniformly on compacts of  $B$  we have*

$$\lim_{k \rightarrow \infty} \|T_{g,\varphi}f_k\|_{\mathcal{B}_\mu} = 0.$$

**Lemma 2.** *Assume that  $\mu : B \rightarrow [0, \infty)$  is normal. A closed set  $K$  in  $\mathcal{B}_{\mu,0}$  is compact if and only if it is bounded and satisfies*

$$\lim_{|z| \rightarrow 1} \sup_{f \in K} \mu(z)|\Re f(z)| = 0.$$

**Lemma 3.** [18] *Assume that  $0 < p, q < \infty$  and  $\nu : B \rightarrow [0, \infty)$  is normal. Then there is a positive constant  $C$  independent of  $f$  such that*

$$|f(z)| \leq C \frac{\|f\|_{H(p,q,\nu)}}{\nu(z)(1-|z|^2)^{\frac{n}{q}}}, \quad z \in B. \quad (3)$$

Now we are in a position to state and prove our main results.

**Theorem 1.** *Assume that  $0 < p, q < \infty$ ,  $g \in H(B)$ ,  $\varphi$  is a holomorphic self-map of  $B$ ,  $\nu : B \rightarrow [0, \infty)$  and  $\mu : B \rightarrow [0, \infty)$  are normal. Then  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_\mu$  is bounded if and only if*

$$M := \sup_{z \in B} \frac{\mu(z)|\Re g(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{n}{q}}} < \infty. \quad (4)$$

Moreover, if  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_\mu$  is bounded then the following asymptotic relation holds

$$\|T_{g,\varphi}\|_{H(p,q,\nu) \rightarrow \mathcal{B}_\mu} \asymp \sup_{z \in B} \frac{\mu(z)|\Re g(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{n}{q}}}. \quad (5)$$

*Proof.* Assume that (4) holds. A calculation with (2) gives the following formula (see, e.g. [3, 22])

$$\Re[T_{g,\varphi}(f)](z) = f(\varphi(z))\Re g(z).$$

Moreover  $(T_{g,\varphi}f)(0) = 0$ . Then by Lemma 3 and the condition (4), we have

$$\begin{aligned} \|T_{g,\varphi}f\|_{\mathcal{B}_\mu} &= |(T_{g,\varphi}f)(0)| + \sup_{z \in B} \mu(z)|\Re g(z)||f(z)| \\ &\leq C \sup_{z \in B} \frac{\mu(z)|\Re g(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{n}{q}}} \|f\|_{H(p,q,\nu)} = CM \|f\|_{H(p,q,\nu)}, \end{aligned} \quad (6)$$

i.e.  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_\mu$  is bounded.

Now assume that  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_\mu$  is bounded. Set

$$f_w(z) = \frac{(1 - |w|^2)^\beta}{\varphi(w)(1 - \langle z, w \rangle)^{\frac{n}{q} + \beta}}, \quad z \in B, \quad (7)$$

where  $\beta > 1$ . By [18] we know that  $\sup_{w \in B} \|f_w\|_{H(p,q,\nu)} \leq C$ . Therefore

$$\begin{aligned} C \|T_{g,\varphi}\|_{H(p,q,\nu) \rightarrow \mathcal{B}_\mu} &\geq \|T_{g,\varphi}f_w\|_{\mathcal{B}_\mu} = \sup_{z \in B} \mu(z)|\Re g(z)||f_w(\varphi(z))| \\ &\geq \mu(w)|\Re g(w)||f_w(\varphi(w))| = \frac{\mu(w)|\Re g(w)|}{\nu(\varphi(w))(1 - |\varphi(w)|^2)^{\frac{n}{q}}}, \end{aligned}$$

from which (4) follows, moreover

$$\sup_{w \in B} \frac{\mu(w)|\Re g(w)|}{\nu(\varphi(w))(1 - |\varphi(w)|^2)^{\frac{n}{q}}} \leq C \|T_{g,\varphi}\|_{H(p,q,\nu) \rightarrow \mathcal{B}_\mu}. \quad (8)$$

From (6) and (8), asymptotic relationship (5) follows.  $\square$

**Theorem 2.** Assume  $0 < p, q < \infty$ ,  $g \in H(B)$ ,  $\varphi$  is a holomorphic self-map of  $B$ ,  $\nu : B \rightarrow [0, \infty)$  and  $\mu : B \rightarrow [0, \infty)$  are normal. Then  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_\mu$  is compact if and only if  $g \in \mathcal{B}_\mu$  and

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{\mu(z)|\Re g(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{n}{q}}} = 0. \quad (9)$$

*Proof.* Suppose that  $g \in \mathcal{B}_\mu$  and (9) holds. From  $g \in \mathcal{B}_\mu$  and (9), it is easy to see that (4) holds. Hence  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_\mu$  is bounded by Theorem 1. From (9) we have that for every  $\varepsilon > 0$ , there is an  $r \in (0, 1)$  such that

$$\frac{\mu(z)|\Re g(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{n}{q}}} < \varepsilon. \quad (10)$$

when  $|\varphi(z)| > r$ . Let  $(f_k)_{k \in \mathbb{N}}$  be a bounded sequence in  $H(p, q, \nu)$  such that  $f_k$  converges to 0 uniformly on compact subsets of  $B$  as  $k \rightarrow \infty$ . Let  $G = \{w \in B : |w| \leq \delta\}$ . From the fact that  $g \in \mathcal{B}_\mu$  and (10), we have

$$\begin{aligned} \|T_{g,\varphi} f_k\|_{\mathcal{B}_\mu} &= \sup_{z \in B} \mu(|z|) |f_k(\varphi(z)) \Re g(z)| \\ &= \left( \sup_{\{z \in B : |\varphi(z)| \leq \delta\}} + \sup_{\{z \in B : \delta < |\varphi(z)| < 1\}} \right) \mu(z) |\Re g(z)| |f_k(\varphi(z))| \\ &= \|g\|_{\mathcal{B}_\mu} \sup_{w \in G} |f_k(w)| + C \|f_k\|_{H(p,q,\nu)} \sup_{\{z \in B : \delta < |\varphi(z)| < 1\}} \frac{\mu(z)|\Re g(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{n}{q}}} \\ &\leq \|g\|_{\mathcal{B}_\mu} \sup_{w \in G} |f_k(w)| + C \|f_k\|_{H(p,q,\nu)} \varepsilon. \end{aligned} \quad (11)$$

Note that  $G$  is a compact subset of  $B$ , we have  $\lim_{k \rightarrow \infty} \sup_{w \in G} |f_k(w)| = 0$ . Using this fact and letting  $k \rightarrow \infty$  in (11), we obtain  $\limsup_{k \rightarrow \infty} \|T_{g,\varphi} f_k\|_{\mathcal{B}_\mu} \leq C \|f_k\|_{H(p,q,\nu)} \varepsilon$ . Since  $\varepsilon$  is an arbitrary positive number, we obtain

$$\limsup_{k \rightarrow \infty} \|T_{g,\varphi} f_k\|_{\mathcal{B}_\mu} = 0.$$

Employing Lemma 1, we get that  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_\mu$  is compact.

Conversely we assume that  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_\mu$  is compact. We need to prove that (9) holds. This can be done by contradiction. We assume that (9) is not true. Then there would be some  $\varepsilon_0 > 0$  and a sequence  $(z_k)_{k \in \mathbb{N}} \subseteq B$  with  $\lim_{k \rightarrow \infty} |\varphi(z_k)| = 1$ , and such that

$$\frac{\mu(z_k)|\Re g(z_k)|}{\nu(\varphi(z_k))(1 - |\varphi(z_k)|^2)^{\frac{n}{q}}} \geq \varepsilon_0, \quad (12)$$

for every  $k \in \mathbb{N}$ . Set

$$f_k(z) = \frac{(1 - |\varphi(z_k)|^2)^\beta}{\nu(\varphi(z_k))(1 - \langle z, \varphi(z_k) \rangle)^{\frac{n}{q} + \beta}}, \quad k \in \mathbb{N}, \quad (13)$$

where  $\beta > 1$ . Then  $\sup_{k \in \mathbb{N}} \|f_k\|_{H(p,q,\nu)} \leq C$ , and  $f_k$  converges to 0 uniformly on compacts of  $B$ . Hence by Lemma 1,

$$\lim_{k \rightarrow \infty} \|T_{g,\varphi} f_k\|_{\mathcal{B}_\mu} = 0. \quad (14)$$

On the other hand, we have

$$\begin{aligned} \|T_{g,\varphi} f_k\|_{\mathcal{B}_\mu} &= \sup_{z \in B} \mu(z) |\Re(T_{g,\varphi} f_k)(z)| \geq \mu(z_k) |\Re g(z_k)| |f_k(\varphi(z_k))| \\ &= \frac{\mu(z_k) |\Re g(z_k)|}{\nu(\varphi(z_k))(1 - |\varphi(z_k)|^2)^{\frac{n}{q}}} \geq \varepsilon_0, \end{aligned}$$

which contradicts to (14). This completes the proof of this theorem.  $\square$

**Theorem 3.** Assume that  $0 < p, q < \infty$ ,  $g \in H(B)$ ,  $\varphi$  is a holomorphic self-map of  $B$ ,  $\nu : B \rightarrow [0, \infty)$  and  $\mu : B \rightarrow [0, \infty)$  are normal. Then  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_{\mu,0}$  is bounded if and only if  $g \in \mathcal{B}_{\mu,0}$  and  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_\mu$  is bounded.

*Proof.* Suppose that  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_{\mu,0}$  is bounded, then  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_\mu$  is bounded. Taking  $f(z) = 1$ , and employing the boundedness of  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_{\mu,0}$ , we get  $g \in \mathcal{B}_{\mu,0}$ , as desired.

Conversely, suppose that  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_\mu$  is bounded and  $g \in \mathcal{B}_{\mu,0}$ . For each polynomial  $p(z)$ , we obtain

$$\mu(z) |\Re(T_{g,\varphi} p)(z)| = \mu(z) |p(\varphi(z))| |\Re g(z)| \leq \|p\|_\infty \mu(z) |\Re g(z)|. \quad (15)$$

From the above inequality, it follows that for each polynomial  $p$ ,  $T_{g,\varphi}(p) \in \mathcal{B}_{\mu,0}$ . Since the set of all polynomials is dense in  $H(p, q, \nu)$ , for every  $f \in H(p, q, \nu)$  there is a sequence of polynomials  $(p_k)_{k \in \mathbb{N}}$  such that  $\|p_k - f\|_{H(p,q,\nu)} \rightarrow 0$  as  $k \rightarrow \infty$ . From the boundedness of  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_\mu$ , we have that

$$\|T_{g,\varphi} p_k - T_{g,\varphi} f\|_{\mathcal{B}_\mu} \leq \|T_{g,\varphi}\|_{H(p,q,\nu) \rightarrow \mathcal{B}_\mu} \|p_k - f\|_{H(p,q,\nu)} \rightarrow 0, \quad \text{as } k \rightarrow \infty. \quad (16)$$

From this and since  $\mathcal{B}_{\mu,0}$  is a closed subset of  $\mathcal{B}_\mu$ , we obtain

$$T_{g,\varphi} f = \lim_{k \rightarrow \infty} T_{g,\varphi} p_k \in \mathcal{B}_{\mu,0}. \quad (17)$$

Therefore  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_{\mu,0}$  is bounded. The proof is completed.  $\square$

**Theorem 4.** Assume that  $0 < p, q < \infty$ ,  $g \in H(B)$ ,  $\varphi$  is a holomorphic self-map of  $B$ ,  $\nu : B \rightarrow [0, \infty)$  and  $\mu : B \rightarrow [0, \infty)$  are normal. Then  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_{\mu,0}$  is compact if and only if

$$\lim_{|z| \rightarrow 1} \frac{\mu(z) |\Re g(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{n}{q}}} = 0. \quad (18)$$

*Proof.* Suppose that  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_{\mu,0}$  is compact. Then  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_{\mu,0}$  is bounded and  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_{\mu}$  is compact. By Theorems 2 and 3 we obtain

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{\mu(z)|\Re g(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{n}{q}}} = 0, \quad (19)$$

and

$$\lim_{|z| \rightarrow 1} \mu(z)|\Re g(z)| = 0. \quad (20)$$

By (19), for every  $\varepsilon > 0$ , there exists a  $\delta \in (0, 1)$ ,

$$\frac{\mu(z)|\Re g(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{n}{q}}} < \varepsilon$$

when  $\delta < |\varphi(z)| < 1$ . By (20), for the above  $\varepsilon$ , there exists a  $r \in (0, 1)$ ,

$$\mu(z)|\Re g(z)| \leq \varepsilon \nu(\delta)(1 - |\delta|^2)^{\frac{n}{q}}$$

when  $r < |z| < 1$ .

Therefore, when  $r < |z| < 1$  and  $\delta < |\varphi(z)| < 1$ , we have that

$$\frac{\mu(z)|\Re g(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{n}{q}}} < \varepsilon. \quad (21)$$

If  $|\varphi(z)| \leq \delta$  and  $r < |z| < 1$ , we obtain

$$\frac{\mu(z)|\Re g(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{n}{q}}} \leq \frac{1}{\nu(\delta)(1 - |\delta|^2)^{\frac{n}{q}}} \mu(|z|)|\Re g(z)| < \varepsilon. \quad (22)$$

Combing (21) with (22) we get (18), as desired.

Conversely, suppose that (18) holds. From Lemma 2, we see that  $T_{g,\varphi} : H(p, q, \nu) \rightarrow \mathcal{B}_{\mu,0}$  is compact if and only if

$$\lim_{|z| \rightarrow 1} \sup_{\|f\|_{H(p,q,\nu)} \leq 1} \mu(z)|\Re(T_{g,\varphi}f)(z)| = 0. \quad (23)$$

For any  $f \in H(p, q, \nu)$  with  $\|f\|_{H(p,q,\nu)} \leq 1$ , by (3) we have

$$\mu(z)|\Re(T_{g,\varphi}f)(z)| \leq C\|f\|_{H(p,q,\nu)} \frac{\mu(z)|\Re g(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{n}{q}}}.$$

Using (18) we get

$$\lim_{|z| \rightarrow 1} \sup_{\|f\|_{H(p,q,\nu)} \leq 1} \mu(z)|\Re(T_{g,\varphi}f)(z)| \leq C \lim_{|z| \rightarrow 1} \frac{\mu(z)|\Re g(z)|}{\nu(\varphi(z))(1 - |\varphi(z)|^2)^{\frac{n}{q}}} = 0,$$

as desired. This completes the proof of the theorem.  $\square$

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## NEW KKM TYPE THEOREMS IN $G$ -CONVEX SPACES WITH APPLICATIONS

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**Abstract.** The purpose of this paper is to establish some new KKM type theorems in  $G$ -convex spaces and, as applications, to obtain some new matching theorems, fixed point theorems, section theorems and minimax theorems in  $G$ -convex spaces. The results presented in this paper improve and generalize the corresponding results in [1-6, 16, 20, 23].

**Key Words:**  $G$ -convex space; KKM type theorem; Matching theorem; Fixed point theorem; Section theorem; Minimax theorem.

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### 1. INTRODUCTION AND PRELIMINARIES

In 1929, Knaster, Kuratowski and Mazurkiewicz [11] deduced the celebrated KKM theorem. In 1961, Ky Fan [7] gave an extended version of the KKM theorem to infinite dimensional spaces and made useful variations such as geometric property of convex sets, best approximation theorem and minimax

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inequality. Since then, many authors have given numerous generalizations of the known results and new applications (see Granas [8], Park [15, 17] and references therein). Later, this theorem was extended to convex spaces by Lassonde [12, 13] and to space having certain families of contractible subsets (simply,  $H$ -spaces) by Horvath [10] and was further developed in  $H$ -spaces ([1-5, 9, 16]). In [18-22], Park et al. introduced and studied generalized convex spaces (simply,  $G$ -convex spaces) and obtained many important results in  $G$ -convex spaces. The concept of  $G$ -convex spaces is a common generalization of the usual convexity in a topological vector space and many abstract convexities which have been mainly developed in connection with the fixed point and the KKM theory.

The purpose of this paper is to establish a new KKM type theorem in  $G$ -convex spaces and as applications, to obtain some new matching theorems, fixed point theorems, section theorems and minimax theorems in  $G$ -convex spaces. The results presented in this paper improve and generalize the corresponding results in [1-6, 16, 20, 23].

**Definition 1.1.** ([14, 18, 19]) (1) A generalized convex space (or a  $G$ -convex space)  $(X, D; \Gamma)$  consists of a topological space  $X$  and a nonempty set  $D$  such that, for each  $A = \{a_0, a_1, \dots, a_n\} \in \langle D \rangle$ , there exists a subset  $\Gamma(A) = \Gamma_A$  of  $X$  and a continuous function  $\phi_A : \Delta_n \rightarrow \Gamma(A)$  such that  $J \subset \{0, 1, \dots, n\}$  implies  $\phi_A(\Delta_J) \subset \Gamma(\{a_j : j \in J\})$ , where  $\langle D \rangle$  denotes the set of all nonempty finite subsets of  $D$ ,  $\Delta_n$  an  $n$ -simplex with vertices  $v_0, v_1, \dots, v_n$  and  $\Delta_J = co\{v_j : j \in J\}$  the face of  $\Delta_n$  corresponding to  $J$ , respectively.

In case to emphasize  $X \supset D$ ,  $(X, D; \Gamma)$  will be denoted by  $(X \supset D; \Gamma)$ ; and if  $X = D$ , then  $(X \supset X; \Gamma)$  by  $(X, \Gamma)$ .

(2) For a  $G$ -convex space  $(X \supset D; \Gamma)$ , a subset  $K$  of  $X$  is said to be  $\Gamma$ -convex if, for each  $N \in \langle D \rangle$ ,  $N \subset K$  implies  $\Gamma_N \subset K$ . If  $X$  is compact, then the  $G$ -convex space  $(X, D; \Gamma)$  is said to be compact.

**Definition 1.2.** ([14]) Let  $(X, D; \Gamma)$  be a  $G$ -convex space and  $I$  a nonempty set. A mapping  $F : I \multimap X$  is called a generalized KKM mapping provided that, for each  $N \in \langle I \rangle$ , there exists a function  $\sigma : N \rightarrow D$  such that  $\Gamma_{\sigma(M)} \subset F(M)$  for each  $M \in \langle N \rangle$ .

Lee [14] studied generalized KKM mapping and proved the following result:

**Theorem 1.1.** Let  $(X, D; \Gamma)$  be a  $G$ -convex space,  $I$  a nonempty set and  $F : I \multimap X$  a generalized KKM mapping with closed values such that  $\bigcap_{z \in M} F(z)$  is compact for some  $M \in \langle I \rangle$ . Then  $\bigcap_{z \in I} F(z) \neq \emptyset$ .

## 2. A NEW KKM TYPE THEOREM

In order to prove the main results of this work, we introduce the following concept:

**Definition 2.1.** Let  $(X, \Gamma)$  and  $(Y, D; \Gamma')$  be two  $G$ -convex spaces. A mapping  $F : X \multimap Y$  is called a quasi-generalized KKM mapping provided that, for each  $N \in \langle X \rangle$ , there exists a function  $\sigma : N \rightarrow D$  such that  $\Gamma'_{\sigma(M)} \subset F(M \cup \Gamma_M)$  for each  $M \in \langle N \rangle$ .

**Example 2.1.** Let  $X = (-\infty, 0)$ ,  $Y = (0, +\infty)$  with Euclidean topologies and, for each  $M \in \langle X \rangle$  and  $N \in \langle Y \rangle$ , let  $\Gamma_M = co(M \cup \{\min M - 2\})$  and  $\Gamma'_N = co(N \cup \{\max N + 1\})$  ( $coA$  denotes the hull of  $A$ ). Then  $(X, \Gamma)$  and  $(Y, \Gamma')$  are two  $G$ -convex spaces. Define a mapping  $F : X \multimap Y$  by

$$F(x) = \{-x, -x + 1\}, \quad \forall x \in X,$$

and, for each  $M \in \langle X \rangle$ , let  $\sigma(x) = -x$  for each  $x \in M$ . Then we have

$$\begin{aligned} \Gamma'_{\sigma(M_0)} &= co(\sigma(M_0) \cup \{\max \sigma(M_0) + 1\}) = co\left(\bigcup_{x \in M_0} \{-x, -x + 1\}\right) \\ &\subset co\left(\bigcup_{x \in co M_0} \{-x, -x + 1\}\right) \subset \bigcup_{x \in \Gamma_{M_0}} \{-x, -x + 1\} \\ &= \bigcup_{x \in M_0 \cup \Gamma_{M_0}} F(x), \quad \forall M_0 \in \langle M \rangle. \end{aligned}$$

This shows that  $F$  is a quasi-generalized KKM mapping.

Let  $X, Y$  be two nonempty sets and  $F : X \multimap Y$  a mapping. Then, for each  $y \in Y$ , we put

$$F^{-1}(y) = \{x \in X : y \in F(x)\}, \quad F^*(y) = X \setminus F^{-1}(y).$$

**Theorem 2.1.** Let  $(X, \Gamma)$ ,  $(Y, D; \Gamma')$  be two  $G$ -convex spaces and  $F : X \multimap Y$  a quasi-generalized KKM mapping with closed values such that  $\bigcap_{x \in M} F(x)$  is compact for some  $M \in \langle X \rangle$ . Suppose that  $F^*(y)$  is  $\Gamma$ -convex for each  $y \in Y$ . Then  $\bigcap_{x \in X} F(x) \neq \emptyset$ .

**Proof.** If  $F$  is not a generalized KKM mapping, then there exists  $N_0 \in \langle X \rangle$  and, for each function  $\sigma : N_0 \rightarrow D$ , there exists  $M_0 \in \langle N_0 \rangle$  such that

$$\Gamma'_{\sigma(M_0)} \not\subset F(M_0).$$

That is, there exists  $y_0 \in \Gamma'_{\sigma(M_0)}$  with  $y_0 \notin F(M_0)$ , i.e.,  $y_0 \notin F(x)$  for each  $x \in M_0$ , which implies that  $x \notin F^{-1}(y_0)$  for each  $x \in M_0$  and so  $M_0 \subset F^*(y_0)$ . Since  $F^*(y_0)$  is  $\Gamma$ -convex, it follows that  $\Gamma_{M_0} \subset F^*(y_0)$ . Thus  $M_0 \cup \Gamma_{M_0} \subset F^*(y_0)$  and so  $x \notin F^{-1}(y_0)$  for each  $x \in M_0 \cup \Gamma_{M_0}$ . This shows that  $y_0 \notin$

$F(M_0 \cup \Gamma_{M_0})$ , which is a contradiction to  $F$ , that is, the quasi-generalized KKM mapping. Therefore, it follows from Theorem 1.1 that  $\bigcap_{x \in X} F(x) \neq \emptyset$ . This completes the proof.

**Corollary 2.1.** Let  $(X, \Gamma)$ ,  $(Y, D; \Gamma')$  be two  $G$ -convex spaces and  $F : X \multimap Y$  a quasi-generalized KKM mapping with closed values. Suppose that  $F^*(y)$  is  $\Gamma$ -convex for each  $y \in Y$  and there exists  $x_0 \in X$  such that  $F(x_0)$  is compact. Then  $\bigcap_{x \in X} F(x) \neq \emptyset$ .

**Remark 2.1.** Theorem 2.2 and Corollary 2.1 generalize Theorems 1 and 2 in Badaro and Ceppitelli [1], Theorem 4 in Badaro and Ceppitelli [2], Theorem 1 in Chang and Ma [4], Lemma 1.3 in Chang and Yang [5], Theorem 3 and Corollary 4 in Wu and Li [23] to  $G$ -convex spaces.

### 3. MATCHING THEOREMS AND FIXED POINT THEOREMS

By using Theorem 2.1, we have the following:

**Theorem 3.1.** Let  $(X, \Gamma)$ ,  $(Y, D; \Gamma')$  be two  $G$ -convex spaces and  $F : X \multimap Y$  a mapping with open values such that  $Y \setminus \bigcup_{x \in M} F(x)$  is compact for some  $M \in \langle X \rangle$  and the following conditions hold:

- (i)  $F(X) = Y$ ;
- (ii) for each  $y \in Y$ ,  $F^{-1}(y)$  is  $\Gamma$ -convex.

Then there exists  $N_0 \in \langle X \rangle$  such that, for each function  $\sigma : N_0 \rightarrow D$ ,

$$\Gamma'_{\sigma(M_0)} \cap \bigcap_{x \in M_0 \cup \Gamma_{M_0}} F(x) \neq \emptyset$$

for some  $M_0 \in \langle N_0 \rangle$ .

**Proof.** If the conclusion of Theorem 3.1 is false, then for each  $N \in \langle X \rangle$ , there exists a function  $\sigma : N \rightarrow D$  such that

$$\Gamma'_{\sigma(N_0)} \subset Y \setminus \bigcap_{x \in N_0 \cup \Gamma_{N_0}} F(x) = \bigcup_{x \in N_0 \cup \Gamma_{N_0}} (Y \setminus F(x)), \quad \forall N_0 \in \langle N \rangle.$$

Let

$$G(x) = Y \setminus F(x), \quad \forall x \in X.$$

Then

$$\Gamma'_{\sigma(N_0)} \subset \bigcup_{x \in N_0 \cup \Gamma_{N_0}} G(x) = G(N_0 \cup \Gamma_{N_0}), \quad \forall N_0 \in \langle N \rangle.$$

So  $G : X \multimap Y$  is a quasi-generalized KKM mapping with closed values and

$$\bigcap_{x \in M} G(x) = \bigcap_{x \in M} (Y \setminus F(x)) = Y \setminus \bigcup_{x \in M} F(x)$$

is a compact set.

Now, we prove that  $G^*(y)$  is  $\Gamma$ -convex for each  $y \in Y$ . Firstly, we prove  $G^*(y) \neq \emptyset$ . Otherwise, then there exists  $\bar{y} \in Y$  such that  $G^*(\bar{y}) = \emptyset$  and thus  $G^{-1}(\bar{y}) = X$ . Therefore,  $x \in G^{-1}(\bar{y})$  for each  $x \in X$ , which implies that  $\bar{y} \in G(x)$  and so  $\bar{y} \notin F(x)$ , that is,  $\bar{y} \notin F(X) = Y$ , which is a contradiction.

For each  $N \in \langle G^*(y) \rangle$ , let  $x \in N$ . Then  $x \notin G^{-1}(y)$  and so  $y \notin G(x)$ . Thus  $y \in F(x)$ . This implies that  $x \in F^{-1}(y)$  for each  $x \in N$ , that is,  $N \subset F^{-1}(y)$ . By also the condition (ii), we have  $\Gamma_N \subset F^{-1}(y)$ . Let  $x \in \Gamma_N$ . Then  $x \in F^{-1}(y)$  and so  $y \in F(x)$ . Thus  $y \notin G(x)$ . Therefore,  $x \notin G^{-1}(y)$ , which implies that  $x \in G^*(y)$ . This shows that  $\Gamma_N \subset G^*(y)$  and so that  $G^*(y)$  is  $\Gamma$ -convex. It follows from Theorem 2.1 that

$$\bigcap_{x \in X} (Y \setminus F(x)) = \bigcap_{x \in X} G(x) \neq \emptyset$$

and

$$Y \setminus F(X) = Y \setminus \bigcup_{x \in X} F(x) = \bigcap_{x \in X} (Y \setminus F(x)) \neq \emptyset.$$

This is a contradiction to the condition (i). This completes the proof.

As an immediate consequence of Theorem 3.1, we have the following:

**Theorem 3.2.** Let  $(X, \Gamma)$  be a  $G$ -convex space,  $(Y, D; \Gamma')$  a compact  $G$ -convex space and  $F : X \multimap Y$  a mapping with open values. Suppose that the following conditions hold:

- (i)  $F(X) = Y$ ;
- (ii) for each  $y \in Y$ ,  $F^{-1}(y)$  is  $\Gamma$ -convex.

Then there exists  $N_0 \in \langle X \rangle$  such that, for each function  $\sigma : N_0 \rightarrow D$ ,

$$\Gamma'_{\sigma(M_0)} \cap \bigcap_{x \in M_0 \cup \Gamma_{M_0}} F(x) \neq \emptyset$$

for some  $M_0 \in \langle N_0 \rangle$ .

**Remark 3.1.** Theorems 3.1 and 3.2 generalize Theorem 2 and Corollary 1 in Chang and Ma [4] and Theorem 5 in Park [16] to  $G$ -convex spaces.

**Corollary 3.1.** Let  $(X, \Gamma)$  be a  $G$ -convex space and  $F : X \multimap X$  a mapping with open values such that  $X \setminus \bigcup_{x \in M} F(x)$  is compact for some  $M \in \langle X \rangle$ . Suppose that the following conditions hold:

- (i)  $F(X) = X$ ;
- (ii) for each  $y \in X$ ,  $F^{-1}(y)$  is  $\Gamma$ -convex.

Then there exist  $M_0 \in \langle X \rangle$  and  $\hat{x} \in \Gamma_{M_0}$  such that

$$\Gamma_{M_0} \cap \bigcap_{x \in M_0} F(x) \neq \emptyset, \quad \Gamma_{M_0} \cap \bigcap_{x \in \Gamma_{M_0}} F(x) \neq \emptyset$$

and

$$\hat{x} \in F(\hat{x}).$$

**Proof.** Let  $D = Y = X$ ,  $\Gamma' = \Gamma$  and  $\sigma$  be an identical mapping as in Theorem 3.1. Then we know that there exists  $M_0 \in \langle X \rangle$  such that

$$\Gamma_{M_0} \cap \bigcap_{x \in M_0} F(x) \supset \Gamma_{M_0} \cap \bigcap_{x \in M_0 \cup \Gamma_{M_0}} F(x) = \Gamma_{\sigma(M_0)} \cap \bigcap_{x \in M_0 \cup \Gamma_{M_0}} F(x) \neq \emptyset$$

and

$$\Gamma_{M_0} \cap \bigcap_{x \in \Gamma_{M_0}} F(x) \supset \Gamma_{M_0} \cap \bigcap_{x \in M_0 \cup \Gamma_{M_0}} F(x) = \Gamma_{\sigma(M_0)} \cap \bigcap_{x \in M_0 \cup \Gamma_{M_0}} F(x) \neq \emptyset.$$

Let  $\hat{x} \in \Gamma_{M_0} \cap \bigcap_{x \in \Gamma_{M_0}} F(x)$ . Then there exists  $\hat{x} \in \Gamma_{M_0}$  such that  $\hat{x} \in F(\hat{x})$ . This completes the proof.

From Corollary 3.1, we have the following:

**Corollary 3.2.** Let  $(X, \Gamma)$  be a compact  $G$ -convex space and  $F : X \multimap X$  a mapping with open values. Suppose that the following conditions hold:

- (i)  $F(X) = X$ ;
- (ii) for each  $y \in X$ ,  $F^{-1}(y)$  is  $\Gamma$ -convex.

Then there exist  $M_0 \in \langle X \rangle$  and  $\hat{x} \in \Gamma_{M_0}$  such that

$$\Gamma_{M_0} \cap \bigcap_{x \in M_0} F(x) \neq \emptyset, \quad \Gamma_{M_0} \cap \bigcap_{x \in \Gamma_{M_0}} F(x) \neq \emptyset$$

and

$$\hat{x} \in F(\hat{x}).$$

**Remark 3.2.** Corollaries 3.1 and 3.2 generalize Theorem 3.1 in Chen et al. [6] to  $G$ -convex spaces and improve Theorems 5 and 8, Corollaries 8.2 and 8.3 in Park [20].

#### 4. SECTION THEOREMS

By using Theorem 3.1, we have the following:

**Theorem 4.1.** Let  $(X, \Gamma)$ ,  $(Y, D; \Gamma')$  be two  $G$ -convex spaces,  $Z$  a nonempty set,  $B \subset Z$  and let  $g : X \times Y \rightarrow Z$  be a mapping satisfying the following conditions:

- (i) for each  $x \in X$ , the set  $\{y \in Y : g(x, y) \in B\}$  is open, and the set  $Y \setminus \bigcup_{x \in M} \{y \in Y : g(x, y) \in B\}$  is compact for some  $M \in \langle X \rangle$ ;
- (ii) for each  $y \in Y$ , the set  $\{x \in X : g(x, y) \in B\}$  is  $\Gamma$ -convex;
- (iii)  $\bigcup_{x \in X} \{y \in Y : g(x, y) \in B\} = Y$ .

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Then there exists  $N_0 \in \langle X \rangle$  such that, for each function  $\sigma : N_0 \rightarrow D$ ,

$$g((M_0 \cup \Gamma_{M_0}) \times \{y_0\}) \subset B$$

for some  $M_0 \in \langle N_0 \rangle$  and  $y_0 \in \Gamma'_{\sigma(M_0)}$ .

**Proof.** Let

$$F(x) = \{y \in Y : g(x, y) \in B\}, \quad \forall x \in X.$$

It follows from Theorem 3.1 that there exists  $N_0 \in \langle X \rangle$  such that, for each function  $\sigma : N_0 \rightarrow D$ ,

$$\Gamma'_{\sigma(M_0)} \cap \bigcap_{x \in M_0 \cup \Gamma_{M_0}} F(x) \neq \emptyset$$

for some  $M_0 \in \langle N_0 \rangle$ . Let  $y_0 \in \Gamma'_{\sigma(M_0)} \cap \bigcap_{x \in M_0 \cup \Gamma_{M_0}} F(x)$ . Then  $y_0 \in \Gamma'_{\sigma(M_0)}$  and  $g(x, y_0) \in B$  for each  $x \in M \cup \Gamma_{M_0}$ , that is,

$$g((M_0 \cup \Gamma_{M_0}) \times \{y_0\}) \subset B.$$

This completes the proof.

As an immediate consequence of Theorem 4.1, we have the following:

**Theorem 4.2.** Let  $(X, \Gamma)$  be a  $G$ -convex space,  $(Y, D; \Gamma')$  a compact  $G$ -convex space,  $Z$  a nonempty set,  $B \subset Z$  and let  $g : X \times Y \rightarrow Z$  be a mapping satisfying the following conditions:

- (i) for each  $x \in X$ , the set  $\{y \in Y : g(x, y) \in B\}$  is open;
- (ii) for each  $y \in Y$ , the set  $\{x \in X : g(x, y) \in B\}$  is  $\Gamma$ -convex;
- (iii)  $\bigcup_{x \in X} \{y \in Y : g(x, y) \in B\} = Y$ .

Then there exists  $N_0 \in \langle X \rangle$  such that, for each function  $\sigma : N_0 \rightarrow D$ ,

$$g((M_0 \cup \Gamma_{M_0}) \times \{y_0\}) \subset B$$

for some  $M_0 \in \langle N_0 \rangle$  and  $y_0 \in \Gamma'_{\sigma(M_0)}$ .

**Remark 4.1.** Theorems 4.1 and 4.2 generalize Theorem 1 and Corollary 2 in Wu and Li [23] to  $G$ -convex spaces.

Let  $D = Y = X$ ,  $\Gamma' = \Gamma$ ,  $Z = X \times X$ ,  $g(x, y) = (x, y)$  and  $\sigma$  be an identical mapping as in Theorem 4.1, then we have the following:

**Corollary 4.1.** Let  $(X, \Gamma)$  be a  $G$ -convex space and  $B \subset X \times X$ . Suppose that the following conditions hold:

- (i) for each  $x \in X$ , the set  $\{y \in X : (x, y) \in B\}$  is open and the set  $X \setminus \bigcup_{x \in M} \{y \in X : (x, y) \in B\}$  is compact for some  $M \in \langle X \rangle$ ;
- (ii) for each  $y \in X$ , the set  $\{x \in X : (x, y) \in B\}$  is  $\Gamma$ -convex;
- (iii)  $\bigcup_{x \in X} \{y \in X : (x, y) \in B\} = X$ .

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Then there exist  $M_0 \in \langle X \rangle$  and  $\hat{x} \in \Gamma_{M_0}$  such that

$$M_0 \times \{\hat{x}\} \subset B, \quad \Gamma_{M_0} \times \{\hat{x}\} \subset B$$

and

$$(\hat{x}, \hat{x}) \in B.$$

From Corollary 4.1, we have the following:

**Corollary 4.2.** Let  $(X, \Gamma)$  be a compact  $G$ -convex space and  $B \subset X \times X$ . Suppose that the following conditions hold:

- (i) for each  $x \in X$ , the set  $\{y \in X : (x, y) \in B\}$  is open;
- (ii) for each  $y \in X$ , the set  $\{x \in X : (x, y) \in B\}$  is  $\Gamma$ -convex;
- (iii)  $\bigcup_{x \in X} \{y \in X : (x, y) \in B\} = X$ .

Then there exist  $M_0 \in \langle X \rangle$  and  $\hat{x} \in \Gamma_{M_0}$  such that

$$M_0 \times \{\hat{x}\} \subset B, \quad \Gamma_{M_0} \times \{\hat{x}\} \subset B$$

and

$$(\hat{x}, \hat{x}) \in B.$$

**Remark 4.2.** Corollary 4.2 generalizes Corollary 1 in Badaro and Ceppitelli [3] to  $G$ -convex spaces.

## 5. MINIMAX THEOREMS

We first discuss the minimax theorem for the vector function.

**Theorem 5.1.** Let  $(X, \Gamma)$ ,  $(Y, D; \Gamma')$  be two  $G$ -convex spaces and  $(E, \leq)$  a complete lattice. Let  $f : X \times Y \rightarrow E$  be a mapping satisfying the following conditions:

- (i) for each  $x \in X$ ,  $f(x, \cdot)$  is bounded below and, for each  $y \in Y$ ,  $f(\cdot, y)$  is bounded above;
- (ii) for each  $\lambda > \sup_{x \in X} \inf_{y \in Y} f(x, y)$  and  $x \in X$ , the set  $\{y \in Y : f(x, y) \leq \lambda\}$  is closed and  $\bigcap_{x \in M} \{y \in Y : f(x, y) \leq \lambda\}$  is compact set for some  $M \in \langle X \rangle$ ;
- (iii) for each  $\lambda > \sup_{x \in X} \inf_{y \in Y} f(x, y)$  and  $y \in Y$ , the set  $\{x \in X : f(x, y) \not\leq \lambda\}$  is  $\Gamma$ -convex;
- (iv) for each  $\lambda > \sup_{x \in X} \inf_{y \in Y} f(x, y)$  and  $N \in \langle X \rangle$ , there exists a function  $\sigma : N \rightarrow D$  such that, for each  $M \in \langle N \rangle$ , there exists  $x_0 \in M \cup \Gamma_M$  such that

$$f(x_0, y) \leq \lambda, \quad \forall y \in \Gamma'_{\sigma(M)}.$$

Then

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).$$

**Proof.** Put  $\lambda_0 = \sup_{x \in X} \inf_{y \in Y} f(x, y)$ . We need only to prove

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \lambda_0.$$

Let

$$B_\lambda = \{(x, y) \in X \times Y : f(x, y) \not\leq \lambda\}, \quad \forall \lambda \in E.$$

Suppose that there exists  $\beta > \lambda_0$  such that

$$\bigcup_{x \in X} \{y \in Y : (x, y) \in B_\beta\} = Y.$$

It follows from the conditions (ii) and (iii) that

$$\{y \in Y : (x, y) \in B_\beta\} = \{y \in Y : f(x, y) \not\leq \beta\}$$

and

$$\{x \in X : (x, y) \in B_\beta\} = \{x \in X : f(x, y) \not\leq \beta\}$$

are open in  $Y$  and  $\Gamma$ -convex in  $X$ , respectively. By the condition (ii), we also know that

$$Y \setminus \bigcup_{x \in M} \{y \in Y : (x, y) \in B_\beta\} = \bigcap_{x \in M} \{y \in Y : f(x, y) \leq \beta\}$$

is a compact set. Let  $Z = X \times Y$  and  $g(x, y) = (x, y)$  in Theorem 4.1. Then there exists  $N_0 \in \langle X \rangle$  such that, for each function  $\sigma : N_0 \rightarrow D$ ,

$$(M_0 \cup \Gamma_{M_0}) \times \{y_0\} \subset B_\beta$$

for some  $M_0 \in \langle N_0 \rangle$  and  $y_0 \in \Gamma'_{\sigma(M_0)}$ . Hence  $f(x, y_0) \not\leq \beta$  for each  $x \in M_0 \cup \Gamma_{M_0}$ , which is a contradiction to the condition (iv). Therefore, for each  $\lambda > \lambda_0$ , there exists

$$y_\lambda \in Y \setminus \bigcup_{x \in X} \{y \in Y : (x, y) \in B_\lambda\} = Y \setminus \bigcup_{x \in X} \{y \in Y : f(x, y) \not\leq \lambda\}.$$

This implies that  $f(x, y_\lambda) \leq \lambda$  for each  $x \in X$ . Since  $E$  is a complete lattice, we have

$$\sup_{x \in X} f(x, y_\lambda) \leq \lambda$$

and so  $\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \lambda$ . Therefore, since  $\lambda > \lambda_0$  and  $\lambda$  is arbitrary, we have

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \lambda_0.$$

This completes the proof.

**Remark 5.1.** Theorem 5.1 generalizes Theorem 5 in Badaro and Ceppitelli [3] and Theorem 4.6 in Chen et al. [6] to  $G$ -convex spaces.



As an immediate consequence of Theorem 5.1, we have the following minimax theorem for real function, which generalizes Theorem 5 and Corollary 6 in Wu and Li [23] to  $G$ -convex spaces.

**Corollary 5.1.** Let  $(X, \Gamma)$ ,  $(Y, D; \Gamma')$  be two  $G$ -convex spaces and  $f : X \times Y \rightarrow \mathbb{R}$  ( $\mathbb{R}$  denotes the set of real numbers) a function satisfying the following conditions:

- (i) for each  $x \in X$ ,  $f(x, \cdot)$  is bounded below and, for each  $y \in Y$ ,  $f(\cdot, y)$  is bounded above;
- (ii) for each  $\lambda > \sup_{x \in X} \inf_{y \in Y} f(x, y)$  and  $x \in X$ , the set  $\{y \in Y : f(x, y) \leq \lambda\}$  is closed and  $\bigcap_{x \in M} \{y \in Y : f(x, y) \leq \lambda\}$  is compact set for some  $M \in \langle X \rangle$ ;
- (iii) for each  $\lambda > \sup_{x \in X} \inf_{y \in Y} f(x, y)$  and  $y \in Y$ , the set  $\{x \in X : f(x, y) > \lambda\}$  is  $\Gamma$ -convex;
- (iv) for each  $\lambda > \sup_{x \in X} \inf_{y \in Y} f(x, y)$  and  $N \in \langle X \rangle$ , there exists a function  $\sigma : N \rightarrow D$  such that, for each  $M \in \langle N \rangle$ , there exists  $x_0 \in M \cup \Gamma_M$  such that

$$f(x_0, y) \leq \lambda, \quad \forall y \in \Gamma'_{\sigma(M)}.$$

Then

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).$$

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# Some approximation properties of $q$ -parametric BBH operators

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## Abstract

The paper introduces  $q$ -parametric Bleimann, Butzer and Hahn ( $q$ -BBH) operators as a rational transformation of  $q$ -Bernstein-Lupaş operators. On their basis, a set of new results on  $q$ -BBH operators can be obtained easily from the corresponding properties of  $q$ -Bernstein-Lupaş operators. Among several other results, a set of identities involving divided differences are obtained. Furthermore, convergence properties of  $q$ -BBH operators are studied.

## 1 Introduction

$q$ -Bernstein polynomials  $B_{n,q}$  were introduced by G. M. Phillips in [26].  $q$ -Bernstein polynomials form an area of an intensive research in the approximation theory, see survey paper [25] and references there in. Nowadays, there are new studies on the  $q$ -parametric operators, see [20]-[31]. Two-parametric generalization of  $q$ -Bernstein polynomials have been considered by P. Lewanowicz and P. Wozny (cf. [20]), an analogue of the Bernstein-Durrmeyer operator and Bernstein-Chlodowsky operator related to the  $q$ -Bernstein basis has been studied by M. - M. Derriennic [9], V. Gupta [16] and V. Gupta and H. Karsh [17], respectively, a  $q$ -version of the Szász-Mirakjan operator has been investigated by A. Aral and V. Gupta in [6]. Also some results on  $q$ -parametric Meyer-König and Zeller operators can be found in [27], [31], [11]. Recently Aral and Doğru [5] introduced a  $q$ -analogue of Bleimann, Butzer, and Hahn operators and they have established some approximation properties of their  $q$ -Bleimann, Butzer, and Hahn operators in the subspace of  $C_B[0, \infty)$ . Also, they showed that these operators are more flexible than classical BBH operators, that is, depending on the selection of  $q$ , rate of convergence of the  $q$ -BBH operators is better than the classical one. Voronovskaja type asymptotic estimate and the monotonicity properties for  $q$ -BBH operators are studied in [10].

The linear operator  $H_n$  defined by

$$H_n(f, x) := \frac{1}{(1+x)^n} \sum_{k=0}^n f\left(\frac{k}{n-k+1}\right) \binom{n}{k} x^k, \quad x \geq 0, \quad n = 1, 2, \dots,$$

where  $f \in \mathbb{R}^{[0, \infty)}$  was introduced by Bleimann, Butzer, and Hahn [7] to approximate continuous functions on the positive semi-axis and has been studied by several authors see, for instance, [8], [19], [3], [7], [4], [2], [18]. In [7], [2] the authors pointed out some formal similarities and differences between  $H_n$  and the classical Bernstein operator  $B_n$ . Connection suggested in [2] can be formulated by means of the following identity

$$H_n = V \circ B_{n+1} \circ U,$$

$V$  and  $U$  are suitable positive linear operators which will be defined below. This idea was used in [22] to define new  $q$ -analogue of the Bleimann, Butzer, and Hahn operators as follows:

$$H_{n,q}(f; x) := (V \circ B_{n+1,q} \circ U)(f; x),$$

where  $B_{n+1,q}$  is a Philips  $q$ -analogue of the Bernstein operators.

On the other hand the Lupaş  $q$ -analogue of the Bernstein polynomials  $(R_{n,q})$  are less known. However, they have an advantage of generating positive linear operators for all  $q > 0$ , whereas Phillips polynomials generate positive linear operators only if  $q \in (0, 1)$ . Lupaş [21] investigated approximating properties of the operators  $R_{n,q}(f, x)$  with respect to the uniform norm of  $C[0, 1]$ . In particular, he obtained some sufficient conditions for a sequence  $\{R_{n,q}(f, x)\}$  to be approximating for any function  $f \in C[0, 1]$  and estimated the rate of convergence in terms of the modulus of continuity. He also investigated behavior of the operators  $R_{n,q}(f, x)$  for convex functions. In [24] several results on convergence properties of the sequence  $\{R_{n,q}(f, x)\}$  is presented. In particular, it is proved that the sequence  $\{R_{n,q}(f, x)\}$  converges uniformly to  $f(x)$  on  $[0, 1]$  if and only if  $q_n \rightarrow 1$ . On the other hand, for any  $q > 0$  fixed,  $q \neq 1$ , the sequence  $\{R_{n,q}(f, x)\}$  converges uniformly to  $f(x)$  if and only if  $f(x) = ax + b$  for some  $a, b \in R$ .

Using the classical connection between Bernstein and BBH operators we propose the following  $q$ -analogue of the Bleimann, Butzer and Hahn operators in  $C_{1+x}^0[0, \infty)$ :

$$H_{n,q}(f; x) := (V \circ R_{n+1,q} \circ U)(f; x), \quad (1)$$

where  $R_{n,q}$  is the Lupaş  $q$ -Bernstein operator on  $C[0, 1]$  defined by

$$R_{n,q}(f, x) = \sum_{k=0}^n f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\frac{k(k-1)}{2}} x^k (1-x)^{n-k}}{(1-x+qx) \dots (1-x+q^{n-1}x)}.$$

Thanks to (1), different properties of  $R_{n+1,q}$  can be transferred to  $H_{n,q}$  with some extra effort. Thus, the limiting behaviour of  $H_{n,q}$  can be immediately derived from (1) and the well known properties of  $R_{n+1,q}$ . In [14], Gadjiev and Çakar gave a Korovkin-type theorem using the test function  $\left(\frac{t}{1+t}\right)^i$  for  $i = 0, 1, 2$ . In [5], for  $q$ -BBH operators  $L_{n,q}$  Korovkin-type approximation properties are investigated by using these test functions. Notice that,

$$H_{n,q}(f; x) = L_{n,q}(f; qx)$$

where  $L_{n,q}$  is the  $q$ -BBH operator defined in [5].

The paper is organized as follows. In Section 2 we give construction of  $q$ -BBH operators and study some elementary properties. Moreover, Arama–Popoviciu-type Formula and Stancu-type form for the remainder of the  $q$ -BBH approximation process are obtained. In Section 3 we investigate convergence properties of  $q$ -BBH operators and we show how to reduce the case  $q \in (1, \infty)$  to the case  $q \in (0, 1)$ .

## 2 Construction and properties

Before introducing the operators, we mention some basic definitions of  $q$  calculus.

Let  $q > 0$ . For any  $n \in \mathbb{N} \cup \{0\}$ , the  $q$ -integer  $[n] = [n]_q$  is defined by

$$[n] := 1 + q + \dots + q^{n-1}, \quad [0] := 0;$$

and the  $q$ -factorial  $[n]! = [n]_q!$  by

$$[n]! := [1][2] \dots [n], \quad [0]! := 1.$$

For integers  $0 \leq k \leq n$ , the  $q$ -binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]! [n-k]!}.$$

Also, we use the following notations:

$$\begin{aligned} (1-x)_q^n &:= \prod_{j=0}^{n-1} (1-q^j x), \quad (1-x)_q^\infty := \prod_{j=0}^{\infty} (1-q^j x), \\ b_{n,k}(q; x) &:= \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\frac{k(k-1)}{2}} x^k (1-x)^{n-k}}{(1-x+qx)\dots(1-x+q^{n-1}x)}, \quad h_{n,k}(q; x) := \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{k(k+1)/2} x^k}{(1+qx)_q^n}, \\ b_{\infty,k}(q; x) &:= \frac{q^{\frac{k(k-1)}{2}} (x/1-x)^k}{(1-q)^k [k]! \prod_{j=0}^{\infty} (1+q^j (x/1-x))}, \quad h_{\infty,k}(q; x) := \frac{q^{k(k+1)/2} x^k}{(1+qx)_q^\infty (1-q)^k [k]!}, \\ h_{n,k}(q; x) &= (1+x) \frac{[n-k+1] q^k}{[n+1]} b_{n+1,k} \left( q; \frac{x}{1+x} \right). \end{aligned}$$

As usual,  $[x_0, x_1, \dots, x_n; f]$  denotes the divided difference of the function  $f$  with respect to distinct nodes  $x_0, x_1, \dots, x_n$  in the domain of  $f$  and can be expressed as the following formula:

$$[x_0, \dots, x_n; f] = \frac{[x_1, \dots, x_n; f] - [x_0, \dots, x_{n-1}; f]}{x_n - x_0}.$$

We shall also use the following notations:

$$\begin{aligned} C_B[0, \infty) &= \{f \in C[0, \infty) \mid f(x) = O(1+x)\}, \\ C[0, \infty) &= \{f \in C[0, \infty) \mid f(x) \text{ has a finite limit at } \infty\}, \\ C_{1+x}^0[0, \infty) &= \{f \in C[0, \infty) \mid f(x) = o(1+x) \quad (x \rightarrow \infty)\}. \end{aligned}$$

It is assumed that  $C_{1+x}^0[0, \infty)$  is endowed with the norm  $\|f\|_{1+x} = \sup_{x \geq 0} \frac{|f(x)|}{1+x}$ .

We consider the operators  $U : \mathbb{R}^{[0, \infty)} \rightarrow \mathbb{R}^{[0, 1]}$ ,

$$U(f, t) := \begin{cases} (1-t) f\left(\frac{t}{1-t}\right), & t \in [0, 1), \\ 0, & t = 1, \end{cases}$$

and  $V : \mathbb{R}^{[0, 1]} \rightarrow \mathbb{R}^{[0, \infty)}$ ,

$$V(g, x) := (1+x) g\left(\frac{x}{1+x}\right), \quad x \in [0, \infty).$$

They were introduced in [2] and, aside from notation, in [13]. Some of properties, used throughout the paper, are gathered in the following theorem.

**Theorem 1** [1] *We have the following relations:*

1.  $V \circ U$  is the identity operator on  $\mathbb{R}^{[0, \infty)}$ .
2.  $f \in C_{1+x}^0[0, \infty)$  if and only if  $Uf \in C[0, 1]$ .
3. If  $f \in C_{1+x}^0[0, \infty)$ , then  $f$  is convex if and only if  $f$  is convex and nonincreasing.

We introduce Bleimann, Butzer and Hahn type operators based on  $q$ -integers as follows.

**Definition 2** For  $f \in \mathbb{R}^{[0, \infty)}$  the  $q$ -Bleimann, Butzer and Hahn operators are given by

$$\begin{aligned} H_{n,q}(f; x) &:= (V \circ R_{n+1,q} \circ U)(f; x) = \sum_{k=0}^n f\left(\frac{[k]}{q^k [n-k+1]}\right) h_{n,k}(q; x) \\ &= \frac{1}{(1+qx)_q^n} \sum_{k=0}^n f\left(\frac{[k]}{q^k [n-k+1]}\right) \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+1)/2} x^k, \quad n \in \mathbb{N}. \end{aligned}$$

**Definition 3** Let  $0 < q < 1$ . The linear operator defined on  $\mathbb{R}^{[0,\infty)}$  given by

$$H_{\infty,q}(f; x) := \sum_{k=0}^{\infty} f\left(\frac{1-q^k}{q^k}\right) h_{\infty,k}(q; x) = \frac{1}{(1+qx)_q^{\infty}} \sum_{k=0}^{\infty} f\left(\frac{1-q^k}{q^k}\right) \frac{q^{k(k+1)/2}}{(1-q)^k [k]!} x^k$$

is called the limit  $q$ -BBH operator.

**Lemma 4**  $H_{n,q}, H_{\infty,q} : C_{1+x}^0[0, \infty) \rightarrow C_{1+x}^0[0, \infty)$  are linear positive operators and

$$\|H_{n,q}(f)\|_{1+x} \leq \|f\|_{1+x}, \quad \|H_{\infty,q}(f)\|_{1+x} \leq \|f\|_{1+x}.$$

**Proof.** We prove the first inequality, since the second one can be done in a like manner. Thanks to the definition we have

$$\begin{aligned} |H_{n,q}(f; x)| &\leq \sum_{k=0}^n \left| f\left(\frac{[k]}{q^k [n-k+1]}\right) \right| h_{n,k}(q; x) \\ &= (1+x) \sum_{k=0}^n \left| f\left(\frac{[k]}{q^k [n-k+1]}\right) \right| \frac{[n-k+1] q^k}{[n+1]} b_{n+1,k}\left(q; \frac{x}{1+x}\right) \\ &= (1+x) \sum_{k=0}^n \left| f\left(\frac{[k]}{q^k [n-k+1]}\right) \right| / \left(1 + \frac{[k]}{q^k [n-k+1]}\right) b_{n+1,k}\left(q; \frac{x}{1+x}\right) \\ &\leq (1+x) \|f\|_{1+x} \sum_{k=0}^{n+1} b_{n+1,k}\left(q; \frac{x}{1+x}\right) = (1+x) \|f\|_{1+x}. \end{aligned}$$

■

**Lemma 5** We have

$$H_{n,q}(1; x) = 1, \quad H_{n,q}(t; x) = x - \frac{q^{n(n+1)/2} x^{n+1}}{(1+qx)_q^n}.$$

The Arama–Popoviciu-type formula was obtained in [4] and in [8] by direct calculation. Formula (2) below is a  $q$ -analogue of the mentioned formula for the  $q$ -BBH operators.

**Theorem 6** (i) If  $f \in C_{1+x}^0[0, \infty)$  is a convex function, then the sequence  $\{H_{n,q}(f; x)\}$  is nonincreasing in  $n$  for each  $q \in (0, 1]$  and  $x \in [0, \infty)$ .

(ii) If  $f \in \mathbb{R}^{[0,\infty)}$ , then the following formula is valid

$$\begin{aligned} H_{n,q}(f; x) - H_{n+1,q}(f; x) &= -\frac{1}{(1+qx)_q^{n+1}} q^{n(n+1)/2} x^{n+1} \left[ \frac{[n]}{q^n}, \frac{[n+1]}{q^{n+1}}; f \right] \\ &\quad + \frac{1}{(1+qx)_q^{n+1}} \sum_{k=0}^{n-1} \frac{q^{n-3k-2}}{[n-k][n-k+1]} \\ &\quad \times \left[ \frac{[k]}{q^k [n-k+1]}, \frac{[k+1]}{q^{k+1} [n-k+1]}, \frac{[k+1]}{q^{k+1} [n-k]}; f \right] \left[ \begin{matrix} n+1 \\ k \end{matrix} \right] q^{(k+1)(k+2)/2} x^{k+1}. \end{aligned} \quad (2)$$

**Proof.** (i) We start by writing

$$\begin{aligned}
& H_{n,q}(f; x) - H_{n+1,q}(f; x) \\
&= \frac{1}{(1+qx)_q^{n+1}} \sum_{k=0}^n f\left(\frac{[k]}{q^k [n-k+1]}\right) \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+1)/2} x^k (1+q^{n+1}x) \\
&\quad - \frac{1}{(1+qx)_q^{n+1}} \sum_{k=0}^{n+1} f\left(\frac{[k]}{q^k [n-k+2]}\right) \begin{bmatrix} n+1 \\ k \end{bmatrix} q^{k(k+1)/2} x^k \\
&= -\frac{1}{(1+qx)_q^{n+1}} q^{(n+1)(n+2)/2} x^{n+1} \left( f\left(\frac{[n+1]}{q^{n+1}}\right) - f\left(\frac{[n]}{q^n}\right) \right) \\
&\quad + \frac{1}{(1+qx)_q^{n+1}} \sum_{k=0}^{n-1} f\left(\frac{[k]}{q^k [n-k+1]}\right) \begin{bmatrix} n \\ k \end{bmatrix} q^{(k+1)(k+2)/2} q^{n-k} x^{k+1} \\
&\quad + \frac{1}{(1+qx)_q^{n+1}} \sum_{k=0}^{n-1} f\left(\frac{[k+1]}{q^{k+1} [n-k]}\right) \begin{bmatrix} n \\ k+1 \end{bmatrix} q^{(k+1)(k+2)/2} x^{k+1} \\
&\quad - \frac{1}{(1+qx)_q^{n+1}} \sum_{k=0}^{n-1} f\left(\frac{[k+1]}{q^{k+1} [n-k+1]}\right) \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} q^{(k+1)(k+2)/2} x^{k+1}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
H_{n,q}(f; x) - H_{n+1,q}(f; x) &= -\frac{1}{(1+qx)_q^{n+1}} q^{(n+1)(n+2)/2} x^{n+1} \left( f\left(\frac{[n+1]}{q^{n+1}}\right) - f\left(\frac{[n]}{q^n}\right) \right) \\
&\quad + \frac{1}{(1+qx)_q^{n+1}} \sum_{k=0}^{n-1} a_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} q^{(k+1)(k+2)/2} x^{k+1},
\end{aligned} \tag{3}$$

where

$$a_k = \frac{q^{n-k} [k+1]}{[n+1]} f\left(\frac{[k]}{q^k [n-k+1]}\right) + \frac{[n-k]}{[n+1]} f\left(\frac{[k+1]}{q^{k+1} [n-k]}\right) - f\left(\frac{[k+1]}{q^{k+1} [n-k+1]}\right). \tag{4}$$

Now from Theorem 1 since  $f$  is nonincreasing the first term is nonnegative. Thus to show monotonicity of  $H_{n,q}$  it suffices to show nonnegativity of  $a_k$ ,  $0 \leq k \leq n$ . Let us write

$$\alpha = \frac{q^{n-k} [k+1]}{[n+1]}, \quad 1-\alpha = \frac{[n-k]}{[n+1]}, \quad x_1 = \frac{[k]}{q^k [n-k+1]}, \quad x_2 = \frac{[k+1]}{q^{k+1} [n-k]}.$$

Then it follows that

$$\begin{aligned}
\alpha x_1 + (1-\alpha) x_2 &= \frac{q^{n-k} [k+1]}{[n+1]} \frac{[k]}{q^k [n-k+1]} + \frac{[n-k]}{[n+1]} \frac{[k+1]}{q^{k+1} [n-k]} \\
&= \frac{[k+1]}{q^{k+1} [n+1]} \frac{q^{n-k+1} [k] + [n-k+1]}{[n-k+1]} = \frac{[k+1]}{q^{k+1} [n+1]} \frac{[n+1]}{[n-k+1]} = \frac{[k+1]}{q^{k+1} [n-k+1]}.
\end{aligned}$$

We see immediately that

$$a_k = \alpha f(x_1) + (1-\alpha) f(x_2) - f(\alpha x_1 + (1-\alpha) x_2) \geq 0$$

which proves part (i).

(ii) For part (ii) we evaluate the second divided difference of  $f$  at the points  $\frac{[k]}{q^k [n-k+1]}, \frac{[k+1]}{q^{k+1} [n-k+1]},$

$\frac{[k+1]}{q^{k+1}[n-k]}$  and we obtain

$$\begin{aligned} & \left[ \frac{[k]}{q^k [n-k+1]}, \frac{[k+1]}{q^{k+1} [n-k+1]}, \frac{[k+1]}{q^{k+1} [n-k]}; f \right] \\ &= \frac{q^{2k+2} [n-k] [n-k+1]^2}{[n+1]} f \left( \frac{[k]}{q^k [n-k+1]} \right) \\ & - \frac{q^{2k+2} [n-k] [n-k+1]^2}{q^{n-k} [k+1]} f \left( \frac{[k+1]}{q^{k+1} [n-k+1]} \right) \\ & + \frac{q^{2k+2} [n-k]^2 [n-k+1]^2}{q^{n-k} [k+1] [n+1]} f \left( \frac{[k+1]}{q^{k+1} [n-k]} \right). \end{aligned} \quad (5)$$

From (4) and (5) we see that

$$\begin{aligned} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} a_k &= \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \frac{q^{n-k} [k+1]}{q^{2k+2} [n-k] [n-k+1]^2} \\ & \times \left[ \frac{[k]}{q^k [n-k+1]}, \frac{[k+1]}{q^{k+1} [n-k+1]}, \frac{[k+1]}{q^{k+1} [n-k]}; f \right] \\ &= \frac{q^{n-k}}{q^{2k+2} [n-k] [n-k+1]} \begin{bmatrix} n+1 \\ k \end{bmatrix} \\ & \times \left[ \frac{[k]}{q^k [n-k+1]}, \frac{[k+1]}{q^{k+1} [n-k+1]}, \frac{[k+1]}{q^{k+1} [n-k]}; f \right]. \end{aligned} \quad (6)$$

Therefore the formula (2) follows from (3) and (6). ■

The following theorem provides a Stancu-type form of the remainder of the  $q$ -BBH approximation process.

**Theorem 7** *If  $f \in \mathbb{R}^{[0,\infty)}$  and  $x \in [0, \infty) \setminus \{[k] / [n-k+1] q^k : k = 0, 1, \dots, n\}$ , then*

$$\begin{aligned} H_{n,q}(f; x) - f(x) &= -\frac{x^{n+1}}{(1+qx)_q^n} \left[ x, \frac{[n]}{q^n}; f \right] q^{n(n+1)/2} \\ &+ \frac{x}{(1+qx)_q^n} \sum_{k=0}^{n-1} \left[ x, \frac{[k]}{[n-k+1] q^k}, \frac{[k+1]}{[n-k] q^{k+1}}; f \right] \frac{q^{\frac{(k-2)(k+1)}{2}}}{[n-k]} \begin{bmatrix} n+1 \\ k \end{bmatrix} x^k. \end{aligned} \quad (7)$$

**Proof.** By the definition of  $H_{n,q}$  we have

$$\begin{aligned} H_{n,q}(f; x) - f(x) &= \frac{1}{(1+qx)_q^n} \sum_{k=0}^n \left( f \left( \frac{[k]}{q^k [n-k+1]} \right) - f(x) \right) \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+1)/2} x^k \\ &= -\frac{1}{(1+qx)_q^n} \sum_{k=0}^n \left( x - \frac{[k]}{q^k [n-k+1]} \right) \left[ x, \frac{[k]}{[n-k+1] q^k}; f \right] \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+1)/2} x^k. \end{aligned}$$

Since

$$\frac{[k]}{[n-k+1]} \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k-1 \end{bmatrix}$$

we have

$$\begin{aligned} H_{n,q}(f; x) - f(x) &= -\frac{1}{(1+qx)_q^n} \sum_{k=0}^n \left[ x, \frac{[k]}{[n-k+1] q^k}; f \right] \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+1)/2} x^{k+1} \\ &+ \frac{1}{(1+qx)_q^n} \sum_{k=1}^n \left[ x, \frac{[k]}{[n-k+1] q^k}; f \right] \begin{bmatrix} n \\ k-1 \end{bmatrix} q^{k(k-1)/2} x^k. \end{aligned}$$



Rearranging the above equality, we can write

$$\begin{aligned} H_{n,q}(f; x) - f(x) &= -\frac{x^{n+1}}{(1+qx)_q^n} \left[ x, \frac{[n]}{q^n}; f \right] q^{n(n+1)/2} \\ &+ \frac{1}{(1+qx)_q^n} \sum_{k=0}^{n-1} \left( \left[ x, \frac{[k+1]}{[n-k]q^{k+1}}; f \right] - \left[ x, \frac{[k]}{[n-k+1]q^k}; f \right] \right) \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+1)/2} x^{k+1}. \end{aligned} \quad (8)$$

Using the equality

$$\frac{[k+1]}{[n-k]q^{k+1}} - \frac{[k]}{[n-k+1]q^k} = \frac{[n+1]}{[n-k][n-k+1]q^{k+1}}$$

we have the following formula for divided differences:

$$\begin{aligned} &\left[ x, \frac{[k+1]}{[n-k]q^{k+1}}; f \right] - \left[ x, \frac{[k]}{[n-k+1]q^k}; f \right] \\ &= \left[ x, \frac{[k]}{[n-k+1]q^k}, \frac{[k+1]}{[n-k]q^{k+1}}; f \right] \frac{[n+1]}{[n-k][n-k+1]q^{k+1}}, \end{aligned} \quad (9)$$

and therefore the formula (7) follows from (8) and (9). ■

We will focus now on the formula for the second central moment of the  $H_{n,q}$ , which is the  $q$ -analogue of classical formula for  $q$ -BBH operators, see [19], [1].

**Lemma 8** *We have*

$$H_{n,q}\left((t-x)^2; x\right) = \frac{x^{n+1}}{(1+qx)_q^n} q^{n(n+1)/2} \left(x - \frac{[n]}{q^n}\right) + \frac{1}{(1+qx)_q^n} \sum_{k=0}^{n-1} \frac{1}{[n-k]} \begin{bmatrix} n+1 \\ k \end{bmatrix} q^{k(k+1)/2} x^{k+1}.$$

**Proof.** Indeed, with  $f(t) = (t-x)\left(t - \frac{[n]}{q^n}\right)$ , equation (7) gives

$$H_{n,q}\left((t-x)\left(t - \frac{[n]}{q^n}\right); x\right) = \frac{1}{(1+qx)_q^n} \sum_{k=0}^{n-1} \frac{1}{[n-k]q^{k+1}} \begin{bmatrix} n+1 \\ k \end{bmatrix} q^{k(k+1)/2} x^{k+1}.$$

Consequently, we obtain

$$\begin{aligned} H_{n,q}\left((t-x)^2; x\right) &= \left(\frac{[n]}{q^n} - x\right) (H_{n,q}(t; x) - x) + H_{n,q}\left((t-x)\left(t - \frac{[n]}{q^n}\right); x\right) \\ &= \frac{x^{n+1}}{(1+qx)_q^n} q^{n(n+1)/2} \left(x - \frac{[n]}{q^n}\right) + \frac{1}{(1+qx)_q^n} \sum_{k=0}^{n-1} \frac{1}{[n-k]q^{k+1}} \begin{bmatrix} n+1 \\ k \end{bmatrix} q^{k(k+1)/2} x^{k+1}. \end{aligned}$$

■

It is known that a function  $f$  is convex on an interval if and only if all second order divided differences of  $f$  are nonnegative. This property, Theorems 7, 1 and 6 together imply the following result.

**Corollary 9** *If  $f \in C_{1+x}^0[0, \infty)$  is convex or if  $f \in R^{[0, \infty)}$  is convex and nonincreasing, then*

$$H_{n,q}(f; x) \geq H_{n+1,q}(f; x) \geq f(x).$$

### 3 Convergence properties

For  $f \in C[0, 1]$ ,  $t > 0$ , the modulus of continuity  $\omega(f, t)$  of  $f$  is defined by

$$\omega(f, t) = \sup_{|x-y| \leq t} |f(x) - f(y)|.$$

**Theorem 10** Let  $q = q_n$  satisfies  $0 < q_n < 1$  and let  $q_n \rightarrow 1$  as  $n \rightarrow \infty$ . For any  $x \in [0, \infty)$  and for any  $f \in C_{1+x}^0[0, \infty)$  the following inequality holds

$$\frac{1}{1+x} |H_{n,q_n}(f; x) - f(x)| \leq 2\omega(Uf, \sqrt{\lambda_n(x)}),$$

$$\text{where } \lambda_n(x) = \frac{x}{(1+x)^2} \frac{1}{[n+1]_{q_n}}.$$

**Proof.** Positivity of  $R_{n+1,q_n}$  implies that for any  $g \in C[0, 1]$

$$|R_{n+1,q_n}(g; x) - g(x)| \leq R_{n+1,q_n}(|g(t) - g(x)|; x). \quad (10)$$

On the other hand

$$\begin{aligned} & |(Uf)(t) - (Uf)(x)| \\ & \leq \omega(Uf, |t-x|) \leq \omega(Uf, \delta) \left(1 + \frac{1}{\delta} |t-x|\right), \quad \delta > 0. \end{aligned}$$

This inequality and (10) imply that

$$|R_{n+1,q_n}(Uf; x) - (Uf)(x)| \leq \omega(Uf, \delta) \left(1 + \frac{1}{\delta} R_{n+1,q_n}(|t-x|; x)\right)$$

and

$$\begin{aligned} & |H_{n,q_n}(f; x) - f(x)| \\ & = (1+x) \left| R_{n+1,q_n} \left( Uf; \frac{x}{1+x} \right) - (Uf) \left( \frac{x}{1+x} \right) \right| \\ & \leq (1+x) \omega(Uf, \delta) \left( 1 + \frac{1}{\delta} R_{n+1,q_n} \left( \left| t - \frac{x}{1+x} \right|; \frac{x}{1+x} \right) \right) \\ & \leq (1+x) \omega(Uf, \delta) \left( 1 + \frac{1}{\delta} \left( R_{n+1,q_n} \left( \left| t - \frac{x}{1+x} \right|^2; \frac{x}{1+x} \right) \right)^{1/2} \right) \\ & = (1+x) \omega(Uf, \delta) \left( 1 + \frac{1}{\delta} \left( \frac{x}{1+x} \frac{1}{[n+1]_{q_n}} + \frac{x}{1+x} \frac{q_n x}{1+q_n x} \left( 1 - \frac{1}{[n+1]_{q_n}} \right) - \left( \frac{x}{1+x} \right)^2 \right)^{1/2} \right) \\ & \leq (1+x) \omega(Uf, \delta) \left( 1 + \frac{1}{\delta} \left( \frac{x}{1+x} \frac{1}{[n+1]_{q_n}} - \left( \frac{x}{1+x} \right)^2 \frac{1}{[n+1]_{q_n}} \right)^{1/2} \right) \\ & = (1+x) \omega(Uf, \delta) \left( 1 + \frac{1}{\delta} \left( \frac{x}{(1+x)^2} \frac{1}{[n+1]_{q_n}} \right)^{1/2} \right), \end{aligned}$$

where we have used the explicit formula for  $R_{n+1,q_n} \left( \left| t - \frac{x}{1+x} \right|^2; \frac{x}{1+x} \right)$ , which can be found in [24].

Now by choosing  $\delta = \sqrt{\lambda_n(x)}$ , we obtain desired result. ■

**Corollary 11** Let  $q = q_n$  satisfies  $0 < q_n < 1$  and let  $q_n \rightarrow 1$  as  $n \rightarrow \infty$ . For any  $f \in C_{1+x}^0[0, \infty)$  it holds that

$$\lim_{n \rightarrow \infty} \|H_{n,q_n}(f; x) - f(x)\|_{1+x} = 0.$$

In [24], it is proved that  $b_{n,k}(q; x) \rightarrow b_{\infty,k}(q; x)$  uniformly in  $x \in [0, 1)$  as  $n \rightarrow \infty$ . In the next lemma we give an estimate for  $\left| b_{n,k}(q; \frac{x}{1+x}) - b_{\infty,k}(q; \frac{x}{1+x}) \right|$  for  $x \in [0, \infty)$ .

**Lemma 12** *Let  $0 < q < 1$ ,  $k \geq 0$ ,  $n \geq 1$ . For any  $x \in [0, \infty)$  we have*

$$\left| b_{n,k}(q; \frac{x}{1+x}) - b_{\infty,k}(q; \frac{x}{1+x}) \right| \leq b_{n,k}(q; \frac{x}{1+x}) \frac{xq^n}{1-q} + b_{\infty,k}(q; \frac{x}{1+x}) \frac{q^{n-k+1}}{1-q}.$$

**Proof.** Standard computations show that

$$\begin{aligned} & \left| b_{n,k}(q; \frac{x}{1+x}) - b_{\infty,k}(q; \frac{x}{1+x}) \right| \\ &= \left| \left[ \begin{matrix} n \\ k \end{matrix} \right] \prod_{j=0}^{k-1} \frac{q^j x}{1+q^j x} \prod_{j=0}^{n-k-1} \left( 1 - \frac{q^{k+j} x}{1+q^{k+j} x} \right) \right. \\ & \quad \left. - \frac{1}{(1-q)^k [k]!} \prod_{j=0}^{k-1} \frac{q^j x}{1+q^j x} \prod_{j=0}^{\infty} \left( 1 - \frac{q^{k+j} x}{1+q^{k+j} x} \right) \right| \\ &= \left| \left[ \begin{matrix} n \\ k \end{matrix} \right] \prod_{j=0}^{k-1} \frac{q^j x}{1+q^j x} \left( \prod_{j=0}^{n-k-1} \left( 1 - \frac{q^{k+j} x}{1+q^{k+j} x} \right) - \prod_{j=0}^{\infty} \left( 1 - \frac{q^{k+j} x}{1+q^{k+j} x} \right) \right) \right. \\ & \quad \left. + \prod_{j=0}^{k-1} \frac{q^j x}{1+q^j x} \prod_{j=0}^{\infty} \left( 1 - \frac{q^{k+j} x}{1+q^{k+j} x} \right) \left( \left[ \begin{matrix} n \\ k \end{matrix} \right] - \frac{1}{(1-q)^k [k]!} \right) \right| \\ &\leq b_{n,k}(q; \frac{x}{1+x}) \left| 1 - \prod_{j=n}^{\infty} \left( 1 - \frac{q^j x}{1+q^j x} \right) \right| + b_{\infty,k}(q; \frac{x}{1+x}) \left| \prod_{j=n-k+1}^n (1-q^j) - 1 \right|. \quad (11) \end{aligned}$$

Now using the inequality

$$1 - \prod_{j=1}^k (1 - a_j) \leq \sum_{j=1}^k a_j, \quad (a_1, a_2, \dots, a_k \in (0, 1), \quad k = 1, 2, \dots, \infty),$$

we get from (11) that

$$\begin{aligned} \left| b_{n,k}(q; \frac{x}{1+x}) - b_{\infty,k}(q; \frac{x}{1+x}) \right| &\leq b_{n,k}(q; \frac{x}{1+x}) \sum_{j=n}^{\infty} \frac{q^j x}{1+q^j x} + b_{\infty,k}(q; \frac{x}{1+x}) \sum_{j=n-k+1}^n q^j \\ &\leq b_{n,k}(q; \frac{x}{1+x}) \frac{xq^n}{1-q} + b_{\infty,k}(q; \frac{x}{1+x}) \frac{q^{n-k+1}}{1-q}. \end{aligned}$$

■

Using Lemma 12 we prove the following quantitative result for the rate of local convergence of  $H_{n,q}(f; x)$  in terms of the first modulus of continuity.

**Theorem 13** *Let  $0 < q < 1$  and  $f \in C_{1+x}^0[0, \infty)$ . Then for all  $0 \leq x < \infty$  we have*

$$|H_{n,q}(f; x) - H_{\infty,q}(f; x)| \leq (1+x)^2 \frac{2}{1-q} \omega(Uf, q^{n+1}).$$

**Proof.** Consider

$$\begin{aligned} \Delta(x) &:= H_{n,q}(f; x) - H_{\infty,q}(f; x) = (V \circ R_{n+1,q} \circ U)(f; x) - (V \circ R_{\infty,q} \circ U)(f; x) \\ &= (V \circ (R_{n+1,q} - R_{\infty,q}) \circ U)(f; x) = (V \circ (R_{n+1,q} - R_{\infty,q}))(Uf; x). \end{aligned}$$

Since  $H_{n,q}(f; x)$  and  $H_{\infty,q}(f; x)$  possess the end point interpolation property,  $\Delta(0) = 0$ . For all  $x \in (0, \infty)$  we rewrite  $\Delta$  in the following form

$$\begin{aligned} \Delta(x) &= V \circ \sum_{k=0}^{n+1} \left[ (Uf) \left( \frac{[k]}{[n+1]} \right) - (Uf)(1 - q^k) \right] b_{n+1,k}(q; x) \\ &\quad + V \circ \sum_{k=0}^{n+1} [(Uf)(1 - q^k) - (Uf)(1)] (b_{n+1,k}(q; x) - b_{\infty,k}(q; x)) \\ &\quad - V \circ \sum_{k=n+2}^{\infty} [(Uf)(1 - q^k) - (Uf)(1)] b_{\infty,k}(q; x) =: I_1 + I_2 + I_3. \end{aligned}$$

We start with estimation of  $I_1$  and  $I_3$ . Since

$$\begin{aligned} \frac{[k]}{[n+1]} - (1 - q^k) &= \frac{1 - q^k}{1 - q^{n+1}} - (1 - q^k) = q^{n+1} \frac{1 - q^k}{1 - q^{n+1}} \leq q^{n+1}, \\ 0 &\leq 1 - (1 - q^k) = q^k \leq q^{n+1}, \quad k > n+1, \end{aligned}$$

we get

$$|I_1| \leq (1+x) \omega(Uf, q^{n+1}) \sum_{k=0}^{n+1} b_{n+1,k}(q; \frac{x}{1+x}) = (1+x) \omega(Uf, q^{n+1}), \quad (12)$$

$$|I_3| \leq (1+x) \omega(Uf, q^{n+1}) \sum_{k=n+2}^{\infty} b_{\infty,k}(q; \frac{x}{1+x}) \leq (1+x) \omega(Uf, q^{n+1}). \quad (13)$$

Finally we estimate  $I_2$ . Using the property of the modulus of continuity

$$\omega(f, \lambda t) \leq (1 + \lambda) \omega(f, t), \quad \lambda > 0$$

and Lemma 12 we get

$$\begin{aligned} |I_2| &\leq (1+x) \sum_{k=0}^{n+1} \omega(Uf, q^k) \left| b_{n+1,k}(q; \frac{x}{1+x}) - b_{\infty,k}(q; \frac{x}{1+x}) \right| \\ &\leq (1+x) \omega(Uf, q^{n+1}) \sum_{k=0}^{n+1} (1 + q^{k-n-1}) \left| b_{n+1,k}(q; \frac{x}{1+x}) - b_{\infty,k}(q; \frac{x}{1+x}) \right| \\ &\leq 2(1+x) \omega(Uf, q^{n+1}) \frac{1}{q^{n+1}} \sum_{k=0}^{n+1} q^k \left| b_{n+1,k}(q; \frac{x}{1+x}) - b_{\infty,k}(q; \frac{x}{1+x}) \right| \\ &\leq 2(1+x) \omega(Uf, q^{n+1}) \frac{1}{q^{n+1}} \sum_{k=0}^{n+1} q^k \left( b_{n+1,k}(q; \frac{x}{1+x}) \frac{xq^{n+1}}{1-q} + b_{\infty,k}(q; \frac{x}{1+x}) \frac{q^{n-k+2}}{1-q} \right) \\ &\leq 2(1+x)^2 \frac{1}{1-q} \omega(Uf, q^{n+1}). \end{aligned} \quad (14)$$

From (12), (13), and (14), we conclude the desired estimation. ■

Finally we give the following theorem which allows us to reduce the case  $q \in (1, \infty)$  to the case  $q \in (0, 1)$ . Let  $f \in C[0, \infty]$ . Define

$$g(x) = \begin{cases} \lim_{x \rightarrow \infty} f(x), & \text{if } x = 0, \\ f(\frac{1}{x}), & \text{if } 0 < x < \infty, \\ f(0), & \text{if } x = \infty. \end{cases}$$

**Theorem 14** For any  $q > 0$  we have

$$H_{n,1/q}(g; \frac{1}{x}) = \begin{cases} \lim_{x \rightarrow \infty} H_{n,q}(f; x), & \text{if } x = 0, \\ H_{n,q}(f; x), & \text{if } 0 < x < \infty, \\ f(0), & \text{if } x = \infty. \end{cases}$$

**Proof.** It is clear that

$$H_{n,q}(f; x) = \sum_{k=0}^n f \left( \frac{[n-k]_q}{q^{n-k} [k+1]_q} \right) h_{n,n-k}(q; x).$$

Consider

$$\begin{aligned} h_{n,n-k}(q; x) &= \frac{1}{(1+qx)_q^n} \begin{bmatrix} n \\ n-k \end{bmatrix}_q q^{(n-k)(n-k+1)/2} x^{n-k} \\ &= \frac{1}{q^{n(n+1)/2} x^n \left(1 + \frac{1}{qx}\right)_{1/q}^n} \frac{q^{n(n+1)/2}}{q^{(n-k)(n-k+1)/2} q^{k(k+1)/2}} \begin{bmatrix} n \\ k \end{bmatrix}_{1/q} q^{(n-k)(n-k+1)/2} x^{n-k} \\ &= \frac{1}{\left(1 + \frac{1}{qx}\right)_{1/q}^n} \begin{bmatrix} n \\ k \end{bmatrix}_{1/q} \left(\frac{1}{q}\right)^{k(k+1)/2} \left(\frac{1}{x}\right)^k = h_{n,k}\left(\frac{1}{q}; \frac{1}{x}\right). \end{aligned}$$

On the other hand,

$$\frac{[n+1-k]_q}{q^{n+1-k} [k]_q} = \frac{1 - q^{n+1-k}}{q^{n+1-k} (1 - q^k)} = \frac{[n-k+1]_{1/q}}{q^k [k]_{1/q}} = \frac{1}{q^k} \frac{1 - \left(\frac{1}{q}\right)^{n-k+1}}{1 - \left(\frac{1}{q}\right)^k} = \frac{1}{q^{n-k+1}} \frac{q^{n-k+1} - 1}{q^k - 1}.$$

Therefore

$$\begin{aligned} H_{n,q}(f; x) &= \sum_{k=0}^n f \left( \frac{[n+1-k]_q}{q^{n+1-k} [k]_q} \right) h_{n,n-k}(q; x) = \sum_{k=0}^n f \left( \frac{[n-k+1]_{1/q}}{q^k [k]_{1/q}} \right) h_{n,k}\left(\frac{1}{q}; \frac{1}{x}\right) \\ &= \sum_{k=0}^n g \left( \frac{q^k [k]_{1/q}}{[n-k+1]_{1/q}} \right) h_{n,k}\left(\frac{1}{q}; \frac{1}{x}\right) = H_{n,1/q}(g; \frac{1}{x}). \end{aligned}$$

■

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# A GENERAL COMPOSITE ITERATIVE METHOD FOR EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS

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**ABSTRACT.** We introduce a new general composite iterative scheme by the viscosity approximation method for finding a common point of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in Hilbert spaces. It is proved that the sequence generated by the iterative scheme converges strongly to a common point of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping, which is the unique solution of a certain variational inequality. Our results substantially develop and improve the corresponding results of Jung [J. S. Jung, Strong convergence of composite iterative methods for equilibrium problems and fixed point problems, Appl. Math. Comput. 213 (2009) 498-505], Plubtieng and Punpaeng [S. Plubtieng and R. Punpaeng, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 336 (2007) 455-469], Shang et al. [M. Shang, X. Qin, Y. Su, A general iterative method for equilibrium problems and fixed point problems, Fixed Point Theory Appl. 2007 (2007), Article ID 64306, 7 pages] and Takahashi and Takahashi [A. Takahashi and W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert space, J. Math. Anal. Appl. 333 (2007) 506-515].

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## 1. Introduction

Let  $H$  be a real Hilbert space and  $C$  be a nonempty closed convex subset of  $H$ . Recall that a mapping  $f : C \rightarrow C$  is a *contraction* on  $C$  if there exists a constant  $k \in (0, 1)$  such that  $\|f(x) - f(y)\| \leq k\|x - y\|$ ,  $x, y \in C$ . We use  $\Sigma_C$  to denote the

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collection of all contractions on  $C$ . That is,  $\Sigma_C = \{f : f : C \rightarrow C \text{ a contraction}\}$ . A mapping  $S : C \rightarrow C$  is called *nonexpansive* if  $\|Sx - Sy\| \leq \|x - y\|$   $x, y \in C$ . We denote by  $F(S)$  the set of fixed points of  $S$ ; that is,  $F(S) = \{x \in C : x = Sx\}$ . If  $C \in H$  is a bounded closed convex subset and  $S$  is a nonexpansive mapping of  $C$  into itself, then  $F(S)$  is nonempty; see [5,15].

Let  $F$  be a bifunction of  $C \times C$  into  $\mathbb{R}$ , where  $\mathbb{R}$  is the set of real numbers. The equilibrium problem for  $F : C \times C \rightarrow \mathbb{R}$  is to find  $x \in C$  such that

$$F(x, y) \geq 0 \quad \text{for all } y \in C. \quad (1.1)$$

The set of solutions of (1.1) is denoted by  $EP(F)$ . Given a mapping  $T : C \rightarrow H$ , let  $F(x, y) = \langle Tx, y - x \rangle$  for all  $x, y \in C$ . Then,  $z \in EP(F)$  if and only if  $\langle Tz, y - z \rangle \geq 0$  for all  $y \in C$ , that is,  $z$  is a solution of the variational inequality. Many problems in physics, optimization, and economics reduce to find a solution of the equilibrium problem (1.1). Some methods have been proposed to solve the equilibrium problem; see, for instance, [2,4]. In 2005, Combettes and Hirstoaga [2] introduced an iterative scheme of finding the best approximation to the initial data when  $EP(F)$  is nonempty and prove a strong convergence theorem.

Let  $A$  be a strongly positive bounded linear operator on  $H$ : that is, there is a constant  $\bar{\gamma} > 0$  with property

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \text{for all } x \in H.$$

Recently iterative methods for nonexpansive mappings have been applied to solve convex minimization problem; see, e.g., [3,17,18,20] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space  $H$ :

$$\min_{x \in F(S)} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \quad (1.2)$$

where  $b$  is a given point in  $H$

In 2000, as a method for approximation of fixed points of a nonexpansive mapping, the viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping was proposed by Moudafi [9]. In 2004, in order to improve Theorem 2.2 of Moudafi [9], Xu [19] considered the the following explicit iterative scheme: for  $S : C \rightarrow C$  nonexpansive mapping,  $f \in \Sigma_C$  and  $\alpha_n \in (0, 1)$ ,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) Sx_n, \quad n \geq 1. \quad (1.3)$$

Moreover, in [19], he also studied the strong convergence of  $\{x_n\}$  generated by (1.3) as  $n \rightarrow \infty$  in a Hilbert space and showed that the strong  $\lim_{n \rightarrow \infty} x_n$  is the unique solution of a certain variational inequality.

In 2003, Xu [18] proved that the sequence  $\{x_n\}$  generated by the following iterative method with the initial guess  $x_0 \in H$  chosen arbitrarily,

$$x_{n+1} = \alpha_n b + (I - \alpha_n A) Sx_n, \quad n \geq 0,$$

converges strongly to the unique solution of the minimization problem (1.2) provided the sequence  $\{\alpha_n\}$  satisfies certain conditions. In 2006, Marino and Xu [8] introduced a new iterative scheme by the viscosity approximation method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) S x_n, \quad n \geq 0, \quad (1.4)$$

and proved that the sequence  $\{x_n\}$  generated by (1.4) converges strongly to the unique solution  $x^*$  of the variational inequality

$$\langle (\gamma f - A)x^*, x^* - x \rangle \geq 0, \quad x \in F(S),$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where  $h$  is a potential function for  $\gamma f$  (that is,  $h'(x) = \gamma f(x)$  for  $x \in H$ ).

On the other hand, in 2007, using the metric projection, Tada and Takahashi [13] introduced the an iterative scheme for finding a common point of the set of solutions of the equilibrium problem (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space as follows:

$$\left\{ \begin{array}{l} u_n \in C \text{ such that } F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \text{for all } y \in C, \\ w_n = (1 - \alpha_n)x_n + \alpha_n S u_n, \\ C_n = \{z \in H : \|w_n - z\| \leq \|x_n - z\|\} \\ D_n = \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap D_n}(x), \quad n \geq 1, \end{array} \right. \quad (1.5)$$

where  $P_K$  denotes the metric projection from  $H$  onto a closed convex subset  $K$  of  $H$ . Using an additional projection to calculate at each iteration step, they showed that the sequence  $\{x_n\}$  generated by (1.5) converges strongly to  $P_{F(S) \cap EP(F)}(x)$  (see also Tada and Takahashi [12]).

In 2007, without using the metric projection, Takahashi and Takahashi [14] considered the following iterative scheme by the the viscosity approximation method for finding a common point of the set of solutions of the equilibrium problem (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space:

$$\left\{ \begin{array}{l} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \text{for all } y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S u_n, \quad n \geq 1, \end{array} \right. \quad (1.6)$$

and proved that the sequence  $\{x_n\}$  generated by (1.6) strongly converges strongly to a point in  $F(S) \cap EP(F)$ , provided  $\{\alpha_n\}$  and  $\{r_n\}$  satisfy

$$(C1) \lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

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(C2)  $\liminf_{n \rightarrow \infty} r_n > 0$ ,  $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ .

Their result was connected with Combettes and Hirstoaga's result [2] and Wittmann's result [16].

In 2009, Jung [7] considered the following composite iterative scheme for finding a common point of the set of solutions of the equilibrium problem (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space:

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \text{for all } y \in C, \\ y_n = \alpha_n f(x_n) + (1 - \alpha_n) S u_n, \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n S y_n, & n \geq 1, \end{cases} \quad (1.7)$$

and established the strong convergence of  $\{x_n\}$  generated by (1.7) to a point in  $F(S) \cap EP(F)$  under conditions (C1) and (C2) on  $\{\alpha_n\}$  and  $\{r_n\}$  and the following condition on  $\{\beta_n\}$ :

(C3)  $\lim_{n \rightarrow \infty} \beta_n = 0$  and  $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ .

In 2007, Plubtieng and Punpaeng [10] (see also Shang et al. [11]) studied the following iterative scheme by the viscosity approximation method for finding a common point of the set of solutions of the equilibrium problem (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space:

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \text{for all } y \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) S u_n, & n \geq 1, \end{cases} \quad (1.8)$$

where  $0 < \gamma < \frac{2}{k}$ , and proved that the sequence  $\{x_n\}$  generated by (1.8) strongly converges to a point in  $F(S) \cap EP(F)$  under the conditions (C1) and (C2) on  $\{\alpha_n\}$  and  $\{r_n\}$ , which is the unique solution of the following variational inequality

$$\langle \gamma f(q) - Aq, q - p \rangle \geq 0, \quad p \in F(S) \cap EP(F),$$

which is also the optimality condition for the minimization problem

$$\min_{x \in F(S) \cap EP(F)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where  $h$  is a potential function for  $\gamma f$ .

In this paper, motivated by above-mentioned results, we introduce a new composite iterative scheme by viscosity approximation method for finding a common point of the set of solutions of the equilibrium problem (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space:

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \text{for all } y \in C, \\ y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A) S u_n, \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n S y_n, & n \geq 1. \end{cases} \quad (\text{IS})$$

If  $\beta_n = 0$ , then (IS) reduces to (1.8). Then, under the conditions (C1), (C2) and (C3) on the sequences  $\{\alpha_n\}$ ,  $\{r_n\}$  and  $\{\beta_n\}$ , we show that the sequence  $\{x_n\}$  generated by (IS) converges strongly to a point in  $F(S) \cap EP(F)$ , which is the unique solution of a certain variational inequality. The main results develop and improve the corresponding results of Jung [7], Plubtieng and Punpaeng [10], Shang et al. [11], and Takahashi and Takahashi [14], as well as Combettes and Hirstoaga [2], Marino and Xu [8], Moudafi [9], Xu [19], and Wittmann [16]. We point out that the iterative scheme (IS) is a new method for finding the solutions of the equilibrium problem (1.1) and the fixed points of nonexpansive mappings.

## 2. Preliminaries and Lemmas

Let  $H$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$ . For the sequence  $\{x_n\}$  in  $H$ , we write  $x_n \rightharpoonup x$  to indicate that the sequence  $\{x_n\}$  converges weakly to  $x$ .  $x_n \rightarrow x$  implies that  $\{x_n\}$  converges strongly to  $x$ . In a real Hilbert space  $H$ , we have

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$$

for all  $x, y \in H$  and  $\lambda \in \mathbb{R}$ . Let  $C$  be a closed convex subset of  $H$ . For every point  $x \in H$ , there exists a unique nearest point in  $C$ , denoted by  $P_C x$ , such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

$P_C$  is called the *metric projection* of  $H$  to  $C$ . It is well known that  $P_C$  satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2$$

for every  $x, y \in H$ . Moreover,  $P_C x$  is characterized by the properties: for  $x \in H$  and  $z \in C$ ,

$$z = P_C x \Leftrightarrow \langle x - z, z - y \rangle \geq 0 \quad \text{for all } y \in C, \quad (2.1)$$

We also know that for any sequence  $\{x_n\} \subset H$  with  $x_n \rightharpoonup x$ , the inequality

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|$$

holds for every  $y \in H$  with  $x \neq y$ ; we refer [5,15] for more details.

For solving the equilibrium problem for a bifunction  $F : C \times C \rightarrow \mathbb{R}$ , let us assume that  $F$  satisfies the following conditions:

- (A1)  $F(x, x) = 0$  for all  $x \in C$ ;
- (A2)  $F$  is monotone, that is,  $F(x, y) + F(y, x) \leq 0$  for all  $x, y \in C$ ;
- (A3) for each  $x, y, z \in C$ ,

$$\lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y);$$

- (A4) for each  $x \in C, y \mapsto F(x, y)$  is convex and lower semicontinuous.

The following lemmas were given in [1,2].

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**Lemma 2.1** ([1]). *Let  $C$  be a nonempty closed convex subset of  $H$  and let  $F$  be a bifunction of  $C \times C$  into  $\mathbb{R}$  satisfying (A1)–(A4). Let  $r > 0$  and  $x \in H$ . Then, there exists  $z \in C$  such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \quad \text{for all } y \in C.$$

**Lemma 2.2** ([2]). *Assume that  $F : C \times C \rightarrow \mathbb{R}$  satisfies (A1)–(A4). For  $r > 0$  and  $x \in H$ , define a mapping  $T_r : H \rightarrow C$  as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \text{ for all } y \in C \right\}$$

*for all  $z \in H$ . Then, the following hold:*

- (1)  $T_r$  is single-valued;
- (2)  $T_r$  is firmly nonexpansive, that is, for any  $x, y \in H$ ,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

- (3)  $F(T_r) = EP(F)$ ;
- (4)  $EP(F)$  is closed and convex.

We also need the following lemmas for the proof of our main results; see also [17] for the proof of Lemma 2.3.

**Lemma 2.3** ([18]). *Let  $\{s_n\}$  be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \beta_n, \quad n \geq 0,$$

*where  $\{\lambda_n\}$  and  $\{\beta_n\}$  satisfy the following conditions:*

- (i)  $\{\lambda_n\} \subset [0, 1]$  and  $\sum_{n=0}^{\infty} \lambda_n = \infty$  or, equivalently,  $\prod_{n=0}^{\infty} (1 - \lambda_n) = 0$ ,
  - (ii)  $\limsup_{n \rightarrow \infty} \frac{\beta_n}{\lambda_n} \leq 0$  or  $\sum_{n=0}^{\infty} |\beta_n| < \infty$ ,
- Then  $\lim_{n \rightarrow \infty} s_n = 0$ .*

**Lemma 2.4** ([5]). (Demiclosedness principle) *Let  $H$  be a real Hilbert space,  $C$  a nonempty closed convex subset of  $H$  and  $T : C \rightarrow E$  a nonexpansive mapping. Then the mapping  $I - T$  is demiclosed on  $C$ , where  $I$  is the identity mapping; that is,  $x_n \rightharpoonup x$  in  $E$  and  $(I - T)x_n \rightarrow y$  imply that  $x \in C$  and  $(I - T)x = y$ .*

**Lemma 2.5.** *In a real Hilbert space  $H$ , there holds the following inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle,$$

*for all  $x, y \in H$*

**Lemma 2.6** ([8]). *Assume that  $A$  is a strongly positive linear bounded operator on a Hilbert space  $H$  with coefficient  $\bar{\gamma} > 0$  and  $0 < \rho \leq \|A\|^{-1}$ . Then  $\|I - \rho A\| \leq 1 - \rho \bar{\gamma}$ .*

### 3. Main results

In this section, as the viscosity approximation method, we introduce a new general composite iterative scheme for finding a common point of the set of solutions of the equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space.

**Theorem 3.1.** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $F$  be a bifunction from  $C \times C$  to  $\mathbb{R}$  satisfying (A1)–(A4) and  $S$  be a nonexpansive mapping of  $C$  into  $H$  such that  $F(S) \cap EP(F) \neq \emptyset$ . Let  $A$  be a strongly positive linear bounded operator on  $H$  with coefficient  $\bar{\gamma} > 0$  and let  $f \in \Sigma_H$  with a coefficient  $k$  ( $0 < k < 1$ ). Assume that  $0 < \gamma < \frac{\bar{\gamma}}{k}$ . Let  $\{x_n\}$  and  $\{u_n\}$  be sequences generated by  $x_1 \in H$  and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \text{for all } y \in C, \\ y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A) S u_n \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n S y_n, & n \geq 1, \end{cases} \quad (\text{IS})$$

where  $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$  and  $\{r_n\} \subset (0, \infty)$ . If  $\{\alpha_n\}, \{\beta_n\}$  and  $\{r_n\}$  satisfy the conditions:

(C1)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ;

(C2)  $\liminf_{n \rightarrow \infty} r_n > 0$ ,  $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ ;

(C3)  $\lim_{n \rightarrow \infty} \beta_n = 0$ ,  $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ ,

then  $\{x_n\}$  and  $\{u_n\}$  converge strongly to  $q \in F(S) \cap EP(F)$ , where  $q = P_{F(S) \cap EP(F)}(\gamma f + (I - A))(q)$ , which is the unique solution of the variational inequality

$$\langle \gamma f(q) - Aq, q - p \rangle \geq 0, \quad p \in F(S) \cap EP(F).$$

*Proof.* By Lemma 2.1,  $\{u_n\}$  and  $\{x_n\}$  are well defined. Since  $\alpha_n \rightarrow 0$  by the condition (C1), we may assume, with no loss of generality, that  $\alpha_n < \|A\|^{-1}$  for all  $n$ . From Lemma 2.6, we know that if  $0 < \rho \leq \|A\|^{-1}$ , then  $\|I - \rho A\| \leq 1 - \rho \bar{\gamma}$ . We will assume that  $\|I - A\| \leq 1 - \bar{\gamma}$ . Let  $Q = P_{F(S) \cap EP(F)}$ . Then  $Q(\gamma f + (I - A))$  is a contraction of  $H$  into itself. Indeed, for  $x, y \in H$ , we have

$$\begin{aligned} & \|Q(\gamma f + (I - A))(x) - Q(\gamma f + (I - A))(y)\| \\ & \leq \|(\gamma f + (I - A))(x) - (\gamma f + (I - A))(y)\| \\ & \leq \gamma \|f(x) - f(y)\| + \|I - A\| \|x - y\| \\ & \leq \gamma k \|x - y\| + (1 - \bar{\gamma}) \|x - y\| \\ & < \|x - y\| \end{aligned}$$

for some  $k \in [0, 1)$ . Since  $H$  is complete, there exists a unique point  $q \in H$  such that  $q = Q(\gamma f + (I - A))(q) = P_{F(S) \cap EP(F)}(\gamma f + (I - A))(q)$ . Such a  $q$  is a point of  $C$ .

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We proceed with following steps:

**Step 1.** We show that  $\{x_n\}$  and  $\{u_n\}$  are bounded.

In fact, let  $v \in F(S) \cap EP(F)$ . Then from  $u_n = T_{r_n}x_n$ , we have

$$\|u_n - v\| = \|T_{r_n}x_n - T_{r_n}v\| \leq \|x_n - v\|, \quad n \geq 1.$$

Put  $M = \max\{\|x_1 - v\|, \frac{\|\gamma f(v) - Av\|}{\bar{\gamma} - \gamma k}\}$ . It is obvious that  $\|x_1 - v\| \leq M$ . Suppose that  $\|x_n - v\| \leq M$ . Then, we have

$$\begin{aligned} \|y_n - v\| &= \|\alpha_n(\gamma f(x_n) - Av) + (I - \alpha_n A)(Su_n - v)\| \\ &\leq \alpha_n \|\gamma f(x_n) - Av\| + \|I - \alpha_n A\| \|u_n - v\| \\ &\leq \alpha_n [\gamma \|f(x_n) - f(v)\| + \|\gamma f(v) - Av\|] + (1 - \alpha_n \bar{\gamma}) \|x_n - v\| \\ &\leq \alpha_n \gamma k \|x_n - v\| + (1 - \alpha_n \bar{\gamma}) \|x_n - v\| + \alpha_n \|\gamma f(v) - Av\| \\ &= (1 - (\bar{\gamma} - \gamma k) \alpha_n) \|x_n - v\| + \alpha_n (\bar{\gamma} - \gamma k) \frac{1}{\bar{\gamma} - \gamma k} \|\gamma f(v) - Av\| \\ &\leq (1 - (\bar{\gamma} - \gamma k) \alpha_n) M + (\bar{\gamma} - \gamma k) \alpha_n M = M, \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - v\| &= \|(1 - \beta_n)(y_n - v) + \beta_n(Sy_n - v)\| \\ &\leq (1 - \beta_n) \|y_n - v\| + \beta_n \|y_n - v\| \\ &= \|y_n - v\| \leq M. \end{aligned}$$

So, we have that  $\|x_n - v\| \leq M$  for  $n \geq 1$  and hence  $\{x_n\}$  is bounded. We also obtain that  $\{u_n\}$ ,  $\{Su_n\}$ ,  $\{y_n\}$ ,  $\{f(x_n)\}$ ,  $\{ASu_n\}$  and  $\{Sy_n\}$  are bounded. Moreover, by condition (C1), we also obtain

$$\|y_n - Su_n\| = \alpha_n \|\gamma f(x_n) - ASu_n\| \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \quad (3.1)$$

**Step 2.** We show that  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ . From (IS), we have

$$\begin{cases} y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A) Su_n \\ y_{n-1} = \alpha_{n-1} \gamma f(x_{n-1}) + (I - \alpha_{n-1} A) Su_{n-1}. \end{cases}$$

Simple calculations show that

$$\begin{aligned} y_n - y_{n-1} &= (I - \alpha_n A)(Su_n - Su_{n-1}) - (\alpha_n - \alpha_{n-1}) ASu_{n-1} \\ &\quad + \gamma [\alpha_n (f(x_n) - f(x_{n-1})) + (\alpha_n - \alpha_{n-1}) f(x_{n-1})]. \end{aligned}$$

So, we have

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|(I - \alpha_n A)(Su_n - Su_{n-1}) - (\alpha_n - \alpha_{n-1}) ASu_{n-1} \\ &\quad + \gamma [\alpha_n (f(x_n) - f(x_{n-1})) + (\alpha_n - \alpha_{n-1}) f(x_{n-1})]\| \\ &\leq (1 - \alpha_n \bar{\gamma}) \|Su_n - Su_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|ASu_{n-1}\| \\ &\quad + \gamma [\alpha_n k \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\|] \\ &\leq (1 - \alpha_n \bar{\gamma}) \|u_n - u_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|ASu_{n-1}\| \\ &\quad + \gamma [\alpha_n k \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\|] \\ &\leq (1 - \alpha_n \bar{\gamma}) \|u_n - u_{n-1}\| + \gamma \alpha_n k \|x_n - x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| L_1 \end{aligned} \quad (3.2)$$

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for every  $n \geq 1$ , where  $L_1 = \sup\{\gamma\|f(x_n)\| + \|ASu_n\| : n \geq 1\}$ .

On the other hand, from  $u_n = T_{r_n}x_n$  and  $u_{n-1} = T_{r_{n-1}}x_{n-1}$ , we have

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \text{for all } y \in C \quad (3.3)$$

and

$$F(u_{n-1}, y) + \frac{1}{r_{n-1}} \langle y - u_{n-1}, u_{n-1} - x_{n-1} \rangle \geq 0, \quad \text{for all } y \in C. \quad (3.4)$$

Putting  $y = u_{n-1}$  in (3.3) and  $y = u_n$  in (3.4), we have

$$F(u_n, u_{n-1}) + \frac{1}{r_n} \langle u_{n-1} - u_n, u_n - x_n \rangle \geq 0$$

and

$$F(u_{n-1}, u_n) + \frac{1}{r_{n-1}} \langle u_n - u_{n-1}, u_{n-1} - x_{n-1} \rangle \geq 0.$$

So, from (A2) we have

$$\left\langle u_n - u_{n-1}, \frac{u_{n-1} - x_{n-1}}{r_{n-1}} - \frac{u_n - x_n}{r_n} \right\rangle \geq 0$$

and hence

$$\left\langle u_n - u_{n-1}, u_{n-1} - u_n + u_n - x_{n-1} - \frac{r_{n-1}}{r_n} (u_n - x_n) \right\rangle \geq 0.$$

Without loss of generality, let us assume that there exists a real number  $b$  such that  $r_n > b > 0$  for all  $n \geq 1$ . Then, we have

$$\begin{aligned} & \|u_n - u_{n-1}\|^2 \\ & \leq \left\langle u_n - u_{n-1}, x_n - x_{n-1} + \left(1 - \frac{r_{n-1}}{r_n}\right)(u_n - x_n) \right\rangle \\ & \leq \|u_n - u_{n-1}\| \left\{ \|x_n - x_{n-1}\| + \left|1 - \frac{r_{n-1}}{r_n}\right| \|u_n - x_n\| \right\} \end{aligned}$$

and hence

$$\begin{aligned} \|u_n - u_{n-1}\| & \leq \|x_n - x_{n-1}\| + \frac{1}{r_n} |r_n - r_{n-1}| \|u_n - x_n\| \\ & \leq \|x_n - x_{n-1}\| + \frac{1}{b} |r_n - r_{n-1}| L_2, \end{aligned} \quad (3.5)$$



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where  $L_2 = \sup\{\|u_n - x_n\| : n \geq 1\}$ . So, from (3.2) we have

$$\begin{aligned}
 \|y_n - y_{n-1}\| &\leq (1 - \alpha_n \bar{\gamma})(\|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}|L_2) \\
 &\quad + \gamma \alpha_n k \|x_n - x_{n-1}\| + L_1 |\alpha_n - \alpha_{n-1}| \\
 &= (1 - (\bar{\gamma} - \gamma k) \alpha_n) \|x_n - x_{n-1}\| + L_1 |\alpha_n - \alpha_{n-1}| \\
 &\quad + \frac{1}{b} |r_n - r_{n-1}| L_2.
 \end{aligned} \tag{3.6}$$

Also, simple calculations show that

$$\begin{aligned}
 x_{n+1} - x_n &= (1 - \beta_n)(y_n - y_{n-1}) + \beta_n(Sy_n - Sy_{n-1}) \\
 &\quad + (\beta_n - \beta_{n-1})(Sy_{n-1} - y_{n-1}).
 \end{aligned}$$

So, from (3.6) it follows that

$$\begin{aligned}
 &\|x_{n+1} - x_n\| \\
 &\leq (1 - \beta_n)\|y_n - y_{n-1}\| + \beta_n\|Sy_n - Sy_{n-1}\| \\
 &\quad + |\beta_n - \beta_{n-1}|\|Sy_{n-1} - y_{n-1}\| \\
 &\leq (1 - \beta_n)\|y_n - y_{n-1}\| + \beta_n\|y_n - y_{n-1}\| \\
 &\quad + |\beta_n - \beta_{n-1}|\|Sy_{n-1} - y_{n-1}\| \\
 &\leq \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|L_3 \\
 &\leq (1 - (\bar{\gamma} - \gamma k) \alpha_n) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|L_1 + \frac{1}{b}|r_n - r_{n-1}|L_2 \\
 &\quad + |\beta_n - \beta_{n-1}|L_3,
 \end{aligned} \tag{3.7}$$

where  $L_3 = \sup\{\|Sy_n - y_n\| : n \geq 1\}$ . From the conditions (C1)-(C3), it is easy to see that

$$\lim_{n \rightarrow \infty} (\bar{\gamma} - \gamma k) \alpha_n = 0, \quad \sum_{n=0}^{\infty} (\bar{\gamma} - \gamma k) \alpha_n = \infty$$

and

$$\sum_{n=0}^{\infty} (|\alpha_{n+1} - \alpha_n|L_1 + \frac{1}{b}|r_{n+1} - r_n|L_2 + |\beta_{n+1} - \beta_n|L_3) < \infty.$$

Applying Lemma 2.3, to (3.7), we have

$$\|x_{n+1} - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Moreover, from (3.5) it follows that  $\|u_{n+1} - u_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . By (3.2), we also have that

$$\|y_{n+1} - y_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{3.8}$$

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**Step 3.** We show that  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ . Indeed, from (C3) we have

$$\|x_{n+1} - y_n\| = \beta_n \|y_n - Sy_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

So, we have from Step 2

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

**Step 4.** We show that  $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$  and  $\lim_{n \rightarrow \infty} \|Su_n - u_n\| = 0$ . To this end, let  $v \in F(S) \cap EP(F)$ . Then, from (2) in Lemma 2.2 we have

$$\begin{aligned} \|u_n - v\|^2 &= \|T_{r_n}x_n - T_{r_n}v\|^2 \\ &\leq \langle T_{r_n}x_n - T_{r_n}v, x_n - v \rangle \\ &= \langle u_n - v, x_n - v \rangle \\ &= \frac{1}{2}(\|u_n - v\|^2 + \|x_n - v\|^2 - \|x_n - u_n\|^2) \end{aligned}$$

and hence

$$\|u_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - u_n\|^2.$$

Therefore, by convexity of  $\|\cdot\|^2$ , we have

$$\begin{aligned} \|y_n - v\|^2 &= \|\alpha_n(\gamma f(x_n) - Av) + (I - \alpha_n A)(Su_n - v)\|^2 \\ &\leq (\alpha_n \|\gamma f(x_n) - Av\| + \|I - \alpha_n A\| \|Su_n - v\|)^2 \\ &\leq (\alpha_n \|\gamma f(x_n) - Av\| + (1 - \alpha_n \bar{\gamma}) \|u_n - v\|)^2 \\ &\leq \alpha_n \|\gamma f(x_n) - Av\|^2 + (1 - \alpha_n \bar{\gamma}) \|u_n - v\|^2 \\ &\quad + 2\alpha_n(1 - \alpha_n \bar{\gamma}) \|\gamma f(x_n) - Av\| \|u_n - v\| \\ &\leq \alpha_n \|\gamma f(x_n) - Av\|^2 + \|x_n - v\|^2 - (1 - \alpha_n \bar{\gamma}) \|x_n - u_n\|^2 \\ &\quad + 2\alpha_n(1 - \alpha_n \bar{\gamma}) \|\gamma f(x_n) - Av\| \|u_n - v\|. \end{aligned}$$

So, we obtain

$$\begin{aligned} &(1 - \alpha_n \bar{\gamma}) \|x_n - u_n\|^2 \\ &\leq \alpha_n \|\gamma f(x_n) - Av\|^2 + (\|x_n - v\| + \|y_n - v\|)(\|x_n - v\| - \|y_n - v\|) \\ &\quad + 2\alpha_n(1 - \alpha_n \bar{\gamma}) \|\gamma f(x_n) - Av\| \|u_n - v\| \\ &\leq \alpha_n \|\gamma f(x_n) - Av\|^2 + (\|x_n - v\| + \|y_n - v\|) \|x_n - y_n\| \\ &\quad + 2\alpha_n(1 - \alpha_n \bar{\gamma}) \|\gamma f(x_n) - Av\| \|u_n - v\|. \end{aligned}$$

Since  $\alpha_n \rightarrow 0$  and  $\|x_n - y_n\| \rightarrow 0$  by the condition (C1) and Step 3, we have  $\|x_n - u_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Moreover, by (3.9)

$$\|y_n - u_n\| \leq \|y_n - x_n\| + \|x_n - u_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.9)$$

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Since

$$\|Su_n - u_n\| \leq \|Su_n - y_n\| + \|y_n - u_n\|,$$

from (3.1) and (3.9) we also have

$$\|Su_n - u_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.10)$$

**Step 5.** We show that  $\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, y_n - q \rangle \leq 0$  for  $q \in F(S) \cap EP(F)$ , where  $q = P_{F(S) \cap EP(F)}(\gamma f + (I - A))(q)$ . To this end, choose a subsequence  $\{u_{n_i}\}$  of  $\{u_n\}$  such that

$$\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle = \lim_{i \rightarrow \infty} \langle \gamma f(q) - Aq, x_{n_i} - q \rangle.$$

Since  $\{u_{n_i}\}$  is bounded, there exists a subsequence  $\{u_{n_{i_j}}\}$  of  $\{u_{n_i}\}$  which converges weakly to  $z$ . We may assume without loss of generality that  $u_{n_i} \rightharpoonup z$ . Then we can obtain  $z \in F(S) \cap EP(F)$ . Indeed, let us first show that  $z \in EP(F)$ . By  $u_n = T_{r_n}x_n$ , we have

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \text{ for all } y \in C.$$

From (A2), we also have

$$\frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq F(y, u_n)$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq F(y, u_{n_i}).$$

Since  $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rightarrow 0$  and  $u_{n_i} \rightharpoonup z$ , from (A4) we have

$$0 \geq F(y, z) \text{ for all } y \in C.$$

For  $0 < t \leq 1$  and  $y \in C$ , let  $y_t = ty + (1 - t)z$ . Since  $y \in C$  and  $z \in C$ , we have  $y_t \in C$  and hence  $F(y_t, z) \leq 0$ . So, from (A1) and (A4) we have

$$\begin{aligned} 0 = F(y_t, y_t) &\leq tF(y_t, y) + (1 - t)F(y_t, z) \\ &\leq tF(y_t, y) \end{aligned}$$

and hence  $0 \leq F(y_t, y)$ . From (A3) we have

$$0 \leq F(z, y) \text{ for all } y \in C$$

and hence  $z \in EP(F)$ . On the other hand, since  $\|Su_n - u_n\| \rightarrow 0$  by (3.10) and  $u_{n_i} \rightarrow z$ , from Lemma 2.4 we obtain  $z \in F(S)$ . Therefore  $z \in F(S) \cap EP(F)$ . Since  $q = P_{F(S) \cap EP(F)}(\gamma f + (I - A))(q)$  and  $x_{n_i} \rightarrow z$  by Step 4, from (2.1) we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle &= \lim_{i \rightarrow \infty} \langle \gamma f(q) - Aq, x_{n_i} - q \rangle \\ &= \langle \gamma f(q) - Aq, z - q \rangle \leq 0. \end{aligned} \quad (3.11)$$

Since  $\|x_n - y_n\| \rightarrow 0$  by Step 3, from (3.11) we have

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, y_n - q \rangle \\ &\leq \limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, y_n - x_n \rangle + \limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle \\ &\leq \limsup_{n \rightarrow \infty} \|\gamma f(q) - Aq\| \|y_n - x_n\| + \limsup_{n \rightarrow \infty} \langle \gamma f(q) - Aq, x_n - q \rangle \\ &\leq 0. \end{aligned}$$

**Step 6.** We show that  $\lim_{n \rightarrow \infty} \|x_n - q\| = 0$  for  $q \in F(S) \cap EP(F)$ , where  $q = P_{F(S) \cap EP(F)}(\gamma f + (I - A))(q)$ . Indeed, from  $\|x_{n+1} - q\| \leq \|y_n - q\|$ ,  $\|u_n - q\| \leq \|x_n - q\|$  and  $y_n - q = \alpha_n(\gamma f(x_n) - Aq) + (I - \alpha_n A)(Su_n - q)$ , by Lemma 2.5 we have

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ &\leq \|y_n - q\|^2 = \|\alpha_n(\gamma f(x_n) - Aq) + (I - \alpha_n A)(Su_n - q)\|^2 \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|Su_n - q\|^2 + 2\alpha_n \langle \gamma f(x_n) - Aq, y_n - q \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|u_n - q\|^2 + 2\alpha_n \gamma \langle f(x_n) - f(q), y_n - q \rangle \\ &\quad + 2\alpha_n \langle \gamma f(q) - Aq, y_n - q \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + 2\alpha_n \gamma k \|x_n - q\| \|y_n - q\| \\ &\quad + 2\alpha_n \langle \gamma f(q) - Aq, y_n - q \rangle \\ &\leq (1 - \alpha_n \bar{\gamma})^2 \|x_n - q\|^2 + 2\alpha_n \gamma k \|x_n - q\| (\|y_n - x_n\| + \|x_n - q\|) \\ &\quad + 2\alpha_n \langle \gamma f(q) - Aq, y_n - q \rangle \\ &\leq (1 - 2(\bar{\gamma} - \gamma k)\alpha_n) \|x_n - q\|^2 + \alpha_n^2 \bar{\gamma}^2 \|x_n - q\|^2 + 2\alpha_n \gamma k \|y_n - x_n\| \\ &\quad + 2\alpha_n \langle \gamma f(q) - Aq, y_n - q \rangle \\ &= (1 - \bar{\alpha}_n) \|x_n - q\|^2 + \bar{\alpha}_n \bar{\beta}_n, \end{aligned}$$

where

$$\begin{aligned} \bar{\alpha}_n &= 2(\bar{\gamma} - \gamma k)\alpha_n, \\ \bar{\beta}_n &= \frac{\alpha_n \bar{\gamma}^2}{2(\bar{\gamma} - \gamma k)} M_1 + \frac{\gamma k}{\bar{\gamma} - \gamma k} \|y_n - x_n\| + \frac{1}{\bar{\gamma} - \gamma k} \langle \gamma f(q) - Aq, y_n - q \rangle, \end{aligned}$$

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and  $M_1 = \sup\{\|x_n - q\|^2 : n \geq 1\}$ . From (C1), Step 3 and Step 5, it is easily seen that  $\bar{\alpha}_n \rightarrow 0$ ,  $\sum_{n=1}^{\infty} \bar{\alpha}_n = \infty$ , and  $\limsup_{n \rightarrow \infty} \bar{\beta}_n \leq 0$ . Thus, by Lemma 2.3, we obtain  $x_n \rightarrow q$ . This completes the proof.  $\square$

As in [10,14], we obtain the following corollaries as immediate consequences of Theorem 3.1.

**Corollary 3.1.** ([7, Theorem 3.1]) Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $F$  be a bifunction from  $C \times C$  to  $\mathbb{R}$  satisfying (A1)–(A4) and let  $S$  be a nonexpansive mapping of  $C$  into  $H$  such that  $F(S) \cap EP(F) \neq \emptyset$ . Let  $f \in \Sigma_H$  with a coefficient  $k$  ( $0 < k < 1$ ). and let  $\{x_n\}$  and  $\{u_n\}$  be sequences generated by  $x_1 \in H$  and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \text{for all } y \in C, \\ y_n = \alpha_n f(x_n) + (1 - \alpha_n) S u_n, \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n S y_n, & n \geq 1, \end{cases}$$

where  $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$  and  $\{r_n\} \subset (0, \infty)$ . If  $\{\alpha_n\}, \{\beta_n\}$  and  $\{r_n\}$  satisfy the conditions:

(C1)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ;

(C2)  $\liminf_{n \rightarrow \infty} r_n > 0$ ,  $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ ;

(C3)  $\lim_{n \rightarrow \infty} \beta_n = 0$ ,  $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ ,

then  $\{x_n\}$  and  $\{u_n\}$  converge strongly to  $q \in F(S) \cap EP(F)$ , where  $q = P_{F(S) \cap EP(F)} f(q)$ .

*Proof.* Taking  $A = I$  and  $\gamma = 1$  in Theorem 3.1, we can obtain the desired result.

**Corollary 3.2.** Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $S$  be a nonexpansive mapping of  $C$  into  $H$  such that  $F(S) \cap EP(F) \neq \emptyset$ . Let  $A$  be a strongly positive linear bounded operator on  $H$  with coefficient  $\bar{\gamma} > 0$  and let  $f \in \Sigma_H$  with a coefficient  $k$  ( $0 < k < 1$ ). Assume that  $0 < \gamma < \frac{\bar{\gamma}}{k}$ . Let  $\{x_n\}$  be sequence generated by  $x_1 \in H$  and

$$\begin{cases} y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A) S P_C x_n, \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n S P_C y_n, & n \geq 1, \end{cases}$$

where  $\{\alpha_n\}$  and  $\{\beta_n\} \subset [0, 1]$ . If  $\{\alpha_n\}$  and  $\{\beta_n\}$  satisfy the conditions:

(C1)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ;

(C3)  $\lim_{n \rightarrow \infty} \beta_n = 0$ ,  $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ ,

then  $\{x_n\}$  converge strongly to  $q \in F(S)$ , where  $q = P_{F(S)}(\gamma f + (I - A))(q)$ .

*Proof.* Put  $F(x, y) = 0$  for all  $x, y \in C$  and  $\{r_n\} = 1$  for all  $n$  in Theorem 3.1. Then we have  $u_n = P_C x_n$ . So the sequence  $\{x_n\}$  converges strongly to  $q \in F(S)$ , where  $q = P_{F(S)}(\gamma f + (I - A))(q)$ .

**Corollary 3.3.** ([7, Corollary 3.1]) *Let  $C$  be a nonempty closed convex subset of  $H$  and  $S$  a nonexpansive mapping of  $C$  into  $H$  such that  $F(S) \neq \emptyset$ . Let  $f \in \Sigma_H$  and let  $\{x_n\}$  be sequence generated by  $x_1 \in H$  and*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1 - \alpha_n) SP_C x_n \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n SP_C y_n, \quad n \geq 1, \end{cases}$$

where  $\{\alpha_n\}$  and  $\{\beta_n\} \subset [0, 1]$ . If  $\{\alpha_n\}$  and  $\{\beta_n\}$  satisfy the conditions:

$$(C1) \lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty;$$

$$(C3) \lim_{n \rightarrow \infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$$

then  $\{x_n\}$  converges strongly to  $q \in F(S)$ , where  $q = P_{F(S)} f(q)$ .

**Corollary 3.4.** ([10, Corollary 3.4]) *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $F$  be a bifunction from  $C \times C$  to  $\mathbb{R}$  satisfying (A1)–(A4) and  $EP(F) \neq \emptyset$ . Let  $A$  be a strongly positive linear bounded operator on  $H$  with coefficient  $\bar{\gamma} > 0$  and let  $f \in \Sigma_H$  with a coefficient  $k$  ( $0 < k < 1$ ). Assume that  $0 < \gamma < \frac{\bar{\gamma}}{k}$ . Let  $\{x_n\}$  and  $\{u_n\}$  be sequences generated by  $x_1 \in H$  and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \text{for all } y \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) u_n, \quad n \geq 1, \end{cases}$$

where  $\{\alpha_n\} \subset [0, 1]$  and  $\{r_n\} \subset (0, \infty)$ . If  $\{\alpha_n\}$  and  $\{r_n\}$  satisfy the conditions:

$$(C1) \lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty;$$

$$(C2) \liminf_{n \rightarrow \infty} r_n > 0, \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty;$$

then  $\{x_n\}$  and  $\{u_n\}$  converge strongly to  $q \in EP(F)$ , where  $q = P_{EP(F)}(\gamma f + (I - A))(q)$ , which is the unique solution of the variational inequality

$$\langle \gamma f(q) - Aq, q - p \rangle \geq 0, \quad p \in EP(F).$$

*Proof.* Put  $Sx = x$  for all  $x \in C$  in Theorem 3.1. Then, from Theorem 3.1 the sequence  $\{x_n\}$  and  $\{u_n\}$  generated in Corollary 3.4 converge strongly to  $q \in EP(F)$ , where  $q = P_{EP(F)}(\gamma f + (I - A))(q)$ .

**Corollary 3.5.** ([7, Corollary 3.2]) *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $F$  be a bifunction from  $C \times C$  to  $\mathbb{R}$  satisfying (A1)–(A4) such that  $EP(F) \neq \emptyset$  and  $f \in \Sigma_H$ . Let  $\{x_n\}$  and  $\{u_n\}$  be sequences generated by  $x_1 \in H$  and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \text{for all } y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) u_n \end{cases}$$

for every  $n \geq 1$ , where  $\{\alpha_n\} \subset [0, 1]$  and  $\{r_n\} \subset (0, \infty)$ . If  $\{\alpha_n\}$  and  $\{r_n\}$  satisfy the conditions:

$$(C1) \lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty;$$

$$(C2) \liminf_{n \rightarrow \infty} r_n > 0, \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty,$$

then  $\{x_n\}$  and  $\{u_n\}$  converge strongly to  $q \in EP(F)$ , where  $q = P_{EP(F)} f(q)$ .

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**Corollary 3.6.** ([8, Theorem 3.4]) *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $S$  be a nonexpansive mapping of  $C$  into itself with  $F(S) \neq \emptyset$ . Let  $A$  be a strongly positive linear bounded operator on  $H$  with coefficient  $\bar{\gamma} > 0$  and let  $f \in \Sigma_H$  with a coefficient  $k$  ( $0 < k < 1$ ). Assume that  $0 < \gamma < \frac{\bar{\gamma}}{k}$ . Let  $\{x_n\}$  be sequence generated by  $x_1 \in H$  and*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) S x_n, \quad n \geq 1,$$

where  $\{\alpha_n\} \subset [0, 1]$ . If  $\{\alpha_n\}$  satisfies the condition:

$$(C1) \lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty;$$

then  $\{x_n\}$  converge strongly to  $q \in F(S)$ , where  $q = P_{F(S)}(\gamma f + (I - A))(q)$ , which is the unique solution of the variational inequality

$$\langle \gamma f(q) - Aq, q - p \rangle \geq 0, \quad p \in EP(F).$$

*Proof.* Putting  $F(x, y) = 0$  for all  $x, y \in C$  and  $r_n = 1$  for all  $n$  in Corollary 3.2, we get  $P_C x_n = x_n$ . So, taking  $\beta_n = 0$  for all  $n$  in Corollary 3.2, we obtain the desired result.

**Remark 3.1.** (1) Theorem 3.1 and Corollary 3.1 improve the corresponding result of Plubtieng and Pungaeng [10], Shang et al. [11] and Takahashi and Takahashi [14]. In particular, if  $\beta_n = 0$  for all  $n \geq 1$ , then Theorem 3.1 and Corollary 3.1 reduce to Theorem 3.3 in [10] (Theorem 3.1 in [11]) and Theorem 3.1 in [14], respectively.

(2) In the case  $\beta_n = 0$  for all  $n$ , Corollary 3.2 is just Theorem 4.1 of Shang et al. [11].

(3) In the case when  $f(x) = u = x_1$  for all  $x \in H$  in Corollary 3.5, We also get Combettes and Hirstoaga's theorem [2].

(4) We have Xu's theorem [18] in the case when  $f(x) = u = x_1$  for all  $x \in H$  in Corollary 3.6.

(5) We obtain a composite iterative scheme for nonexpansive mapping if  $S : C \rightarrow C$  and  $f \in \Sigma_C$  in Corollary 3.3 as follows:

$$\begin{cases} x_1 \in C \\ y_n = \alpha_n f(x_n) + (1 - \alpha_n) S x_n \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n S y_n, \end{cases} \quad (3.12)$$

(see also [6]). If  $\beta_n = 0$  for all  $n \geq 1$  and  $f(x) = u = x_1$  for all  $x \in C$  in (3.12), we obtain Wittmann's result [16].

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# On the roots of the twisted $(h, q)$ -Euler polynomials

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**Abstract** Recently several authors studied the  $q$ -extension of Euler numbers and polynomials (see [1-11]). In this paper we observe the behavior of complex roots of the twisted  $(h, q)$ -Euler polynomials  $E_{n,q,w}^{(h)}(x)$ , using numerical investigation. By means of numerical experiments, we demonstrate a remarkably regular structure of the complex roots of twisted  $(h, q)$ -Euler polynomials  $E_{n,q,w}^{(h)}(x)$ . Finally, we give tables for the solutions of twisted  $(h, q)$ -Euler polynomials.

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**Key words**- Euler numbers and polynomials,  $q$ -Euler numbers and polynomials,  $(h, q)$ -Euler numbers and polynomials, twisted  $(h, q)$ -Euler numbers and polynomials

## 1. Introduction

Throughout this paper we use the following notations. By  $\mathbb{Z}_p$  we denote the ring of  $p$ -adic rational integers,  $\mathbb{Q}$  denotes the field of rational numbers,  $\mathbb{Q}_p$  denotes the field of  $p$ -adic rational numbers,  $\mathbb{C}$  denotes the complex number field, and  $\mathbb{C}_p$  denotes the completion of algebraic closure of  $\mathbb{Q}_p$ . Let  $\nu_p$  be the normalized exponential valuation of  $\mathbb{C}_p$  with  $|p|_p = p^{-\nu_p(p)} = p^{-1}$ . When one talks of  $q$ -extension,  $q$  is considered in many ways such as an indeterminate, a complex number  $q \in \mathbb{C}$ , or  $p$ -adic number  $q \in \mathbb{C}_p$ . If  $q \in \mathbb{C}$  one normally assume that  $|q| < 1$ . If  $q \in \mathbb{C}_p$ , we normally assume that  $|q - 1|_p < p^{-\frac{1}{p-1}}$  so that  $q^x = \exp(x \log q)$  for  $|x|_p \leq 1$ .

$$[x]_q = [x : q] = \frac{1 - q^x}{1 - q}, \text{ cf. [1-11] .}$$

Hence,  $\lim_{q \rightarrow 1} [x] = x$  for any  $x$  with  $|x|_p \leq 1$  in the present  $p$ -adic case. Let  $d$  be a fixed integer and let  $p$  be a fixed prime number. For any positive integer  $N$ , we set

$$X = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \quad X^* = \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where  $a \in \mathbb{Z}$  lies in  $0 \leq a < dp^N$ . For any positive integer  $N$ ,

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}$$

is known to be a distribution on  $X$ , cf.[1,2,3, 4, 5, 6,7,8,9,10].

For

$$g \in UD(\mathbb{Z}_p) = \{g|g : \mathbb{Z}_p \rightarrow \mathbb{C}_p \text{ is uniformly differentiable function} \},$$

the  $p$ -adic  $q$ -integral was defined by [1-11]

$$I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]} \sum_{0 \leq x < p^N} g(x) q^x.$$

Let

$$T_p = \cup_{m \geq 1} C_{p^m} = \lim_{m \rightarrow \infty} C_{p^m},$$

where  $C_{p^m} = \{w|w^{p^m} = 1\}$  is the cyclic group of order  $p^m$ . For  $w \in T_p$ , we denote by  $\phi_w : \mathbb{Z}_p \rightarrow \mathbb{C}_p$  the locally constant function  $x \mapsto w^x$ . In [8], we introduced analogue of  $q$ -Euler numbers and polynomials, which is called twisted Euler numbers and polynomials. By using  $p$ -adic  $q$ -integral, we defined the twisted  $q$ -Euler numbers as follows:

$$E_{n,q,w} = \int_{\mathbb{Z}_p} \phi_w(x) [x]_q^n d\mu_{-q}(x), \text{ for } n \in \mathbb{N}.$$

By using the above equation, we have the generating function of twisted  $q$ -Euler numbers  $E_{n,q,w}$  as follows:

$$F_{q,w}(t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^n e^{[n]_q t} = \sum_{n=0}^{\infty} E_{n,q,w} \frac{t^n}{n!}.$$

By using  $p$ -adic  $q$ -integral, we defined the twisted  $q$ -Euler polynomials as follows:

$$E_{n,q,w}(z) = \int_{\mathbb{Z}_p} \phi_w(x) [x+z]_q^n d\mu_{-q}(x), \text{ for } n \in \mathbb{N}.$$

Similarly, we have the generating function of twisted  $q$ -Euler polynomials  $E_{n,q,w}(z)$  as follows:

$$F_{q,w}(t, z) = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m w^m e^{[m+z]_q t}.$$

see [8], for details.

For  $h \in \mathbb{Z}$ ,  $q \in \mathbb{C}_p$  with  $|1-p|_p \leq p^{-\frac{1}{p-1}}$ , the twisted  $(h, q)$ -Euler numbers  $E_{n,q,w}^{(h)}$  are defined by

$$E_{n,q,w}^{(h)} = \int_{\mathbb{Z}_p} \phi_w(x) q^{x(h-1)} [x]_q^n d\mu_{-q}(x). \quad (1.1)$$

By using  $p$ -adic  $q$ -integral, we obtain,

$$\int_{\mathbb{Z}_p} \phi_w(x) q^{x(h-1)} [x]_q^n d\mu_{-q}(x) = [2]_q \left( \frac{1}{1-q} \right)^n \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1+wq^{(h+l)}}. \quad (1.2)$$

Here is the list of the first the twisted  $(h, q)$ -Euler numbers  $E_{n,q,w}^{(h)}$ .

$$\begin{aligned} E_{0,q,w}^{(h)} &= \frac{(1+q)}{(1+wq^h)}, \\ E_{1,q,w}^{(h)} &= -\frac{q^h(1+q)w}{(1+wq^h)(1+wq^{1+h})}, \\ E_{2,q,w}^{(h)} &= \frac{q^h(1+q)w(-1+wq^{1+h})}{(1+wq^h)(1+wq^{1+h})(1+wq^{2+h})}, \\ E_{3,q,w}^{(h)} &= -\frac{q^h(1+q)w(1-2wq^{1+h}-2wq^{2+h}+w^2q^{3+2h})}{(1+wq^h)(1+wq^{1+h})(1+wq^{2+h})(1+wq^{3+h})}, \dots \end{aligned}$$

By (1.1), we have

$$E_{n,q,w}^{(h)} = [2]_q \left( \frac{1}{1-q} \right)^n \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1+wq^{(h+l)}}.$$

We set

$$F_{q,w}^{(h)}(t) = \sum_{n=0}^{\infty} E_{n,q,w}^{(h)} \frac{t^n}{n!}.$$

By using above equation and (1.2), we have

$$\begin{aligned} F_{q,w}^{(h)}(t) &= \sum_{n=0}^{\infty} E_{n,q,w}^{(h)} \frac{t^n}{n!} \\ &= [2]_q \sum_{n=0}^{\infty} \left( \left( \frac{1}{1-q} \right)^n \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1+wq^{h+1}} \right) \frac{t^n}{n!} \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m w^m q^{hm} e^{[m]_q t}. \end{aligned} \quad (1.3)$$

Thus twisted  $(h, q)$ -Euler numbers  $E_{n,q,w}^{(h)}$  are defined by means of the generating function

$$F_{q,w}^{(h)}(t) = [2]_q \sum_{n=0}^{\infty} (-1)^n w^n q^{hn} e^{[n]_q t}. \quad (1.4)$$

Note that, if  $h = 1$ , then  $F_{q,w}^{(h)}(t) = F_{q,w}(t)$ . By using (1.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,q,w}^{(h)} \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \phi_w(x) q^{x(h-1)} [x]_q^n d\mu_{-q}(x) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \phi_w(x) q^{x(h-1)} e^{[x]_q t} d\mu_{-q}(x). \end{aligned} \quad (1.5)$$

By (1.3), (1.5), we have

$$\int_{\mathbb{Z}_p} \phi_w(x) q^{x(h-1)} e^{[x]_q t} d\mu_{-q}(x) = [2]_q \sum_{m=0}^{\infty} (-1)^m w^m q^{hm} e^{[m]_q t}.$$

Next, we introduce the twisted  $(h, q)$ -Euler polynomials  $E_{n,q,w}^{(h)}(z)$ . The twisted  $(h, q)$ -Euler polynomials  $E_{n,q,w}^{(h)}(z)$  are defined by

$$E_{n,q,w}^{(h)}(z) = \int_{\mathbb{Z}_p} \phi_w(x) q^{x(h-1)} [z+x]_q^n d\mu_{-q}(x). \quad (1.6)$$

By using  $p$ -adic  $q$ -integral, we obtain

$$E_{n,q,w}^{(h)}(z) = [2]_q \left( \frac{1}{1-q} \right)^n \sum_{l=0}^n \binom{n}{l} (-1)^l q^{zl} \frac{1}{1+wq^{h+l}}. \quad (1.7)$$

Here is the list of the first the  $(h, q)$ -extension of the  $q$ -Euler polynomials  $E_{n,q}^{(h)}(z)$ .

$$\begin{aligned} E_{0,q,w}^{(h)}(z) &= \frac{(1+q)}{(1+wq^h)}, \\ E_{1,q,w}^{(h)}(z) &= -\frac{(1+q)(1-q^z + wq^{1+h} - wq^{h+z})}{(-1+q)(1+wq^h)(1+wq^{1+h})}, \dots \end{aligned}$$

We set

$$F_{q,w}^{(h)}(t, z) = \sum_{n=0}^{\infty} E_{n,q,w}^{(h)}(z) \frac{t^n}{n!}. \quad (1.8)$$

By using (1.7) and (1.8), we obtain

$$F_{q,w}^{(h)}(t, z) = \sum_{n=0}^{\infty} E_{n,q,w}^{(h)}(z) \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^{hm} w^m e^{[m+z]_q t}.$$

Similarly, the generating function  $F_{q,w}^{(h)}(t, z)$  of the  $q$ -Euler polynomials  $E_{n,q,w}^{(h)}(z)$  is defined analogously as follows:

$$F_{q,w}^{(h)}(t, z) = \sum_{n=0}^{\infty} E_{n,q,w}^{(h)}(z) \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^{hn} w^n e^{[n+z]_q t}. \quad (1.9)$$

By using (1.9), we easily see that

$$\int_{\mathbb{Z}_p} \phi_w(x) q^{x(h-1)} e^{[z+x]_q t} d\mu_{-q}(x) = [2]_q \sum_{n=0}^{\infty} (-1)^n w^n q^{hn} e^{[n+z]_q t}.$$

Since  $[x+z]_q = [z]_q + q^z[x]_q$ , we have

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,q,w}^{(h)}(z) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} \phi_w(x) q^{x(h-1)} e^{[x+z]_q t} d\mu_{-q}(x) \\ &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \binom{n}{l} q^{lz} [z]_q^{n-l} \int_{\mathbb{Z}_p} \phi_w(x) q^{x(h-1)} [x]_q^l d\mu_{-q}(x) \right) \frac{t^n}{n!}. \end{aligned}$$

By using comparing coefficients  $\frac{t^n}{n!}$ , we easily see that

$$E_{n,q,w}^{(h)}(z) = \sum_{l=0}^n \binom{n}{l} q^{lz} [z]_q^{n-l} E_{n,q,w}^{(h)}.$$

Observe that, if  $q \rightarrow 1, w \rightarrow 1$ , then  $E_{n,q,w}^{(h)}(z) \rightarrow E_n(z)$ , where  $E_n(z)$  are the Euler polynomials. Note that, if  $q \rightarrow 1$ , then  $E_{n,q,w}^{(h)}(z) \rightarrow E_{n,w}(z)$ , where  $E_{n,w}(z)$  are the twisted Euler polynomials.

## 2. Distribution and Structure of the Zeros

In this section, we investigate the zeros of the twisted  $(h, q)$ -Euler polynomials  $E_{n,q,w}^{(h)}(z)$  by using computer. We plot the zeros of  $E_{n,q,w}^{(h)}(z), z \in \mathbb{C}$  for  $n = 10, 20, 30, 40, q = 1/2, w = e^{\pi i}$  and  $h = 2$ . (Figures 1, 2, 3, and 4). Next, we plot the zeros of  $E_{n,q,w}^{(h)}(z), z \in \mathbb{C}$  for

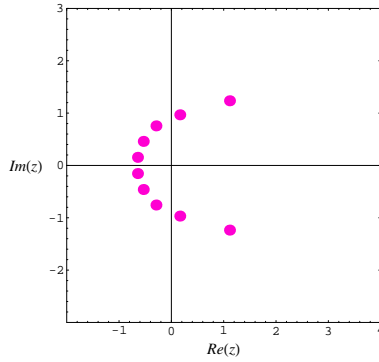


Figure 1: Zeros of  $E_{10,q,w}^{(2)}$

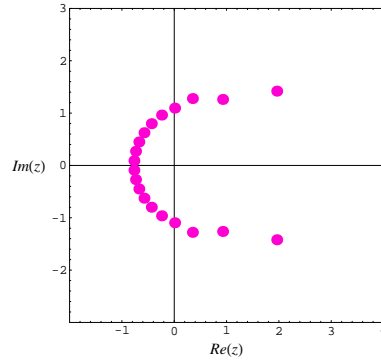
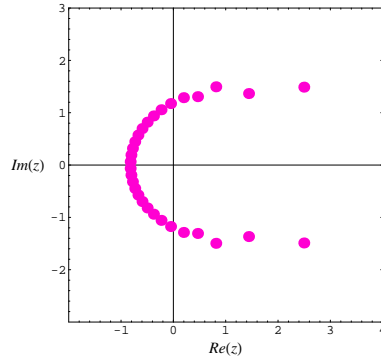
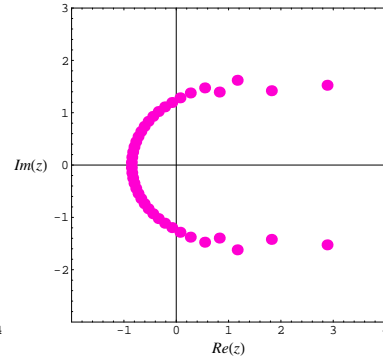
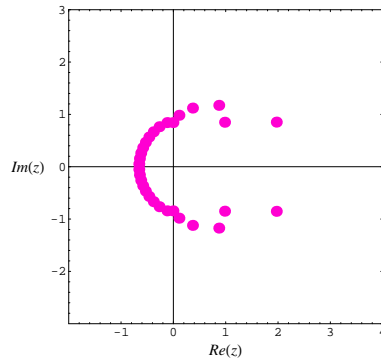
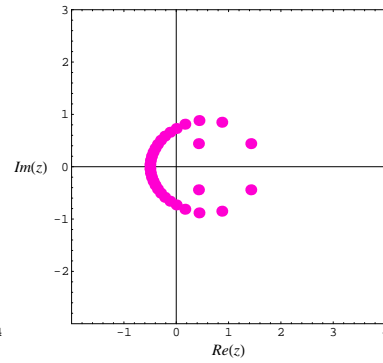
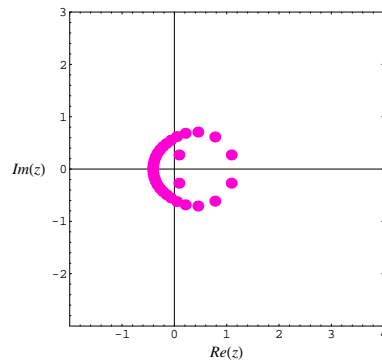
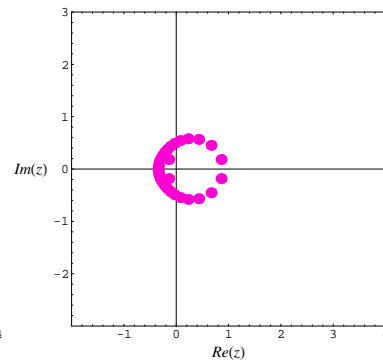


Figure 2: Zeros of  $E_{20,q,w}^{(2)}$

$n = 30, h = 5, 10, 15, 20, w = e^{\pi i}, q = 1/2$ . (Figures 5, 6, 7, and 8). In Figures 1, 2, 3, 4, 5, 6,


 Figure 3: Zeros of  $E_{30,q,w}^{(2)}$ 

 Figure 4: Zeros of  $E_{30,q,w}^{(2)}$ 

 Figure 5: Zeros of  $E_{30,q,w}^{(5)}$ 

 Figure 6: Zeros of  $E_{30,q,w}^{(10)}$ 

7, and 8,  $E_{n,q,w}^{(h)}(z)$ ,  $z \in \mathbb{C}$ , has  $Im(z) = 0$  reflection symmetry. This translates to the following open problem: Prove or disprove that  $E_{n,q,w}^{(h)}(z)$ ,  $z \in \mathbb{C}$ , has  $Im(z) = 0$  reflection symmetry. Our numerical results for numbers of real and complex zeros of  $E_{n,q,w}^{(h)}(z)$  are displayed in Table 1. In general, how many roots does  $E_{n,q,w}^{(h)}(z)$  have? Prove or disprove:  $E_{n,q,w}^{(h)}(z)$  has  $n$  distinct solutions. Find the numbers of complex zeros  $C_{E_{n,q,w}^{(h)}(z)}$  of  $E_{n,q,w}^{(h)}(z)$ ,  $Im(z) \neq 0$ . Prove or give a counterexample: *Conjecture:* Since  $n$  is the degree of the polynomial  $E_{n,q,w}^{(h)}(z)$ , the number of real zeros  $R_{E_{n,q,w}^{(h)}(z)}$  lying on the real plane  $Im(z) = 0$  is then  $R_{E_{n,q,w}^{(h)}(z)} = n - C_{E_{n,q,w}^{(h)}(z)}$ , where  $C_{E_{n,q,w}^{(h)}(z)}$  denotes complex zeros. See Table 1 for tabulated values of  $R_{E_{n,q,w}^{(h)}(z)}$  and  $C_{E_{n,q,w}^{(h)}(z)}$ . We calculated an approximate solution satisfying  $E_{n,q,w}^{(h)}(z)$ ,  $q = 1/2$ ,  $w = e^{\pi i}$ ,  $z \in \mathbb{R}$ . The results are given in Table 2 and Table 3.

Figure 7: Zeros of  $E_{30,q,w}^{(15)}$ Figure 8: Zeros of  $E_{30,q,w}^{(20)}$ **Table 1.** Numbers of real and complex zeros of  $E_{n,q,w}^{(2)}(z)$ 

degree $n$	$w = e^{\pi i}$		$w = e^{\pi i/3}$	
	real zeros	complex zeros	real zeros	complex zeros
1	1	0	0	1
2	0	2	0	2
3	1	2	0	3
4	0	4	0	4
5	1	4	0	5
6	0	6	0	6
7	1	6	0	6
8	0	8	0	8
9	1	8	0	9
10	0	10	0	10
11	1	10	0	11
12	0	12	0	12
13	1	12	0	13

**Table 2.** Approximate solutions of  $E_{n,q,w}^{(2)}(z) = 0, w = e^{\pi i}, z \in \mathbb{R}$ 

degree $n$	$z$
1	-0.222392
3	-0.425458
5	-0.525835
7	-0.593454
9	-0.641321
11	-0.677809
13	-0.706818
15	-0.730465
17	-0.750137
19	-0.7668

**Table 3.** Approximate solutions of  $E_{n,q,w}^{(5)}(z) = 0, w = e^{\pi i}, z \in \mathbb{R}$ 

degree $n$	$z$
1	-0.0230836
3	-0.191203
5	-0.281468
7	-0.344607
9	-0.400306
11	-0.446693
13	-0.48419
15	-0.515564
17	-0.542866
19	-0.56714

Finally, we shall consider the more general problems. Find the equation of envelope curves bounding the real zeros lying on the plane, and the equation of a trajectory curve running through the complex zeros on any one of the arcs. We can draw a plot of zeros of the  $E_{n,q,w}^{(h)}(z)$ , respectively (Figures 1, 2, 3, 4, 5, 6, 7, and 8). These figures give mathematicians an unbounded capacity to create visual mathematical investigations of the behavior of roots of the  $E_{n,q,w}^{(h)}(z)$ . Moreover, it is possible to create a new mathematical ideas and analyze them in ways that generally are not possible by hand. The author has no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the  $E_{n,q,w}^{(h)}(z)$  to appear in mathematics and physics. For related topics the interested reader is referred to [8], [9], [10], [11].

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# A numerical computation of the roots of $q$ -Euler polynomials

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## Abstract

Recently several authors studied the  $q$ -extension of Euler numbers and polynomials(see [1-8]). In this paper we construct the  $q$ -Euler numbers  $E_{n,q}$  and Polynomials  $E_{n,q}(x)$ . We also observe the behavior of complex roots of the  $q$ -Euler polynomials  $E_{n,q}(x)$ , using numerical investigation. By means of numerical experiments, we demonstrate a remarkably regular structure of the complex roots of the  $q$ -Euler polynomials  $E_{n,q}(x)$ . Finally, we give a table for the solutions of the  $q$ -Euler polynomials  $E_{n,q}(x)$ .

## 1. Introduction

In the 21st century, the computing environment would make more and more rapid progress. Using computer, a realistic study for new analogs of  $q$ -Euler numbers and polynomials is very interesting. It is the aim of this paper to observe an interesting phenomenon of ‘scattering’ of the zeros of the  $q$ -Euler polynomials  $E_{n,q}(x)$ . The outline of this paper is as follows. In Section 2, we study  $q$ -Euler polynomials  $E_{n,q}(x)$ . In Section 3, we describe the beautiful zeros of the  $q$ -Euler polynomials  $E_{n,q}(x)$  using a numerical investigation. Also we display distribution and structure of the zeros of the  $q$ -Euler polynomials  $E_{n,q}(x)$  by using computer. By using the results of our paper the readers can observe the regular behavior of the roots of the  $q$ -Euler polynomials  $E_{n,q}(x)$ . Finally, we carried out computer experiments for doing demonstrate a remarkably regular structure of the complex roots of the  $q$ -Euler polynomials  $E_{n,q}(x)$ . Throughout this paper we use the following notations. By  $\mathbb{Z}_p$  we denote the ring of  $p$ -adic rational integers,  $\mathbb{Q}$  denotes the field of rational numbers.  $\mathbb{C}$  denotes the complex number field. Let  $\nu_p$  be the normalized exponential valuation of  $\mathbb{C}_p$  with  $|p|_p = p^{-\nu_p(p)} = p^{-1}$ . When one talks of  $q$ -extension,  $q$  is considered in many ways such as an indeterminate, a complex number  $q \in \mathbb{C}$ , or  $p$ -adic number  $q \in \mathbb{C}_p$ . If  $q \in \mathbb{C}$  one normally assume that  $|q| < 1$ . If  $q \in \mathbb{C}_p$ , we normally assume that  $|q - 1|_p < p^{-\frac{1}{p-1}}$  so that  $q^x = \exp(x \log q)$  for  $|x|_p \leq 1$ .

$$[x]_q = [x : q] = \frac{1 - q^x}{1 - q}, \text{ cf. [1,4,5] .}$$

Hence,  $\lim_{q \rightarrow 1} [x] = x$  for any  $x$  with  $|x|_p \leq 1$  in the present  $p$ -adic case. Let  $d$  be a fixed integer and let  $p$  be a fixed prime number. For any positive integer  $N$ , we set

$$\begin{aligned} X &= \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \\ X^* &= \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned}$$

where  $a \in \mathbb{Z}$  lies in  $0 \leq a < dp^N$ . For any positive integer  $N$ ,

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}$$

is known to be a distribution on  $X$ , cf.[3,4,5]. For

$$g \in UD(\mathbb{Z}_p) = \{g|g : \mathbb{Z}_p \rightarrow \mathbb{C}_p \text{ is uniformly differentiable function } \},$$



the  $p$ -adic  $q$ -integral was defined by [3,5,6]

$$I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]} \sum_{0 \leq x < p^N} g(x) q^x.$$

Note that

$$I_1(g) = \lim_{q \rightarrow 1} I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_1(x) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{0 \leq x < p^N} g(x)$$

(see [3,5,6]). Now, we consider the case  $q \in (-1, 0)$  corresponding to  $q$ -deformed fermionic certain and annihilation operators and the literature given therein [3,4,5,6]. The expression for the  $I_q(g)$  remains same, so it is tempting to consider the limit  $q \rightarrow -1$ . That is,

$$I_{-1}(g) = \lim_{q \rightarrow -1} I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{0 \leq x < p^N} g(x) (-1)^x. \quad (1.1)$$

If we take  $g_1(x) = g(x+1)$  in (1.1), then we easily see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0). \quad (1.2)$$

First, we introduce the Euler numbers and Euler polynomials. The Euler numbers  $E_n$  are defined by the generating function:

$$F(t) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \text{ cf. [1,4,5]} \quad (1.3)$$

where we use the technique method notation by replacing  $E^n$  by  $E_n (n \geq 0)$  symbolically. For  $x \in \mathbb{R}$  (= the field of real numbers), we consider the Euler polynomials  $E_n(x)$  as follows:

$$F(x, t) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \quad (1.4)$$

Note that  $E_n(x) = \sum_{k=0}^n \binom{n}{k} E_k x^{n-k}$ . In the special case  $x = 0$ , we define  $E_n(0) = E_n$ .

## 2. The $q$ -Euler numbers and polynomials

In [5], we defined the  $q$ -Euler numbers and polynomials. In this section, we introduce another  $q$ -Euler numbers  $E_{n,q}$  and polynomials  $E_{n,q}(x)$  and investigate their properties. In (1.2), if we take  $g(x) = q^x e^{xt}$ , then we easily see that

$$I_{-1}(q^x e^{xt}) = \int_{\mathbb{Z}_p} q^x e^{xt} d\mu_{-1}(x) = \frac{2}{qe^t + 1}.$$

Let us define the twisted Euler numbers  $E_{n,q}$  and polynomials  $E_{n,q}(x)$  as follows:

$$I_{-1}(q^y e^{yt}) = \int_{\mathbb{Z}_p} q^y e^{yt} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}, \quad (2.1)$$

$$I_{-1}(q^y e^{(y+x)t}) = \int_{\mathbb{Z}_p} q^y e^{(y+x)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \quad (2.2)$$

By (2.1) and (2.2), we obtain the following Witt's formula.

**Theorem 1.** For  $n \in \mathbb{N}$ , we have

$$\int_{\mathbb{Z}_p} q^x x^n d\mu_{-1}(x) = E_{n,q},$$

$$\int_{\mathbb{Z}_p} q^y (x+y)^n d\mu_{-1}(y) = E_{n,q}(x).$$

Let  $q$  be a complex number with  $|q| < 1$ . By the meaning of (1.3) and (1.4), let us define the  $q$ -Euler numbers  $E_{n,q}$  and polynomials  $E_{n,q}(x)$  as follows:

$$F_q(t) = \frac{2}{qe^t + 1} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}, \quad (2.3)$$

$$F_q(x, t) = \frac{2}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \quad (2.4)$$

We have the following remark.

**Remark.** Note that

- (1)  $E_{n,q}(0) = E_{n,q}$ ,
- (2) If  $q \rightarrow 1$ , then  $E_{n,q}(x) = E_n(x)$ ,  $E_{n,q} = E_n$ ,
- (3) If  $q \rightarrow 1$ , then  $F_q(x, t) = F(x, t)$ ,  $F_q(t) = F(t)$ .

Here is the list of the first  $q$ -Euler numbers  $E_{n,q}$ .

$$\begin{aligned} E_{0,q} &= \frac{2}{1+q}, & E_{2,q} &= -\frac{2q}{(1+q)^2}, \\ E_{3,q} &= \frac{2(-1+q)q}{(1+q)^3}, \\ E_{4,q} &= -\frac{2q(1-4q+q^2)}{(1+q)^4}, \\ E_{5,q} &= \frac{2(-1+q)q(1-10q+q^2)}{(1+q)^5}, \\ E_{6,q} &= -\frac{2q(1-26q+66q^2-26q^3+q^4)}{(1+q)^6} \dots, \end{aligned}$$

By the above definition, we obtain

$$\begin{aligned} \sum_{l=0}^{\infty} E_{l,q}(x) \frac{t^l}{l!} &= \frac{2}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!} \sum_{m=0}^{\infty} x^m \frac{t^m}{m!} \\ &= \sum_{l=0}^{\infty} \left( \sum_{n=0}^l E_{n,q} \frac{t^n}{n!} x^{l-n} \frac{t^{l-n}}{(l-n)!} \right) \\ &= \sum_{l=0}^{\infty} \left( \sum_{n=0}^l \binom{l}{n} E_{n,q} x^{l-n} \right) \frac{t^l}{l!}. \end{aligned}$$

By using comparing coefficients  $\frac{t^l}{l!}$ , we have the following theorem.

**Theorem 2.** For any positive integer  $n$ , we have

$$E_{n,q}(x) = \sum_{k=0}^n \binom{n}{k} E_{k,q} x^{n-k}.$$

Here is the list of the first  $q$ -Euler polynomials  $E_{n,q}(x)$ .

$$\begin{aligned} E_{0,q}(x) &= \frac{2}{1+q}, \\ E_{1,q}(x) &= \frac{2(-q+x+qx)}{(1+q)^2}, \\ E_{2,q}(x) &= \frac{2(-q+q^2-2qx-2q^2x+x^2+2qx^2+q^2x^2)}{(1+q)^3}, \\ E_{3,q}(x) &= \frac{2(-q+4q^2-q^3-3qx+3q^3x-3qx^2-6q^2x^2-3q^3x^2)}{(1+q)^4} \\ &\quad + \frac{2(x^3+3qx^3+3q^2x^3+q^3x^3)}{(1+q)^4}, \dots, \end{aligned}$$

Because

$$\frac{\partial}{\partial x} F_q(x, t) = t G_q(x, t) = \sum_{n=0}^{\infty} \frac{d}{dx} E_{n,q}(x) \frac{t^n}{n!},$$

it follows the important relation

$$\frac{d}{dx} E_{n,q}(x) = n E_{n-1,q}(x).$$

We also obtain the following integral formula

$$\int_a^b E_{n-1,q}(x) dx = \frac{1}{n} (E_{n,q}(b) - E_{n,q}(a)).$$

Over five decades ago, Carlitz [1] defined  $q$ -extensions of the classical Bernoulli numbers  $B_n$  and Bernoulli polynomials  $B_n(x)$  and proved properties analogues to those satisfied by  $B_n$  and  $B_n(x)$ . Carlitz's  $q$ -Bernoulli numbers  $\beta_n = \beta_{n,q}$  can be determined inductively by [1]

$$\beta_0 = 1, \quad q(q\beta + 1)^k - \beta_k = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}$$

with the usual convention about replacing  $\beta^k$  by  $\beta_k$ . For the  $q$ -Euler numbers, we obtain the following theorem.

**Theorem 3.** The  $q$ -Euler numbers  $E_{n,q}$  are defined respectively by

$$q(E_q + 1)^n + E_{n,q} = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases}$$

with the usual convention about replacing  $(E_q)^n$  by  $E_{n,q}$  in the binomial expansion.

**Proof.** From (2.3), we obtain

$$\frac{2}{qe^t + 1} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!} = \sum_{n=0}^{\infty} (E_q)^n \frac{t^n}{n!} = e^{E_q t}$$

which yields

$$2 = (qe^t + 1)e^{E_q t} = qe^{(E_q+1)t} + e^{E_q t}.$$

Using Taylor expansion of exponential function, we have

$$\begin{aligned} 2 &= \sum_{n=0}^{\infty} \{q(E_q + 1)^n + (E_q)^n\} \frac{t^n}{n!} \\ &= q(E_q + 1)^0 + (E_q)^0 + \sum_{n=1}^{\infty} \{q(E_q + 1)^n + (E_q)^n\} \frac{t^n}{n!}. \end{aligned}$$

The result follows by comparing the coefficients.

Since

$$\begin{aligned}
 \sum_{l=0}^{\infty} E_{l,q}(x+y) \frac{t^l}{l!} &= \frac{2}{qe^t + 1} e^{(x+y)t} \\
 &= \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} \sum_{m=0}^{\infty} y^m \frac{t^m}{m!} \\
 &= \sum_{l=0}^{\infty} \left( \sum_{n=0}^l E_{n,q}(x) \frac{t^n}{n!} y^{l-n} \frac{t^{l-n}}{(l-n)!} \right) \\
 &= \sum_{l=0}^{\infty} \left( \sum_{n=0}^l \binom{l}{n} E_{n,q}(x) y^{l-n} \right) \frac{t^l}{l!},
 \end{aligned}$$

we have the following addition theorem.

**Theorem 4.** The twisted Euler polynomials  $E_{n,w}(x)$  satisfies the following relation:

$$E_{l,q}(x+y) = \sum_{n=0}^l \binom{l}{n} E_{n,q}(x) y^{l-n}.$$

It is easy to see that

$$\begin{aligned}
 \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!} &= \frac{2}{qe^t + 1} e^{xt} = \frac{2t}{q^n e^{mt} + 1} e^{xt} \sum_{a=0}^{m-1} (-1)^a q^a e^{at} \\
 &= \frac{1}{m} \sum_{a=0}^{m-1} (-1)^a q^a \frac{2m}{q^m e^{mt} + 1} e^{\frac{a+x}{m} (mt)} \\
 &= \frac{1}{m} \sum_{a=0}^{m-1} (-1)^a q^a \sum_{n=0}^{\infty} E_{n,q^m} \left( \frac{a+x}{m} \right) \frac{(mt)^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left( m^{n-1} \sum_{a=0}^{m-1} (-1)^a q^a E_{n,q^m} \left( \frac{a+x}{m} \right) \right) \frac{t^n}{n!}.
 \end{aligned}$$

Hence we have the below theorem.

**Theorem 5.** For any positive integer  $m$ (=odd), we have

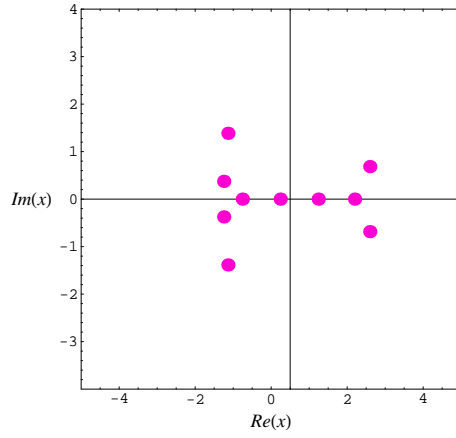
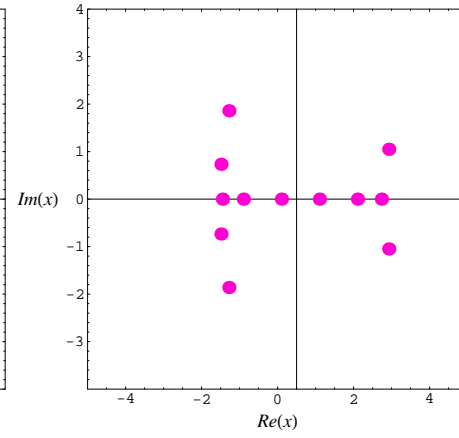
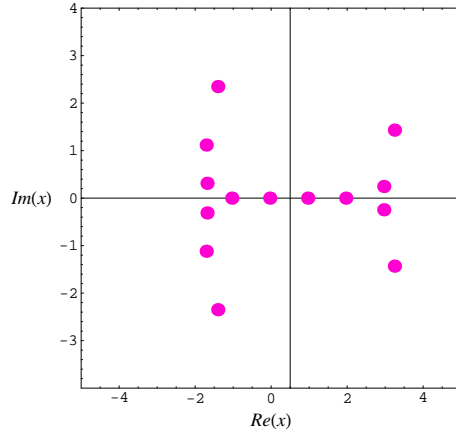
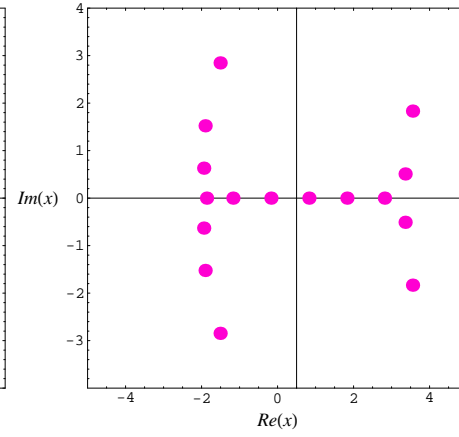
$$E_{n,q}(x) = m^{n-1} \sum_{i=0}^{m-1} (-1)^i q^i E_{n,q^m} \left( \frac{i+x}{m} \right), \text{ for } n \geq 0.$$

### 3. Distribution and Structure of the zeros

In this section, we investigate the zeros of  $q$ -Euler polynomials  $E_{n,q}(x)$  by using computer. Let  $q$  be a complex number with  $0 < q < 1$ . We plot the zeros of  $E_{n,q}(x)$ ,  $x \in \mathbb{C}$  for  $q = 1/2$ . (Figures 1, 2, 3, and 4). Next, we plot the zeros of  $E_{16,q}(x)$ ,  $x \in \mathbb{C}$  for  $q = 1/10, 1/50, 1/100, 1/150$ . (Figures 5, 6, 7, and 8).

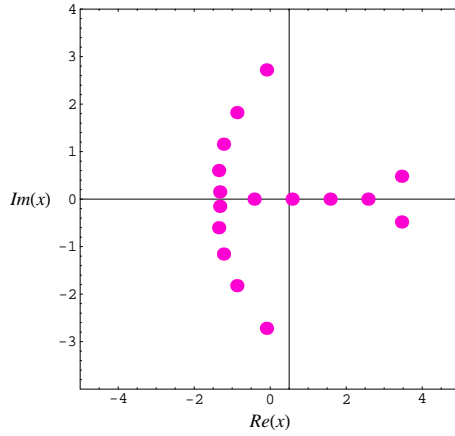
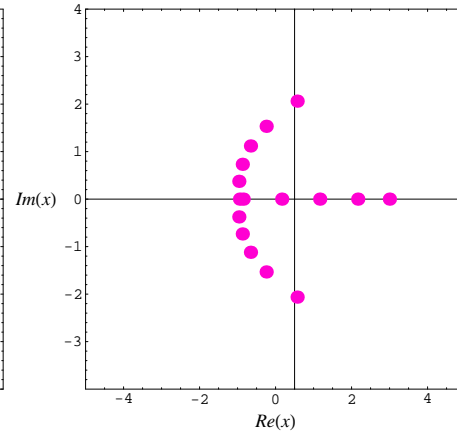
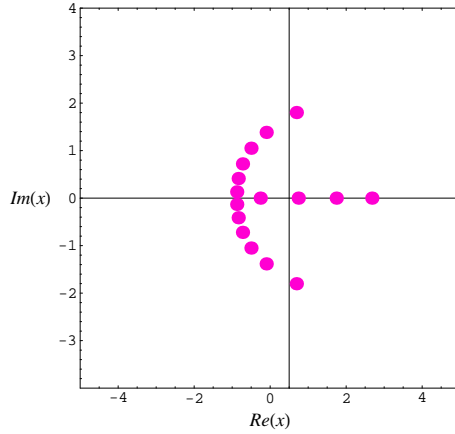
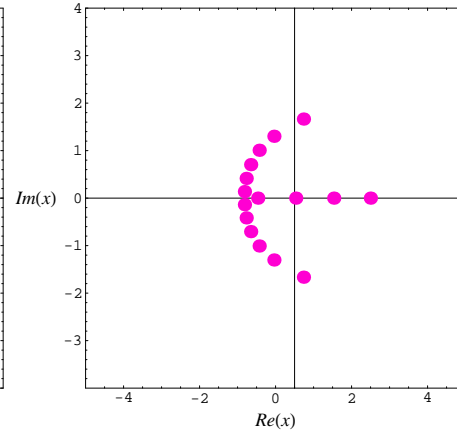
In Figures 1-8,  $E_{n,q}(x)$ ,  $x \in \mathbb{C}$ , has  $Im(x) = 0$  reflection symmetry. This translates to the following open problem: Prove or disprove:  $E_{n,q}(x)$ ,  $x \in \mathbb{C}$ , has  $Im(x) = 0$  reflection symmetry.

Our numerical results for numbers of real and complex zeros of  $E_{n,q}(x)$  are displayed in Table 1.

Figure 1: Zeros of  $E_{10,q}(x)$ Figure 2: Zeros of  $E_{12,q}(x)$ Figure 3: Zeros of  $E_{14,q}(x)$ Figure 4: Zeros of  $E_{16,q}(x)$ **Table 1.** Numbers of real and complex zeros of  $E_{n,q}(x)$ 

degree $n$	$q = 1/2$		$q = 1/5$	
	real zeros	complex zeros	real zeros	complex zeros
1	1	0	1	0
2	2	0	2	0
3	3	0	3	0
4	2	2	2	2
5	3	2	3	2
6	0	6	0	6
7	3	4	3	4
8	4	4	4	4
9	3	6	5	4
10	4	6	4	6

We shall consider the more general open problem. In general, how many roots does  $E_{n,q}(x)$  have? Prove or disprove:  $E_{n,q}(x)$  has  $n$  distinct solutions. Find the numbers of complex zeros  $C_{E_{n,q}(x)}$

Figure 5: Zeros of  $E_{16,q}(x)$ Figure 6: Zeros of  $E_{16,q}(x)$ Figure 7: Zeros of  $E_{16,q}(x)$ Figure 8: Zeros of  $E_{16,q}(x)$ 

of  $E_{n,q}(x)$ ,  $Im(x) \neq 0$ . Prove or give a counterexample: *Conjecture:* Since  $n$  is the degree of the polynomial  $E_{n,q}(x)$ , the number of real zeros  $R_{E_{n,q}(x)}$  lying on the real plane  $Im(x) = 0$  is then  $R_{E_{n,q}(x)} = n - C_{E_{n,q}(x)}$ , where  $C_{E_{n,q}(x)}$  denotes complex zeros. See Table 1 for tabulated values of  $R_{E_{n,q}(x)}$  and  $C_{E_{n,q}(x)}$ . We plot the  $E_{n,q}(x)$ , respectively (Figures 1-9). These figures give mathematicians an unbounded capacity to create visual mathematical investigations of the behavior of the roots of the  $E_{n,q}(x)$ . Moreover, it is possible to create a new mathematical ideas and analyze them in ways that generally are not possible by hand. The author has no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the  $q$ -Euler polynomials  $E_{n,q}(x)$  to appear in mathematics and physics. For related topics the interested reader is referred to [4], [6], [7]. We calculated an approximate solution satisfying  $E_{n,q}(x)$ ,  $q = 1/2, 1/5$ ,  $x \in \mathbb{R}$ .

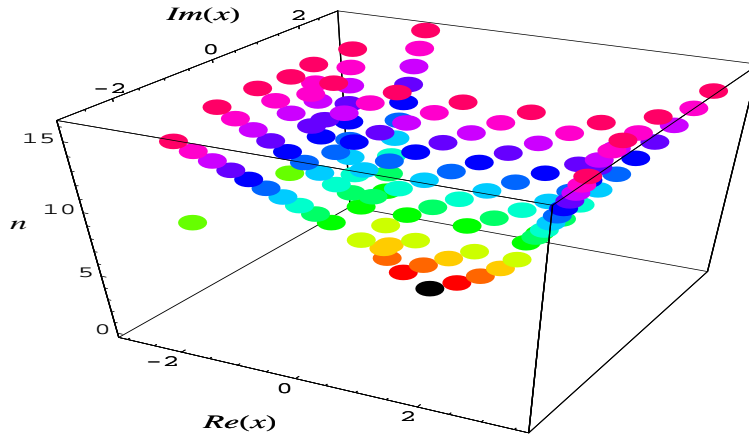


Figure 9: Stacks of zeros  $E_{n,q}(x)$ ,  $q = 1/2$  for  $1 \leq n \leq 16$

The results are given in Table 2 and Table 3.

**Table 2.** Approximate solutions of  $E_{n,q}(x) = 0$ ,  $q = 1/2$

degree $n$	$x$
1	0.3333
2	-0.13807, 0.8047
3	-0.42060, 0.22004, 1.2006
4	0.6547, 1.5273
5	0.08542, 1.0854, 1.7866

**Table 3.** Approximate solutions of  $E_{n,q}(x) = 0$ ,  $q = 1/5$

degree $n$	$x$
1	0.16667
2	-0.20601, 0.53934
3	-0.3009, -0.10220, 0.9031
4	0.24627, 1.2384
5	-0.3717, 0.5951, 1.5439

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## Generalized distance and some common fixed point theorems

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**Abstract.** In this paper, we consider  $w$ -distance on a complete metric spaces and prove some common fixed point theorem for commuting maps.

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**Key Words and Phrases:** Fixed point theorem; metric spaces;  $w$ -distance; commuting maps; common fixed point theorem.

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### 1. INTRODUCTION AND PRELIMINARIES

Recently, Kada, Suzuki and Takahashi [3] introduced the concept of  $w$ -distance on a metric space and proved some fixed point theorems. In the sequel, we state the definition of  $w$ -distance and we state a lemma which we will use in Sections 2 and 3; for more information we refer the reader to [3], [4] and [5].

**Definition 1.1.** ([3]) Let  $X$  be a metric space with metric  $d$ . Then a function  $p : X \times X \longrightarrow [0, \infty)$  is called a  $w$ -distance on  $X$  if the following are satisfied:

- (a)  $p(x, z) \leq p(x, y) + p(y, z)$  for any  $x, y, z \in X$ ;
- (b) for any  $x \in X$ ,  $p(x, \cdot) : X \longrightarrow [0, \infty)$  is lower semi-continuous;
- (c) for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $p(z, x) \leq \delta$  and  $p(z, y) \leq \delta$  imply  $d(x, y) \leq \varepsilon$ .

**Lemma 1.2.** ([3, 5]) Let  $X$  be a metric space with metric  $d$  and  $p$  be a  $w$ -distance on  $X$ . Let  $\{x_n\}$  and  $\{y_n\}$  be sequences in  $X$ , let  $\{\alpha_n\}$  and  $\{\beta_n\}$  be sequences in  $[0, \infty)$  converging to zero, and let  $x, y, z \in X$ . Then the following hold:

- (1) If  $p(x_n, y) \leq \alpha_n$  and  $p(x_n, z) \leq \beta_n$  for any  $n \in \mathbb{N}$ , then  $y = z$ . In particular, if  $p(x, y) = 0$  and  $p(x, z) = 0$ , then  $y = z$ ;
- (2) if  $p(x_n, y_n) \leq \alpha_n$  and  $p(x_n, z) \leq \beta_n$  for any  $n \in \mathbb{N}$ , then  $d(y_n, z) \rightarrow 0$ ;
- (3) if  $p(x_n, x_m) \leq \alpha_n$  for any  $n, m \in \mathbb{N}$  with  $m > n$ , then  $\{x_n\}$  is a Cauchy sequence;
- (4) if  $p(y, x_n) \leq \alpha_n$  for any  $n \in \mathbb{N}$ , then  $\{x_n\}$  is a Cauchy sequence.

## 2. COMMON FIXED POINT THEOREM FOR COMMUTING MAPS

The following theorem is Jungck's [2] generalization of the contraction principle for metric spaces.

**Theorem 2.1.** *Let  $f$  be a continuous mapping of a complete metric space  $(X, d)$  into itself and let  $g : X \rightarrow X$  be a map that satisfy the following conditions:*

- (a)  $g(X) \subseteq f(X)$ ;
- (b)  $g$  commutes with  $f$ ;
- (c)  $d(g(x), g(y)) \leq kd(f(x), f(y))$  for all  $x, y \in X$  and for some  $0 < k < 1$ .

*Then  $f$  and  $g$  have a unique common fixed point.*

In the next example we show that if the function  $f$  is continuous and

$$p(g(x), g(y)) \leq kp(f(x), f(y))$$

for all  $x, y \in X$  and  $0 < k < 1$ , in general,  $g$  may be not continuous.

**Example 2.2.** Let  $(\mathbb{R}, |\cdot|)$  be a normed linear space. Then the function  $p : \mathbb{R}^2 \rightarrow [0, \infty)$  defined by

$$p(x, y) = |y| \text{ for every } x, y \in \mathbb{R},$$

is a  $w$ -distance on  $X$  (see [3] Example 2). Consider the functions  $f$  and  $g$  defined by  $f(x) = 4$  and

$$g(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then

$$p(g(x), g(y)) = |g(y)| \leq 1 \leq \left(\frac{1}{3}\right)p(f(x), f(y)) = \frac{|f(y)|}{3} = \frac{4}{3}.$$

Note that the function  $f$  is continuous but  $g$  is not.

**Theorem 2.3.** *Let  $(X, d)$  be a complete metric space,  $p$  a  $w$ -distance on  $X$ , and let  $f, g : X \rightarrow X$  be maps that satisfy the following conditions:*

- (a)  $g(X) \subseteq f(X)$ ;
- (b)  $g$  commutes with  $f$  and  $f, g$  are continuous;
- (c)  $p(g(x), g(y)) \leq kp(f(x), f(y))$  for all  $x, y \in X$  and  $0 < k < 1$ .

*Then  $f$  and  $g$  have common fixed point provided  $f$  and  $g$  commute. Moreover, if  $g(v) = g(g(v))$  for all  $v \in X$ , then  $p(g(v), g(v)) = 0$ .*

*Proof.* We claim that for every  $x \in X$

$$\inf\{p(f(x), g(x)) + p(f(x), z) + p(g(x), z) + p(g(x), g(g(x)))\} > 0$$

for every  $z \in X$  with  $g(z) \neq g(g(z))$ . For the moment suppose the claim is true. Let  $x_0 \in X$ . By (a) we can find  $x_1$  such that  $f(x_1) = g(x_0)$ . By induction, we can define a sequence  $\{x_n\}_n$  such that  $f(x_n) = g(x_{n-1})$ . By induction again,

$$\begin{aligned} p(f(x_n), f(x_{n+1})) &= p(g(x_{n-1}), g(x_n)) \\ &\leq kp(f(x_{n-1}), f(x_n)) \\ &\leq \cdots \leq k^n p(f(x_0), f(x_1)) \end{aligned}$$

for  $n = 1, 2, \dots$ , which implies that, for  $m > n$ ,

$$\begin{aligned} & p(f(x_n), f(x_m)) \\ & \leq p(f(x_{m-1}), f(x_m)) + p(f(x_{m-2}), f(x_{m-1})) + \dots + p(f(x_n), f(x_{n+1})) \\ & \leq p(f(x_0), f(x_1)) \sum_{j=n}^{m-1} k^j \leq \frac{k^n}{1-k} p(f(x_0), f(x_1)). \end{aligned}$$

Thus  $\{f(x_n)\}$  is a Cauchy sequence. Since  $X$  is complete, there exists  $y \in X$  such that  $\lim_{n \rightarrow \infty} f(x_n) = y$ . As a result  $g(x_{n-1}) = f(x_n)$  tends to  $y$ , so  $\{g(f(x_n))\}_n$  converges to  $g(y)$ . However,  $g(f(x_n)) = f(g(x_n))$  by the commutativity of  $f$  and  $g$  and so  $f(g(x_n))$  converges to  $f(y)$ . Because limits are unique,  $f(y) = g(y)$ , so  $f(f(y)) = f(g(y))$ . On the other hand, by lower semi-continuity of  $p(x, \cdot)$  we have

$$p(f(x_n), y) \leq \liminf_{m \rightarrow \infty} p(f(x_n), f(x_m)) \leq \frac{k^n}{1-k} p(f(x_0), f(x_1)),$$

$$p(g(x_n), y) \leq \liminf_{m \rightarrow \infty} p(f(x_{n+1}), f(x_m)) \leq \frac{k^{n+1}}{1-k} p(f(x_0), f(x_1)),$$

and

$$\begin{aligned} p(g(x_n), g(g(x_n))) & \leq kp(f(x_n), f(g(x_n))) \\ & = kp(g(x_{n-1}), g(g(x_{n-1}))) \\ & \leq k^2 p(f(x_{n-1}), f(g(x_{n-1}))) \\ & = k^2 p(g(x_{n-2}), g(g(x_{n-2}))) \\ & \leq \dots \leq k^n p(f(x_1), g(f(x_1))). \end{aligned}$$

Now, we show that  $g(y) = g(g(y))$ . Suppose  $g(y) \neq g(g(y))$  and since we assume the claim above we have

$$\begin{aligned} 0 & < \inf\{p(f(x), g(x)) + p(f(x), y) + p(g(x), y) + p(g(x), g(g(x))) : x \in X\} \\ & \leq \inf\{p(f(x_n), g(x_n)) + p(f(x_n), y) + p(g(x_n), y) + p(g(x_n), g(g(x_n))) : n \in \mathbb{N}\} \\ & = \inf\{p(f(x_n), f(x_{n+1})) + p(f(x_n), y) + p(g(x_n), y) + p(g(x_n), g(g(x_n))) : n \in \mathbb{N}\} \\ & \leq \inf_n \{k^n p(f(x_0), f(x_1)) + \frac{k^n}{1-k} p(f(x_0), f(x_1)) + \frac{k^{n+1}}{1-k} p(f(x_0), f(x_1)) \\ & \quad + k^n p(f(x_1), g(f(x_1))) : n \in \mathbb{N}\} = 0. \end{aligned}$$

This is a contradiction. Therefore  $g(y) = g(g(y))$ . Thus,  $g(y) = g(g(y)) = f(g(y))$ , and so  $g(y)$  is a common fixed point of  $f$  and  $g$ .

Furthermore, if  $g(y)$  is a common fixed point of  $f$  and  $g$  and  $g(v) = g(g(v))$  for all  $v \in X$ , then we have

$$\begin{aligned} p(g(y), g(y)) & = p(g(g(y)), g(g(y))) \\ & \leq kp(f(g(y)), f(g(y))) \\ & = kp(g(y), g(y)), \end{aligned}$$

which implies that,  $p(g(y), g(y)) = 0$ .

Now it remains to prove the claim. Assume that there exists  $y \in X$  with  $g(y) \neq g(g(y))$  and

$$\inf\{p(f(x), g(x)) + p(f(x), y) + p(g(x), y) + p(g(x), g(g(x))) : x \in X\} = 0.$$

Then there exists  $\{x_n\}$  such that

$$\lim_{n \rightarrow \infty} \{p(f(x_n), g(x_n)) + p(f(x_n), y) + p(g(x_n), y) + p(g(x_n), g(g(x_n)))\} = 0.$$

Since  $p(f(x_n), g(x_n)) \rightarrow 0$  and  $p(f(x_n), y) \rightarrow 0$ , by Lemma 1.2, we have

$$(2.1) \quad \lim_{n \rightarrow \infty} g(x_n) = y.$$

Also, since  $p(g(x_n), y) \rightarrow 0$  and  $p(g(x_n), g(g(x_n))) \rightarrow 0$  by Lemma 1.2, we have

$$(2.2) \quad \lim_{n \rightarrow \infty} g(g(x_n)) = y.$$

By (3.1), (3.2) and continuity of  $g$  we have

$$g(y) = g(\lim_n g(x_n)) = \lim_n g(g(x_n)) = y.$$

Therefore,  $g(y) = g(g(y))$ , which is a contradiction. Hence, if  $g(y) \neq g(g(y))$ , then

$$\inf\{p(f(x), g(x)) + p(f(x), y) + p(g(x), y) + p(g(x), g(g(x))) : x \in X\} > 0.$$

□

### 3. COMMON FIXED POINT THEOREM FOR FOUR MAPPINGS

**Definition 3.1.** ([1]) Let  $G$  be the family of all continuous functions  $g$  where  $g : [0, \infty)^5 \rightarrow [0, \infty)$  satisfies the following properties:

- (g1)  $g$  is non-decreasing in the 1th, 2th and 4th variable;
- (g2) If  $u, v \in [0, \infty)$  such that  $u \leq g(v, v, u, u + v, 0)$  or  $u \leq g(v, 0, v, u + v, u)$  or  $u \leq g(v, v, u, 0, u + v, 0)$ , then  $u < hv$  where  $0 < h < 1$  is a given constant;
- (g3) If  $u \in [0, \infty)$  such that  $u \leq g(u, u, 0, 0, u)$  or  $u \leq g(0, 0, u, u, u)$  or  $u \leq g(u, 0, u, 0, u, u)$ , then  $u = 0$ .

**Definition 3.2.** Let  $(X, d)$  be a metric space. Let  $A$  and  $S$  be mappings from  $X$  into itself and let function  $p$  be a  $w$ -distance on  $X$ . Then  $A$  and  $S$  are said to be  $w$ -compatible mappings on  $X$  if

$$\lim_{n \rightarrow \infty} \max[p(ASx_n, x_n), p(SAx_n, x_n)] = 0.$$

**Definition 3.3.** Let  $(X, d)$  be a metric space. Let  $A$  and  $S$  be mappings from  $X$  into itself and let function  $p$  be a  $w$ -distance on  $X$ . Then  $A$  and  $S$  are said to be  $wR$ -weakly commuting on  $X$  at  $a$  ( $a \in X$ ) if given  $x$  in  $X$  there exists  $R > 0$  such that

$$\max[p(SAx, a), p(ASx, a)] \leq R[|p(Ax, a) - p(Sx, a)|].$$

Let  $(X, d)$  be a metric space. Let  $A, B, S$  and  $T$  be mappings from  $X$  into itself and  $p$  be a  $w$ -distance on  $X$ . Let  $(A, S)$  and  $(B, T)$  be  $wR$ -weakly commuting pairs at  $a$  of self mappings of a complete metric space  $(X, d)$  such that

$$(3.1) \quad A(X) \subseteq T(X), \quad B(X) \subseteq S(X)$$

and

$$(3.2) \quad \begin{aligned} & \max\{p(Ax, a), p(By, a)\} \\ & \leq g(|p(Sx, a) - p(Ty, a)|, |p(Sx, a) - p(Ax, a)|, \max\{p(Ty, a), p(By, a)\} \\ & \quad , |p(Ax, a) - p(Ty, a)|, \max\{p(Sx, a), p(By, a)\}) \end{aligned}$$

for all  $x, y \in X$ , where  $g \in G$ .

Then for an arbitrary point  $x_0$  in  $X$ , by (3.1), we can choose a point  $x_1$  such that  $Tx_1 = Ax_0$  and for this point  $x_1$ , there exists a point  $x_2$  in  $X$  such that  $Sx_2 = Bx_1$  and so on. Continuing in this manner, we can find a sequence  $\{y_n\}$  in  $X$  such that

$$(3,3) \quad y_{2n} = Ax_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+1}, \quad n = 1, 2, 3, \dots$$

**Lemma 3.4.** *Let  $(X, d)$  be a metric space. Let  $p$  be a  $w$ -distance on  $X$  and let  $A, B, S$  and  $T$  be mappings from  $X$  into itself satisfying the conditions (3.1) and (3.2). Then the sequence  $\{y_n\}$  defined by (3.3) is a Cauchy sequence in  $X$ .*

*Proof.* From (3.2) we have

$$\begin{aligned} & \max\{p(Ax_{2n}, a), p(Bx_{2n+1}, a)\} \\ \leq & \quad g(|p(Sx_{2n}, a) - p(Tx_{2n+1}, a)|, |p(Sx_{2n}, a) - p(Ax_{2n}, a)|, \\ & \max\{p(Tx_{2n+1}, a), p(Bx_{2n+1}, a)\}, |p(Ax_{2n}, a) - p(Tx_{2n+1}, a)|, \\ & \max\{p(Sx_{2n}, a), p(Bx_{2n+1}, a)\}) \end{aligned}$$

and

$$\begin{aligned} & \max\{p(y_{2n}, a), p(y_{2n+1}, a)\} \\ \leq & \quad g(|p(y_{2n-1}, a) - p(y_{2n}, a)|, |p(y_{2n-1}, a) - p(y_{2n}, a)|, \\ & \max\{p(y_{2n}, a), p(y_{2n+1}, a)\}, |p(y_{2n}, a) - p(y_{2n}, a)|, \\ & \max\{p(y_{2n-1}, a), p(y_{2n+1}, a)\}) \\ \leq & \quad g(\max\{p(y_{2n-1}, a), p(y_{2n}, a)\}, \max\{p(y_{2n-1}, a), p(y_{2n}, a)\}, \\ & \max\{p(y_{2n}, a), p(y_{2n+1}, a)\}, 0, \\ & \max\{p(y_{2n-1}, a), p(y_{2n}, a)\} + \max\{p(y_{2n}, a), p(y_{2n+1}, a)\}). \end{aligned}$$

By (g2) of Definition 3.1, we obtain,

$$\max\{p(y_{2n}, a), p(y_{2n+1}, a)\} < h \max\{p(y_{2n-1}, a), p(y_{2n}, a)\}$$

in which  $h \in ]0, 1[$ . Therefore

$$\begin{aligned} \max\{p(y_n, a), p(y_{n+1}, a)\} & < h \max\{p(y_{n-1}, a), p(y_n, a)\} \\ & < h^2 \max\{p(y_{n-2}, a), p(y_{n-1}, a)\} \\ & < h^n \max\{p(y_0, a), p(y_1, a)\}. \end{aligned}$$

Moreover, for every integer  $m > n$ , we get

$$\begin{aligned} & \max\{p(y_n, a), p(y_m, a)\} \\ < & \max\{p(y_n, a), p(y_{n+1}, a)\} + \dots + \max\{p(y_{m-1}, a), p(y_m, a)\} \\ < & \frac{h^n}{1-h} \max\{p(y_0, a), p(y_1, a)\}. \end{aligned}$$

By Lemma 2 of [6] the sequence  $\{y_n\}$  is Cauchy.  $\square$

**Theorem 3.5.** *Let  $(X, d)$  be a metric space. Let  $p$  be a  $w$ -distance on  $X$  and let  $A, B, S$  and  $T$  be mappings from  $X$  into itself satisfying the conditions (3.1) and (3.2). Suppose that  $(A, S)$  or  $(B, T)$  is a  $w$ -compatible pair of reciprocally continuous mappings. Then  $A, B, S$  and  $T$  have a unique common fixed point in  $X$ .*

*Proof.* By Lemma 3.4,  $\{y_n\}$  is a Cauchy sequence in  $X$ , Since  $X$  is complete. So there exists a point  $z$  in  $X$  such that  $\lim_{n \rightarrow \infty} y_n = z$ ,  $\lim_{n \rightarrow \infty} Ax_{2n} = \lim_{n \rightarrow \infty} Tx_{2n+1} = z$  and  $\lim_{n \rightarrow \infty} Bx_{2n+1} = \lim_{n \rightarrow \infty} Sx_{2n+2} = z$ . Suppose  $A$  and  $S$  are  $w$ -compatible and reciprocally continuous. Then by reciprocally continuous of  $A$  and  $S$ , we have  $\lim_{n \rightarrow \infty} ASx_{2n} = Az$  and  $\lim_{n \rightarrow \infty} SAx_{2n} = Sz$ . Also, by  $w$ -compatibility of  $A$  and  $S$ ,  $Az = Sz$ , Since  $A(X) \subseteq T(X)$ , so there exists a point  $v$  in  $X$  such that  $Az = Tv$ .

$$\begin{aligned} & \max\{p(Az, a), p(Bv, a)\} \\ \leq & g(|p(Sz, a) - p(Tv, a)|, |p(Sz, a) - p(Az, a)|, \max\{p(Tv, a), p(Bv, a)\} \\ & , |p(Az, a) - p(Tv, a)|, \max\{p(Sz, a), p(Bv, a)\}) \\ \leq & g(0, 0, \max\{p(Az, a), p(Bv, a)\}, 0, \max\{p(Az, a), p(Bv, a)\}). \end{aligned}$$

By (g3) of Definition 3.1,

$$(3.4) \quad \max\{p(Az, a), p(Bv, a)\} = 0$$

so  $Az = Bv$ . Thus  $Az = Sz = Tv = Bv$ .

Since  $A$  and  $S$  are  $wR$ -weak commutativity, there exist  $R > 0$  such that

$$(3.5) \quad \max\{p(SAz, a), p(ASz, a)\} \leq R[|p(Az, a) - p(Sz, a)|].$$

It follows that  $ASz = SAz$  and  $AAz = ASz = SAz = SSz$ . Also,  $B$  and  $T$  are  $\tau R$ -weak commutative, so we have  $BBv = BTv = TBv = TTv$ .

By (3.4) we have  $p(Az, a) = 0$  and by (3.5) we have  $p(ASz, a) = p(AAz, a) = 0$ , hence  $AAz = Az$ . Thus  $Az$  is a common fixed point of  $A$  and  $S$ . Similarly, we can prove that  $Bv (= Az)$  is a common fixed point of  $B$  and  $T$ .

Finally, in order to prove uniqueness of  $Az$ , suppose that  $Az$  and  $Aw$ ,  $Az \neq Aw$ , are common fixed points of  $A, B, S$  and  $T$ . Then by (3.2), we obtain

$$\begin{aligned} \max\{p(Az, a), p(Aw, a)\} &= \max\{p(AAz, a), p(BAw, a)\} \\ &\leq g(|p(SAz, a) - p(TAw, a)|, |p(SAz, a) - p(AAz, a)|, \\ &\quad \max\{p(TAw, a), p(BAw, a)\} \\ &\quad , |p(AAz, a) - p(TAw, a)|, \max\{p(SAz, a), p(BAw, a)\}) \\ &\leq g(\max\{p(Az, a), p(Aw, a)\}, 0, \\ &\quad \max\{p(Az, a), p(Aw, a)\}, 2\max\{p(Az, a), p(Aw, a)\}, \\ &\quad \max\{p(Az, a), p(Aw, a)\}). \end{aligned}$$

By (g2) of Definition 3.1,  $\max\{p(Az, a), p(Aw, a)\} = 0$  which implies that  $Az = Aw$ . This completes the proof.  $\square$

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# Lagrange Interpolations by Bivariate $C^1$ Cubic Splines on Powell-Sabin's Triangulations<sup>★</sup>

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## Abstract

In this paper, by treating Powell-Sabin's type-1 refined triangulation  $\Delta_{PS1}$  as a triangulated quadrangulation plus a few triangles near the boundary and treating Powell-Sabin's type-2 refined triangulation  $\Delta_{PS2}$  as a triangulated quadrangulation, Lagrange interpolation schemes for both bivariate  $C^1$  cubic spline spaces  $S_3^1(\Delta_{PS1})$  and  $S_3^1(\Delta_{PS2})$  are respectively constructed and Lagrange interpolation sets are given.

*Keywords:* bivariate  $C^1$  cubic splines; Powell-Sabin's refined triangulations; Lagrange interpolation schemes; Lagrange interpolation set

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## 1 Introduction

Let  $\Delta$  be a regular triangulation of a simply connected polygonal domain  $\Omega$  in  $\mathbb{R}^2$ , i.e.,  $\Delta$  is a set of closed triangles whose union coincides with  $\Omega$  such that the intersection of any two triangles in  $\Delta$  is either empty, a common edge or a vertex. For convenience, let  $V$ ,  $V_I$ ,  $V_B$ ,  $E$ ,  $E_I$ ,  $E_B$  and  $T$  denote the set of vertices, interior vertices, boundary vertices, edges, interior edges, boundary edges and triangles in  $\Delta$ , respectively. It is well-known from Euler Theorem that

$$|T| = |E_I| - |V_I| + 1, \quad |E| = 2|E_I| - 3|V_I| + 3, \quad |V| = |E_I| - 2|V_I| + 3, \quad (1)$$

where  $|\cdot|$  is the cardinality of the set.

For given integers  $r$  and  $d$  with  $0 \leq r \leq d - 1$ , the space of bivariate splines with degree  $d$  and smoothness order  $r$  with respect to  $\Delta$  is defined by

$$S_d^r(\Delta) = \{s \in C^r(\Omega) : s|_t \in \mathcal{P}_d, \forall t \in T\}, \quad (2)$$

where  $\mathcal{P}_d$  is the space of bivariate polynomials of total degree being at most  $d$ .

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There are usually two methods of interpolation by splines, one is the Hermite interpolation scheme and the other is the Lagrange interpolation scheme. The latter is very useful for the construction and reconstruction of surfaces and for scattered data fitting. It does not require knowing or approximating values of derivatives and requires only function values of the Lagrange interpolation points. This stands in contrast to Hermite interpolation schemes where derivatives have to be computed approximately. In [6], Davydov and Nürnberger developed an algorithm for constructing point sets admissible for Lagrange interpolation by  $S_d^1(\Delta)$  when  $d \geq 4$ . In [7]-[12], the authors studied the Lagrange interpolation by bivariate  $C^1$  splines of various degrees on either triangulations or triangulated quadrangulations, all of these results depend on certain colorings of the triangulations or quadrangulations. Recently, Nürnberger *et al.* [13] discussed the Lagrange interpolation by  $C^2$  splines of degree seven on triangulations, which is also based on a coloring algorithm.

In this paper, we will construct two Lagrange interpolation sets, one is for  $S_3^1(\Delta_{PS1})$  on Powell-Sabin's type-1 refined triangulation and the other is for  $S_3^1(\Delta_{PS2})$  on Powell-Sabin's type-2 refined triangulation.

## 2 Preliminaries

**Definition 1** Given a triangulation  $\Delta$  of  $\Omega$ , splitting each triangle  $t$  of  $\Delta$  into three subtriangles at the incenter  $v_t$  of  $t$  and then connecting the incenters of any two neighboring triangles, if  $t$  is a triangle with an edge  $e$  on the boundary of  $\Omega$ , connecting the incenter of  $t$  to the center of  $e$ , then we will get a Powell-Sabin's type-1 triangulation (see Fig. 1).

**Definition 2** Given a triangulation  $\Delta$  of  $\Omega$ , connecting three central lines of each triangle  $t$  and then connecting each midpoint  $v_e$ , we will obtain a Powell-Sabin's type-2 triangulation (see Fig. 1).

The Bernstein-Bézier technique (B-net, B-form) plays an important role in the study of both curves fitting and multivariate spline approximation. In 1980, Farin [1] first used the B-net technique in the study of bivariate splines. The more details about this technique can be found in [2], [3] and most of the papers in the references list.

Throughout the paper, let  $t := \langle u, v, w \rangle$  be a triangle, we denote the three vertices of  $t$  in counterclockwise direction by  $u, v$  and  $w$ , then every polynomial  $p \in \mathcal{P}_d$  associated with  $t$  can be written uniquely in the B-net representation

$$p = \sum_{i+j+k=d} c_{ijk}^t \mathcal{B}_{ijk}^d, \quad (3)$$

where  $\{\mathcal{B}_{ijk}^d\}_{i+j+k=d}$  is the Bernstein polynomials of degree  $d$  on the triangle  $t$ , and  $c_{ijk}^t$  are called the B-net coefficients of  $p$  associated with the domain points  $\xi_{ijk}^t = (iu + jv + kw)/d$ .

It is clear that there is a one to one correspondence between the set of domain points  $D_\Delta := \bigcup_{t \in \Delta} D_t = \bigcup_{t \in \Delta} \{\xi_{ijk}^t : i + j + k = d\}$  and the set of B-net coefficients  $\{c_\xi\}_{\xi \in D_\Delta}$  for  $s \in S_d^0(\Delta)$ . For each vertex  $v \in V$ , we define the set

$$R_m(v) := \{\text{points which are distance } m \text{ from } v\},$$



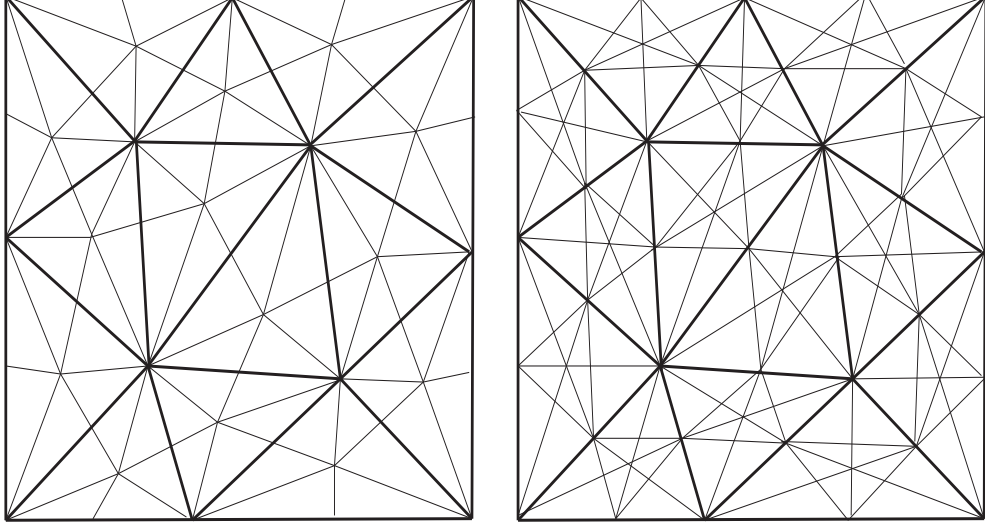


Fig. 1: Powell-Sabin's type-1 triangulation  $\Delta_{PS1}$  (left) and type-2 triangulation  $\Delta_{PS2}$  (right).

and the  $m$ -th disk around  $v$  defined by

$$D_m(v) := \bigcup_{i=0}^m R_i(v).$$

Let  $S$  be a spline space. Following Nürnberger and Zeilfelder [14], a set  $\{z_1, \dots, z_d\}$  in  $\Omega$ , where  $d = \dim S$ , is called a *Lagrange interpolation set* for  $S$  if for any function  $f \in C(\Omega)$ , a unique  $s \in S$  exists and satisfies

$$s(z_i) = f(z_i), \quad i = 1, \dots, d. \quad (4)$$

The following remarks and lemma can be found in Nürnberger *et al.* [10].

**Remark 1** A univariate cubic polynomial  $s(x)$  on an interval  $[a, b]$  which satisfies  $s(a) = z_0$  and  $s(b) = z_3$  can be written in the Bernstein-Bézier form

$$s(x) = \frac{1}{h^3} [z_0(b-x)^3 + 3c_1(x-a)(b-x)^2 + 3c_2(x-a)^2(b-x) + z_3(x-a)^3], \quad (5)$$

where  $h = b - a$ . Then the B-net coefficients  $c_1$  and  $c_2$  of the unique  $s(x)$  that interpolates given values at the points  $t_1 := a + h/3$  and  $t_2 := a + 2h/3$ , can be determined by solving a  $2 \times 2$  linear system whose matrix is

$$\frac{2}{9} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad (6)$$

independent of the interval  $[a, b]$ . Moreover, if  $c_1$  is given, then we can make  $s(x)$  to interpolate a prescribed value  $z_2$  at  $t_2$  by simply setting

$$c_2 = \frac{27z_2 - z_0 - 6c_1 - 8z_3}{12}. \quad (7)$$

**Remark 2** Suppose  $s(x)$  is a cubic polynomial, and that for a given triangle  $t$  we know all of its B-net coefficients except for the one associated with the domain point  $\xi_{111}^t$ . If we know the function value of  $\xi_{111}^t$ , then we can immediately calculate  $c_{111}^t$  by (3).

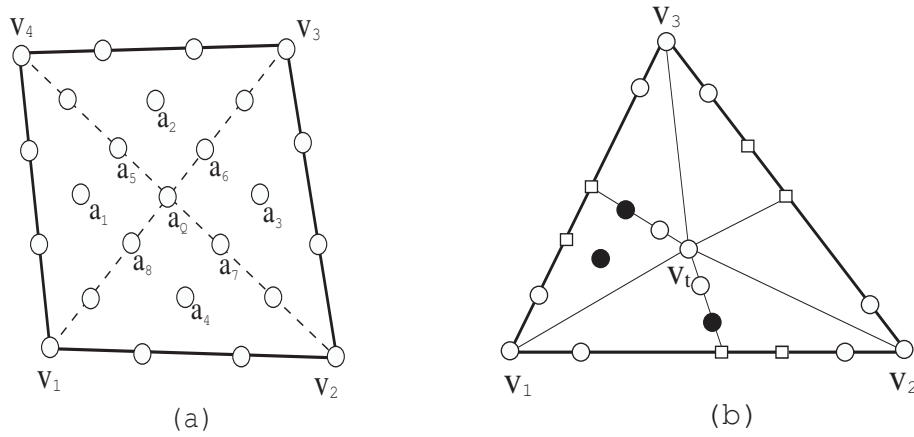


Fig. 2: (a) Domain points for  $S_3^1(\Sigma)$ ; (b) The point set  $P$  produced by Algorithm 3

**Lemma 1** Suppose  $\Sigma$  is the triangulation obtained by inserting both diagonals in a quadrangle  $Q := \langle v_1, v_2, v_3, v_4 \rangle$ , and suppose  $t_i = \langle v_Q, v_i, v_{i+1} \rangle, i = 1, \dots, 4$ , are the four triangles of  $\Sigma$ . As shown in Fig. 2(a), let  $\Gamma_E$  be the set of 12 domain points situated on the edges of  $Q$ , and let

$$\begin{aligned} M_1 &= \{a_1, a_2, a_3, a_4\}, \\ M_2 &= \{a_1, a_2, a_3, a_Q\}, \\ M_3 &= \{a_1, a_3, a_Q, a_8\}, \\ M_4 &= \{a_1, a_2, a_Q, a_8\}, \\ M_5 &= \{a_1, a_Q, a_8, a_5\}. \end{aligned}$$

Then each of the following five sets

$$\Gamma_l = \Gamma_E \bigcup M_l, l = 1, \dots, 5,$$

is a minimal determining set for  $S_3^1(\Sigma)$ .

**Algorithm 1** Let  $\diamond$  be a quadrangulation with all the quadrangles being convex, and  $\diamond$  is the related triangulated quadrangulation. We construct a point set  $P$  for  $S_3^1(\diamond)$  according to the following steps.

1) For each vertex  $v$  of  $\diamond$ , choose 3 points in  $D_1(v)$  which are marked by  $\circ$  in Fig. 4. Note that these points must be chosen on the edges of  $\diamond$ , and the B-net coefficients associated with these points can be computed by Remark 1.

2) Beginning with an arbitrary quadrangle  $Q$ , we label all the quadrangles  $Q_1, Q_2, Q_3, \dots$ , where the quadrangle  $Q_{i+1}$  must share at least one common edge with  $\bigcup_{k=1}^i Q_k$ . Choose 4 points in the first quadrangle  $Q_1$ , see Fig. 3(e), and choose 3 points in the second quadrangle  $Q_2$ , see Fig. 3(a). Furthermore, if the quadrangle  $Q_j$  ( $j=3, 4, \dots$ ) shares 1, 2, 3 or 4 common edges with  $\bigcup_{k=1}^{j-1} Q_k$ , then we respectively choose 3, 2, 1 or 0 points in the quadrangle  $Q_j$ , see white quadrangles in Fig. 3.

**Theorem 1** Let  $P := \{\xi_i\}_{i=1}^{|P|}$  be a point set constructed by Algorithm 1, then  $P$  forms a Lagrange interpolation set for  $S_3^1(\diamond)$ .

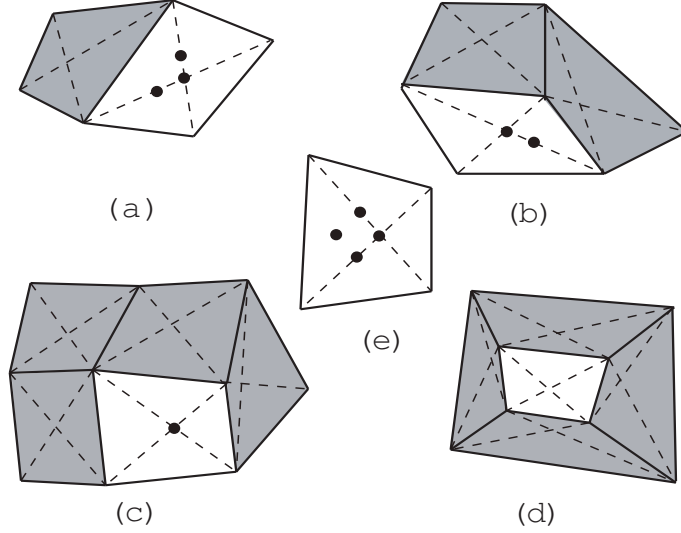


Fig. 3: Choice of points in step 2 of Algorithm 1.

**Proof** We show that for any given data  $Z := \{z_i\}_{i=1}^{|P|}$ , there exists a unique  $s \in S_3^1(\diamond)$  satisfying the interpolation condition (4). Suppose  $s$  is expressed in B-net representation as described in the Introduction, we need to show that each of the B-net coefficients of  $s \in S_3^1(\diamond)$  is uniquely determined by the data.

At first, for each domain point  $\xi$  lying at a vertex of  $\diamond$ , the corresponding B-net coefficient is equal to the data value associated with that point. Next, for each edge  $e$  of  $\diamond$ , it follows from step 2) of Algorithm 1 that  $P$  includes one or two domain points in the interior of  $e$ , the B-net coefficients associated with those domain points are computed according to Remark 1, then all B-net coefficients of  $s$  in  $D_1(v)$  are uniquely determined by  $C^1$  smoothness conditions. Finally, we compute B-net coefficients associated with domain points lying inside quadrangles. According to Algorithm 1, there are 4 points in  $Q_1$  belongs to the point set  $P$ , see Fig. 4, and we can apply Remark 1 to compute the B-net coefficients with respect to the domain points  $a$  and  $c$ , and the B-net coefficient associated with domain point  $d$  can be determined by Remark 2. Lemma 1 ensures that all the B-net coefficients associated with domain points lying inside the first quadrangle  $Q_1$  are determined. Then the B-net coefficient associated with domain point  $\xi_{111}^t$  in triangle  $t$  which neighbors on  $Q_1$  is computed by  $C^1$  smoothness conditions. By repeating the above steps for  $Q_2, Q_3, \dots$ , we can compute all the B-net coefficients of  $s$  associated with the domain points lying inside all quadrangles. Then all the B-net coefficients corresponding to the domain points of  $D_\diamond$  are uniquely determined by the data. Finally, from Algorithm 1 and [15], we know that the cardinality of the point set  $P$  is equal to the dimension of spline space  $S_3^1(\diamond)$ , i.e.,  $\dim S_3^1(\diamond) = |P| = 3|V| + |E|$ .

### 3 Lagrange interpolation set

In this section, Lagrange interpolation schemes are constructed for bivariate  $C^1$  cubic splines for  $S_3^1(\Delta_{PS1})$  and  $S_3^1(\Delta_{PS2})$ . Firstly, we construct a Lagrange interpolation set for  $S_3^1(\Delta_{PS1})$  by using Theorem 1.

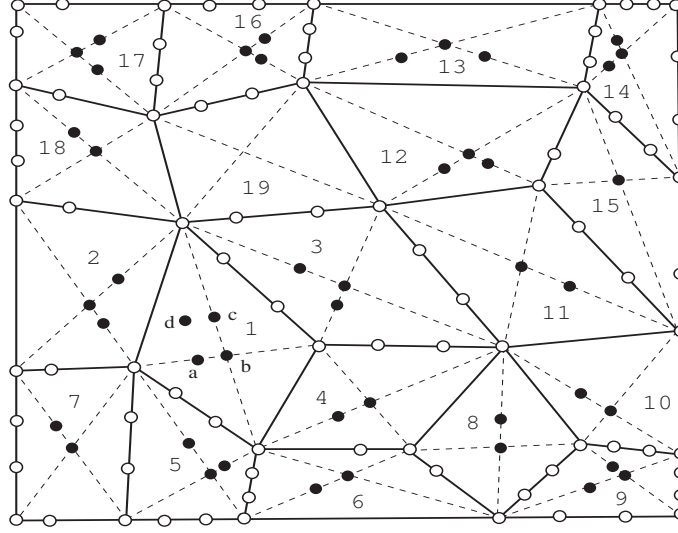


Fig. 4: A Lagrange interpolation set (○ and ●) for  $S_3^1(\diamond)$  obtained by using Algorithm 1.

**Algorithm 2** For Powell-Sabin's type-1 refined triangulation  $\Delta_{PS1}$ , we construct a point set  $P$  for  $S_3^1(\Delta_{PS1})$  according to the following steps.

- 1) Choose some points from gray part in Fig. 5 according to Algorithm 1.
- 2) For each boundary edge  $e$  of  $\Delta$ , choose two points on  $e$  which are marked by □ in Fig. 5. If  $t$  is a triangle with two edges on the boundary of  $\Omega$ , then choose three points in  $D_1(v)$  which are marked by ○ (where  $v$  is a vertex linked by these two boundary edges), and choose one point on  $t$  which is marked by ■ in Fig. 5.

**Theorem 2** Let  $P := \{\xi_i\}_{i=1}^{|P|}$  be a point set constructed by Algorithm 2, then  $P$  forms a Lagrange interpolation set for  $S_3^1(\Delta_{PS1})$ .

**Proof** We show that for any given data  $Z := \{z_i\}_{i=1}^{|P|}$ , there exists a unique  $s \in S_3^1(\Delta_{PS1})$  satisfying the interpolation condition (4). we suppose  $s$  is expressed in B-net form, we need to show that each of the B-net coefficients of  $S_3^1(\Delta_{PS1})$  is uniquely determined by the data. Firstly, we note that for each domain point  $\xi$  lying at a vertex of  $\Delta_{PS1}$ , the corresponding B-net coefficient is equal to the data value associated with that point.

We now show how to compute the remaining B-net coefficients of  $s$ , the B-net coefficients associated with domain points in gray part of Fig. 5 can be computed following Theorem 1, then the remaining B-net coefficients can be determined by Remark 1, Remark 2 and  $C^1$  smoothness conditions. Furthermore, according to [4], we have

$$\dim S_3^1(\Delta_{PS1}) = |P| = 3|V| + 6|T| + 2|V_B|.$$

We now prove such  $P$  is a Lagrange interpolation set for  $S_3^1(\Delta_{PS1})$ .

**Remark 3** When there is a single triangle  $t$  in the original triangulation  $\Delta$ , then Algorithm 2 and Theorem 2 can not construct a Lagrange interpolation set for  $S_3^1(t_{PS1})$ , and another Algorithm must be constructed in this situation.

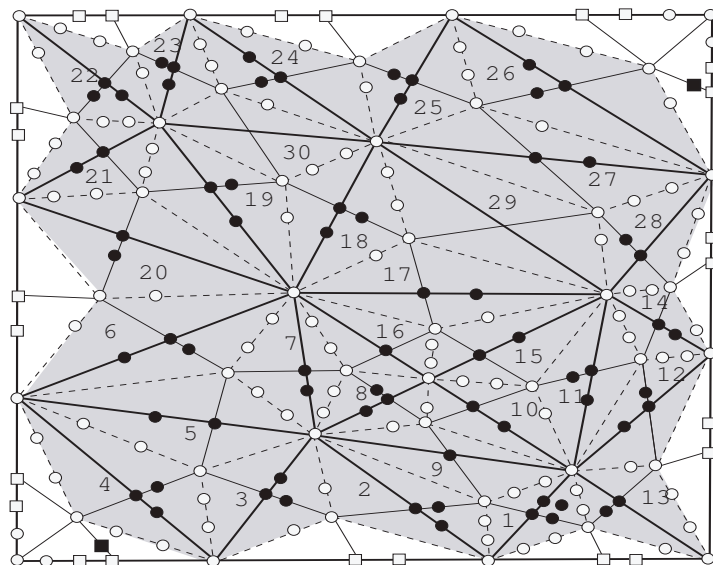


Fig. 5: A Lagrange interpolation set ( $\circ$  and  $\bullet$ ) for  $S_3^1(\Delta_{PS1})$  obtained by using Algorithm 2.

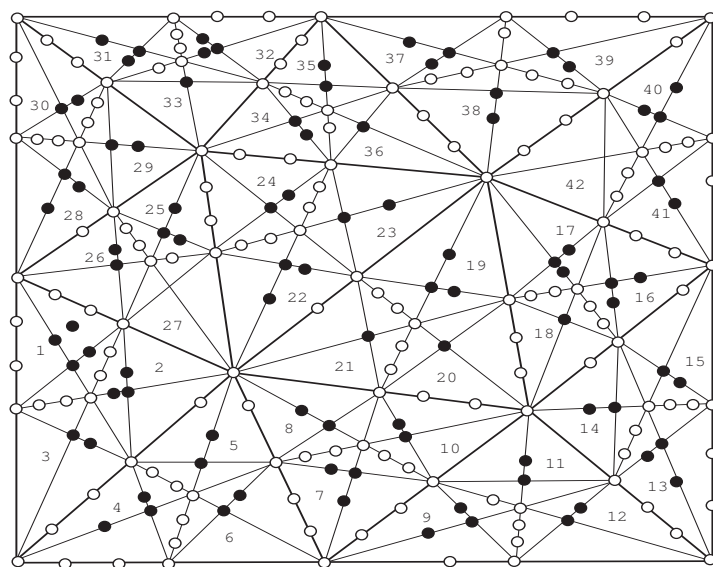


Fig. 6: A Lagrange interpolation set ( $\circ$  and  $\bullet$ ) for  $S_3^1(\Delta_{PS2})$  about Theorem 4.

**Algorithm 3** When  $\triangle$  is a single triangle  $t$ , then we construct a point set  $P$  for  $S_3^1(t_{PS1})$  according to the following steps.

- 1) For each vertex  $v$  and incenter  $v_t$  in  $t$ , choose three points in  $D_1(v)$  and  $D_1(v_t)$  which are marked by  $\circ$  in Fig. 2.
- 2) For each boundary edges  $e$  of  $t$ , choose two points on  $e$  which are marked by  $\square$  in Fig. 2.
- 3) Choose three points in triangle  $t$  which are marked by  $\bullet$  in Fig. 2.

**Theorem 3** Let  $P := \{\xi_i\}_{i=1}^{21}$  be a point set constructed by Algorithm 3, then  $P$  forms a Lagrange interpolation set for  $S_3^1(t_{PS1})$ .

**Proof** It follows from [4] that  $\dim S_3^1(t_{PS1}) = 21$ . Given any data  $Z := \{z_i\}_{i=1}^{21}$ , by using Remark 1, Remark 2 and  $C^1$  smoothness conditions, each of the B-net coefficients of  $S_3^1(t_{PS1})$  is uniquely determined by the data. This completes the proof of this theorem.

Note that triangulation  $\triangle_{PS2}$  can be regarded as a triangulated quadrangulation  $\diamond$  if we make each original vertex  $v$ ,  $v_t$  and center  $v_e$  as the vertex of quadrangle in  $\triangle_{PS2}$ , then we can obtain

**Theorem 4** Let  $P := \{\xi_i\}_{i=1}^{|P|}$  be a point set constructed by Algorithm 1 on  $S_3^1(\triangle_{PS2})$ , then  $P$  forms a it Lagrange interpolation set for  $S_3^1(\triangle_{PS2})$ .

**Proof** We know  $\dim S_3^1(\diamond) = 3|V| + |E|$ , then

$$|P| = 3(|V| + |E| + |T|) + (3|T| + 2|E|) = 3|V| + 5|E| + 6|T|.$$

Using Euler's theorem and [5] we obtain  $\dim S_3^1(\triangle_{PS2}) = |P|$ . Then we can give a similar proof as we did in the Theorem 1.

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# Sums of products of $q$ -Euler numbers

By

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**Abstract.** By using multivariate fermionic  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$ , the author introduced the  $q$ -Euler polynomials of higher order (see [1]). From these  $q$ -Euler polynomials of higher order, we derive the formula for the sums of products of the  $q$ -Euler polynomials of the form  $\sum_{i_1, \dots, i_\ell \geq 0}^{r=i_1+\dots+i_\ell} \sum_{k_1=0}^{r-i_1} \dots \sum_{k_{\ell-1}=0}^{r-i_1-i_2-\dots-i_{\ell-1}} \binom{r}{i_1, \dots, i_\ell} \binom{r-i_1}{k_1} \dots \binom{r-i_1-i_2-\dots-i_{\ell-1}}{k_{\ell-1}} E_{k_1+i_1, q}(\alpha_1) \dots E_{k_{\ell-1}+i_{\ell-1}, q}(\alpha_{\ell-1}) E_{i_\ell, q}(\alpha_\ell) (q-1)^{k_1+\dots+k_{\ell-1}}$ , where  $E_{m, q}(\alpha)$  are the  $m$ -th  $q$ -Euler polynomials and  $\binom{n}{\alpha_1, \dots, \alpha_n} = \frac{n!}{\alpha_1! \dots \alpha_n!}$ .

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**Key Words and Phrases:** Fermionic  $p$ -adic integral,  $q$ -Euler polynomials

## §1. Introduction

Let  $p$  be a fixed odd prime number. Throughout this paper  $\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p$  and  $\mathbb{C}_p$  will respectively, denote the ring of rational integer, the ring of  $p$ -adic rational integers, the field of  $p$ -adic rational numbers and the completion of algebraic closure of  $\mathbb{Q}_p$ . Let  $v_p$  be the normalized exponential valuation of  $\mathbb{C}_p$  with  $|p|_p = p^{-v_p(p)} = 1/p$ . When one talks of  $q$ -extension,  $q$  is variously considered as an indeterminate, a complex number  $q \in \mathbb{C}$  or a  $p$ -adic number  $q \in \mathbb{C}_p$ , (see [1-6]). If  $q \in \mathbb{C}_p$ , we normally assume  $|1-q|_p < 1$ . In this paper, we use the notations as follows.

$$[x]_q = \frac{1-q^x}{1-q}, \quad \text{and} \quad [x]_{-q} = \frac{1-(-q)^x}{1+q}.$$

Let  $d$  be a fixed positive integer with  $d \equiv 1 \pmod{2}$ . Then we set

$$X = X_d = \varprojlim_N \mathbb{Z}/dp^N \mathbb{Z}, \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{\substack{0 < a < dp, \\ (a, p) = 1}} (a + dp\mathbb{Z}_p),$$

$$a + dp^N \mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where  $a \in \mathbb{Z}$  lies  $0 \leq a < dp^N$ , ( see [1-6]). We say that  $f$  is uniformly differential function at a point  $a \in \mathbb{Z}_p$  and denote this property by  $f \in UD(\mathbb{Z}_p)$  if the difference quotients  $F_f(x, y) = \frac{f(x) - f(y)}{x - y}$  have a limit  $\ell = f'(a)$  as  $(x, y) \longrightarrow (a, a)$ . For  $f \in UD(\mathbb{Z}_p)$ , the fermionic  $p$ -adic  $q$ -integral on  $X$  is defined as

$$I_{-q}(f) = \int_X f(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1}{[dp^N]_{-q}} \sum_{x=0}^{dp^N-1} f(x)(-q)^x, \quad (\text{see [1-7]}). \quad (1)$$

The  $q$ -Euler numbers (see [6]) are defined by

$$E_{n,q} = \int_X [x]_q^n d\mu_{-q}(x) = \int_{\mathbb{Z}_p} [x]_q^n d\mu_{-q}(x) = \frac{[2]_q}{(1-q)^n} \sum_{\ell=0}^n \frac{\binom{n}{\ell} (-1)^\ell}{1+q^{\ell+1}}, \quad (2)$$

and the  $q$ -Euler polynomials (see [5]) are defined as

$$E_{n,q}(x) = \int_X [x+y]_q^n d\mu_{-q}(y) = \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-q}(y) = \frac{[2]_q}{(1-q)^n} \sum_{\ell=0}^n \frac{\binom{n}{\ell} (-1)^\ell q^{\ell x}}{1+q^{\ell+1}}.$$

In this paper, we consider the  $q$ -Euler polynomials of higher order using an integral by the  $q$ -analogue of fermionic  $p$ -adic invariant measure and give the formula for the sums of products of the  $q$ -Euler polynomials.

## §2. $q$ -Euler numbers of higher order

In [2], the  $m$ -th  $q$ -Euler polynomials in the variable  $x$  in  $\mathbb{C}_p$  with  $|x|_p \leq 1$  are defined as

$$E_{m,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^m d\mu_{-q}(y) = \frac{[2]_q}{(1-q)^m} \sum_{\ell=0}^m \binom{m}{\ell} \frac{(-1)^\ell}{1+q^{\ell+1}} q^{\ell x}.$$

We use the notation:

$$\sum_{k_1, \dots, k_n=0}^m = \sum_{k_1=0}^m \sum_{k_2=0}^m \cdots \sum_{k_n=0}^m.$$

We define the  $q$ -Euler numbers of higher order,  $E_{n,q}^{(r)} \in \mathbb{C}_p$ , by making use multivariate fermionic  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  as follows.

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_r]_q^n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) = E_{n,q}^{(r)}. \quad (3)$$

Note that  $\lim_{q \rightarrow 1} E_{n,q}^{(r)} = E_n^{(r)}$ , where  $E_n^{(r)}$  are the  $n$ -th Euler numbers of order  $r$ .

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From (3), we note that

$$\begin{aligned} E_{n,q}^{(r)} &= \frac{[2]_q^r}{(1-q)^n} \sum_{\ell=0}^n \binom{n}{\ell} (-1)^\ell \left( \frac{1}{1+q^{\ell+1}} \right)^r \\ &= [2]_q^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m q^m [m]_q^n. \end{aligned}$$

Let us define the  $n$ -th  $q$ -Euler polynomials of order  $r$  as follows.

$$E_{n,q}^{(r)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_r]_q^n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r). \quad (4)$$

From (4), we note that

$$\begin{aligned} E_{n,q}^{(r)}(x) &= \frac{[2]_q^r}{(1-q)^n} \sum_{\ell=0}^n \binom{n}{\ell} (-1)^\ell \left( \frac{1}{1+q^{\ell+1}} \right)^r q^{\ell x} \\ &= [2]_q^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m [m+x]_q^n, \quad (\text{cf. [2]}). \end{aligned}$$

By (4), we easily see that

$$\begin{aligned} E_{n,q^{-1}}^{(r)}(r-x) &= \frac{(-1)^n q^{n+r}}{(1-q)^n} [2]_q^r \sum_{\ell=0}^n \binom{n}{\ell} (-1)^\ell \left( \frac{1}{1+q^{\ell+1}} \right)^r q^{\ell x} \\ &= (-1)^n q^{n+r} E_{n,q}^{(r)}(x), \end{aligned} \quad (5)$$

and

$$E_{n,q^{-1}}^{(r)}(r) = (-1)^n q^{n+r} E_{n,q}^{(r)}.$$

The Eq.(5) seems to be interesting and new. If  $q \rightarrow 1$ , then we can derive the following formula from (5).

$$E_n^{(r)}(r-x) = (-1)^n E_n^{(r)}(x), \quad \text{and} \quad E_n^{(r)}(r) = (-1)^n E_n^{(r)}.$$

For  $d \in \mathbb{Z}_+$  with  $d \equiv 1 \pmod{2}$ , it is easy to see that

$$[d]_q^n \frac{[2]_q}{[2]_{q^d}} \sum_{a=0}^{d-1} (-1)^a q^a E_{n,q^d} \left( \frac{x+a}{d} \right) = E_{n,q}(x). \quad (6)$$

In the sense of the extension of (6), we consider the following multivariate integral.

$$\begin{aligned} &\int_X \cdots \int_X [x_1 + \cdots + x_r + x]_q^n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\ &= \frac{[d]_q^n}{[d]_{-q}^r} \sum_{a_1, \dots, a_r=0}^{d-1} (-1)^{a_1+\cdots+a_r} q^{a_1+\cdots+a_r} \\ &\times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[ \frac{a_1 + \cdots + a_r + x}{d} + x_1 + \cdots + x_r \right]_{q^d}^n d\mu_{-q^d}(x_1) \cdots d\mu_{-q^d}(x_r). \end{aligned} \quad (7)$$

From (7) and (4), we note that

$$E_{n,q}^{(r)}(x) = \frac{[d]_q^n}{[d]_{-q}^r} \sum_{a_1, \dots, a_r=0}^{d-1} (-1)^{a_1+\dots+a_r} q^{a_1+\dots+a_r} E_{n,q^d}^{(r)}\left(\frac{a_1+\dots+a_r+x}{d}\right). \quad (8)$$

We observe that

$$\begin{aligned} & \left( \frac{1 - q^{\alpha_1+\dots+\alpha_m+x_1+\dots+x_r}}{1-q} \right)^n \\ &= \sum_{\substack{i_1, \dots, i_n \geq 0, \\ i_1+\dots+i_r=n}} \binom{n}{i_1, \dots, i_r} \sum_{k_1=0}^{i_2+\dots+i_r} \binom{i_2+\dots+i_r}{k_1} [x_1 + \alpha_1]_q^{k_1+i_1} \\ & \times \sum_{k_2=0}^{i_3+\dots+i_r} \binom{i_3+\dots+i_r}{k_2} [x_2 + \alpha_2]_q^{k_2+i_2} \dots \\ & \times \sum_{k_{r-1}=0}^{i_r} \binom{i_r}{k_{r-1}} [x_{r-1} + \alpha_{r-1}]_q^{k_{r-1}+i_{r-1}} (q-1)^{k_1+\dots+k_{r-1}} [x_r + \alpha_r]_q^{i_r}. \end{aligned} \quad (9)$$

Thus, we have

$$\begin{aligned} & [\alpha_1 + \dots + \alpha_r + x_1 + \dots + x_r]_q^n \\ &= \sum_{\substack{i_1, \dots, i_r \geq 0, \\ i_1+\dots+i_r=n}} \sum_{k_1=0}^{n-i_1} \sum_{k_2=0}^{n-i_1-i_2} \dots \sum_{k_{r-1}=0}^{n-i_1-\dots-i_{r-1}} \binom{n}{i_1, \dots, i_r} \binom{n-i_1}{k_1} \binom{n-i_1-i_2}{k_2} \dots \binom{n-i_1-\dots-i_{r-1}}{k_{r-1}} \\ & \times (q-1)^{k_1+\dots+k_{r-1}} [x_1 + \alpha_1]_q^{k_1+i_1} [x_2 + \alpha_2]_q^{k_2+i_2} \dots [x_{r-1} + \alpha_{r-1}]_q^{k_{r-1}+i_{r-1}} [x_r + \alpha_r]_q^{i_r}. \end{aligned} \quad (10)$$

From (10), we note that

$$\begin{aligned} & E_{n,q}^{(r)}(\alpha_1 + \dots + \alpha_r) \\ &= \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} [\alpha_1 + \dots + \alpha_r + x_1 + \dots + x_r]_q^n d\mu_{-q}(x_1) \dots d\mu_{-q}(x_r) \\ &= \sum_{\substack{i_1, \dots, i_r \geq 0, \\ i_1+\dots+i_r=n}} \sum_{k_1=0}^{n-i_1} \sum_{k_2=0}^{n-i_1-i_2} \dots \sum_{k_{r-1}=0}^{n-i_1-\dots-i_{r-1}} \binom{n}{i_1, \dots, i_r} \binom{n-i_1}{k_1} \binom{n-i_1-i_2}{k_2} \dots \\ & \binom{n-i_1-\dots-i_{r-1}}{k_{r-1}} E_{k_1+i_1,q}(\alpha_1) E_{k_2+i_2,q}(\alpha_2) \dots E_{k_{r-1}+i_{r-1},q}(\alpha_{r-1}) E_{k_r,q}(\alpha_r) (q-1)^{k_1+\dots+k_{r-1}}, \end{aligned}$$

where  $\binom{n}{i_1, \dots, i_r} = \frac{n!}{i_1! \dots i_r!}$ .

Therefore, we obtain the following theorem.



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THEOREM . For  $\alpha_1, \dots, \alpha_r \in \mathbb{C}_p$ ,  $n \in \mathbb{Z}_+$ , and  $r \in \mathbb{N}$ , we have

$$\begin{aligned} & E_{n,q}^{(r)}(\alpha_1 + \dots + \alpha_r) \\ &= \sum_{\substack{i_1, \dots, i_r \geq 0, \\ i_1 + \dots + i_r = n}} \sum_{k_1=0}^{n-i_1} \sum_{k_2=0}^{n-i_1-i_2} \dots \sum_{k_{r-1}=0}^{n-i_1-\dots-i_{r-1}} \binom{n}{i_1, \dots, i_r} \binom{n-i_1}{k_1} \binom{n-i_1-i_2}{k_2} \dots \\ & \binom{n-i_1-\dots-i_{r-1}}{k_{r-1}} E_{k_1+i_1,q}(\alpha_1) E_{k_2+i_2,q}(\alpha_2) \dots E_{k_{r-1}+i_{r-1},q}(\alpha_{r-1}) E_{k_r,q}(\alpha_r) (q-1)^{k_1+\dots+k_{r-1}}, \end{aligned}$$

where  $\binom{n}{i_1, \dots, i_r}$  is the multinomial coefficient.

REMARK2. By (4), we see that

$$\begin{aligned} & qE_{n,q}^{(r)}(x+1) + E_{n,q}^{(r)}(x) \\ &= \frac{q[2]_q^r}{(1-q)^n} \sum_{j=0}^n \binom{n}{j} (-1)^j \left( \frac{1}{1+q^{j+1}} \right)^r q^{j(x+1)} + \frac{[2]_q^r}{(1-q)^n} \sum_{j=0}^n \binom{n}{j} (-1)^j \left( \frac{1}{1+q^{j+1}} \right)^r q^{jx} \\ &= \frac{[2]_q^r}{(1-q)^n} \sum_{j=0}^n \binom{n}{j} (-1)^j q^{jx} \left( \frac{1}{1+q^{j+1}} \right)^r (1+q^{j+1}) \\ &= \frac{[2]_q^r}{(1-q)^n} \sum_{j=0}^n \binom{n}{j} (-1)^j q^{jx} \left( \frac{1}{1+q^{j+1}} \right)^{r-1} \\ &= [2]_q E_{n,q}^{(r-1)}(x). \end{aligned}$$

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## On the existence of periodic solutions to nonlinear third order ordinary differential equations with delay

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**Abstract:** In this paper, we are concerned with the existence of periodic solutions to nonlinear third order delay differential equation:

$$x'''(t) + \psi(x'(t))x''(t) + g(x'(t-r)) + f(x(t)) = p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t)),$$

when  $p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t))$  is a periodic function of period  $T$ ,  $T > 0$ . With use of Lyapunov's functional approach, we establish some new sufficient conditions which guarantee that there exists a periodic solution of this equation of period  $T$ ,  $T > 0$ . For illustrations, an example is also given on the existence of periodic solutions.

### 1. Introduction

It is well-known that functional differential equations, in particular, delay differential equations can be used as models to describe many physical, biological systems, etc. In practice, many actual systems have the property aftereffect, i.e. the future states depend not only on the present, but also on the past history. Aftereffect is believed to occur in mechanics, control theory, physics, chemistry, biology, medicine, economics, atomic energy, information theory, etc. Therefore, it is very important to study the qualitative behaviors of solutions of delay differential equations or more generally functional differential equations.

In 1992, Zhu [5] considered the following nonlinear third order differential equation with constant delay,  $r$ :

$$x'''(t) + ax''(t) + \phi(x'(t-r)) + f(x) = p(t), \quad (1)$$

and he discussed the existence of periodic solutions of this equation when  $p(t)$  is a periodic function of period  $T$ ,  $T > 0$ .

In this paper, we consider nonlinear third order differential equation with constant delay:

$$\begin{aligned} x'''(t) + \psi(x'(t))x''(t) + g(x'(t-r)) + f(x(t)) \\ = p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t)). \end{aligned} \quad (2)$$

Obviously, our equation, (2), includes equation (1), and it can be stated as the following system:

$$\begin{aligned} x'(t) &= y(t), \quad y'(t) = z(t), \\ z'(t) &= -\psi(y(t))z(t) - g(y(t)) - f(x(t)) + \int_{t-r}^t g'(y(s))z(s)ds \\ &\quad + p(t, x(t), x(t-r), y(t), y(t-r), z(t)), \end{aligned} \quad (3)$$

where  $r$  is a positive constant, that is,  $r$  is constant delay, which will be determined later;

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the functions  $\psi$ ,  $g$ ,  $f$  and  $p$  depend only on the arguments displayed explicitly. It is assumed as basic that  $\psi(y)$ ,  $g(y)$ ,  $f(x)$  and  $p(t, x, x(t-r), y, y(t-r), z)$  are continuous in their respective arguments on  $\mathfrak{R}$ ,  $\mathfrak{R}$ ,  $\mathfrak{R}$  and  $\mathfrak{R}^+ \times \mathfrak{R}^5$ , respectively;  $g(0) = f(0) = 0$  and  $p(t, x, x(t-r), y, y(t-r), z)$  is periodic in  $t$ , of period  $T$ ,  $T \geq r$ , that is, this function satisfies

$$p(t+T, x, x(t-r), y, y(t-r), z) = p(t, x, x(t-r), y, y(t-r), z). \text{ The derivative } g'(y) \equiv \frac{d}{dy} g(y),$$

exists and is also continuous; throughout the paper  $x(t)$ ,  $y(t)$  and  $z(t)$  are abbreviated as  $x$ ,  $y$  and  $z$ , respectively.

It should also be noted that to the best of our knowledge from the literature, so far, the existence of periodic solutions of nonlinear third order differential equations with delay has only been investigated by three authors; Chukwu [2], Tejumola & Tchegnani [3] and Zhu [5]. Perhaps, the possible difficulty raised to this case is due to the construction of Laypuov functionals for delay differential equations of higher order. But, here, we would not like the details of difficulties. It is also worth mentioning that all the papers afore mentioned, Chukwu [2], Tejumola & Tchegnani [3] and Zhu [5], have been published without including any explanatory example on the existence of periodic solutions of equations taken under consideration. Here, we will give an example on the topic. Our assumptions and Lyapunov functional, which will be introduced, are completely different than that in Chukwu [2], Tejumola & Tchegnani [3].

## 2. Preliminaries

In order to reach the main result of this paper, we will give some basic information for general non-autonomous delay periodic differential system. We now consider the delay periodic system:

$$\dot{x}(t) = f(t, x_t), \quad x_t = x(t + \theta), \quad -r \leq \theta \leq 0, \quad t \geq 0, \quad (4)$$

where  $f: [0, \infty) \times C_H \rightarrow \mathfrak{R}^n$  is a continuous mapping,  $f(t+T, \varphi) = f(t, \varphi)$  for all  $\varphi \in C$  and for some constant  $T > 0$ . We suppose that  $f$  takes closed bounded sets into bounded sets of  $\mathfrak{R}^n$ . Here  $(C, \|\cdot\|)$  is the Banach space of continuous function  $\phi: [-r, 0] \rightarrow \mathfrak{R}^n$  with supremum norm,  $r > 0$ ;  $C_H$  is the open  $H$ -ball in  $C$ ;  $C_H := \{\phi \in (C[-r, 0], \mathfrak{R}^n) : \|\phi\| < H\}$ .

**Definition 1.** (See Burton [1].) A continuous function  $W: [0, \infty) \rightarrow [0, \infty)$  with  $W(0) = 0$ ,  $W(s) > 0$  if  $s > 0$ , and  $W$  strictly increasing is a wedge. (We denote wedges by  $W$  or  $W_i$ , where  $i$  is an integer.)

**Definition 2.** (See Burton [1].) Let  $D$  be an open set in  $\mathfrak{R}^n$  with  $0 \in D$ . A function  $V: [0, \infty) \times D \rightarrow [0, \infty)$  is called positive definite if  $V(t, 0) = 0$  and if there is a wedge  $W_1$  with  $V(t, x) \geq W_1(|x|)$ , and is called decrescent if there is a wedge  $W_2$  with  $V(t, x) \leq W_2(|x|)$ .

**Definition 3.** (See Burton [1].) Let  $V(t, \phi)$  be a continuous functional defined for  $t \geq 0$ ,  $\phi \in C_H$ . The derivative of  $V$  along solutions of (4) will be denoted by  $\dot{V}$  and is defined by the following relation:

$$\dot{V}(t, \phi) = \limsup_{h \rightarrow 0} \frac{V(t+h, x_{t+h}(t_0, \phi)) - V(t, x_t(t_0, \phi))}{h},$$

where  $x(t_0, \phi)$  is the solution of (4) with  $x_{t_0}(t_0, \phi) = \phi$ .

**Theorem 1.** (Yoshizawa [4].) Suppose that  $f(t, \phi) \in \overline{C}_0(\phi)$  and  $f(t, \phi)$  is periodic in  $t$  of period  $T$ ,  $T \geq r$ , and consequently for any  $\alpha > 0$  there exists an  $L(\alpha) > 0$  such that  $\phi \in C_\alpha$  implies  $|f(t, \phi)| \leq L(\alpha)$ . Moreover, suppose that there exist a continuous Lyapunov functional  $V(t, \phi)$  defined on  $t \in I$ ,  $\phi \in S^*$ ,  $S^*$  is the set of  $\phi \in C$  such that with  $|\phi(0)| \geq H$  ( $H$  may be large), and that  $V(t, \phi)$  satisfies the following conditions:

(i) There exist continuous increasing functions  $a(s)$  and  $b(s)$ , satisfying  $a(s) > 0$ ,  $b(s) > 0$  for  $s \geq H$ , and  $a(s) \rightarrow \infty$  as  $s \rightarrow \infty$ , such that

$$a(|\phi(0)|) \leq V(t, \phi) \leq b(|\phi|), \text{ when } |\phi(0)| \geq H.$$

(ii) There exists a continuous and positive function  $w(s)$  such that

$$\dot{V}(t, \phi) \leq -w(|\phi(0)|) \text{ for } s \geq H.$$

(iii) Suppose that there exists an  $H_1 > 0$ ,  $H_1 > H$  such that

$$rL(\gamma^*) < H_1 - H,$$

where  $\gamma^* > 0$  is a constant which is determined in the following way: By the condition on  $V(t, \phi)$ , there exist  $\alpha > 0$ ,  $\beta > 0$ ,  $\gamma > 0$  such that  $b(H_1) \leq a(\alpha)$ ,  $b(\alpha) \leq a(\beta)$ ,  $b(\beta) \leq a(\gamma)$ .  $\gamma^*$  is defined by  $b(\gamma) \leq a(\gamma^*)$ . Under the above conditions, there exists a periodic solution of system (4) of period  $T$ . In particular, the relation  $rL(\gamma^*) < H_1 - H$  can be always satisfied if  $r$  is sufficiently small.

### 3. Main result

The main result of this paper is the following theorem:

**Theorem 2.** We assume that there are positive constants  $a, b, c, m, \delta, \gamma, \mu, T$  and  $L$  such that the following conditions hold:

(i)  $ab - c > 0$ ,  $f(0) = 0$ ,  $f(x) \operatorname{sgn} x > 0$  for all  $x \neq 0$ ,  $\sup\{f'(x)\} = c$  and

$$f(x) \operatorname{sgn} x \rightarrow \infty \text{ as } |x| \rightarrow \infty.$$

(ii)  $g(0) = 0$ ,  $\frac{g(y)}{y} \geq b$ , ( $y \neq 0$ ), and  $|g'(y)| \leq L$ ,  $0 \leq \psi(y) - a \leq \delta$ .

(iii)  $p(t, x, x(t-r), y, y(t-r), z)$  is periodic in  $t$ , of period  $T$  and

$$|p(t, x, x(t-r), y, y(t-r), z)| \leq m.$$

Then equation (2) has a periodic solution of period  $T$  provided  $r$  is small enough,

$$r < \min \left\{ \frac{ab - c}{b(L + 2\gamma)}, \frac{ab - c}{\mu L} \right\}$$

with  $\mu = \frac{ab + c}{2b}$  and  $\gamma = \frac{L}{2}(1 + \mu)$ .

**Proof.** Our main tool for the proof of Theorem 2 is a functional  $V(x_t, y_t, z_t)$  defined by:

$$V(x_t, y_t, z_t) = V_1(x_t, y_t, z_t) + V_2(x, y, z) + 1 + L \int_{-r}^0 \int_{t+s}^t |z(\theta)| d\theta ds, \quad (5)$$

where

$$\begin{aligned} V_1(x_t, y_t, z_t) = & \mu \int_0^x f(\xi) d\xi + f(x)y + \mu \int_0^y \psi(\eta) \eta d\eta + \int_0^y g(\eta) d\eta \\ & + \mu yz + \frac{1}{2} z^2 + \gamma \int_{-r}^0 \int_{t+s}^t z^2(\theta) d\theta ds, \end{aligned} \quad (6)$$

$$V_2(x, y, z) = \begin{cases} \frac{z}{M} \operatorname{sgn} x, & |x| \geq 1, |z| \leq M \\ \operatorname{sgn} z \cdot \operatorname{sgn} x, & |x| \geq 1, |z| \geq M \\ \frac{xz}{M}, & |x| \leq 1, |z| \leq M \\ x \cdot \operatorname{sgn} z, & |x| \leq 1, |z| \geq M, \end{cases} \quad (7)$$

$L, M$  ( $M > 1$ ),  $\mu$  and  $\gamma$  are some positive constants; the constants  $\mu$  and  $\gamma$  is defined in Theorem 2.

Now, since  $g(0) = f(0) = 0$ , it is immediate that  $V_1(0,0,0) = 0$ . In view of the assumptions  $\psi(y) \geq a$ ,  $y^{-1}g(y) \geq b$ , ( $y \neq 0$ ),  $f(x) \operatorname{sgn} x > 0$ , ( $x \neq 0$ ), and  $\sup\{f'(x)\} = c$ , we see that

$$\begin{aligned} V_1(x_t, y_t, z_t) & \geq \mu \int_0^x f(\xi) d\xi + f(x)y + \frac{1}{2} \mu a y^2 + \int_0^y \frac{g(\eta)}{\eta} \eta d\eta + \mu yz + \frac{1}{2} z^2 + \gamma \int_{-r}^0 \int_{t+s}^t z^2(\theta) d\theta ds \\ & \geq \mu \int_0^x f(\xi) d\xi + f(x)y + \frac{1}{2} \mu a y^2 + \frac{b}{2} y^2 + \mu yz + \frac{1}{2} z^2 + \gamma \int_{-r}^0 \int_{t+s}^t z^2(\theta) d\theta ds \\ & = \frac{1}{2b} [by + f(x)]^2 + \mu \int_0^x f(\xi) d\xi + \frac{1}{2} \mu a y^2 - \frac{1}{2b} f^2(x) + \mu yz + \frac{1}{2} z^2 + \gamma \int_{-r}^0 \int_{t+s}^t z^2(\theta) d\theta ds \\ & = \frac{1}{2b} [by + f(x)]^2 + \frac{1}{2by^2} \left[ 4 \int_0^x f(\xi) \left\{ \int_0^y (\mu b - f'(\xi)) \eta d\eta \right\} d\xi \right] \\ & \quad + \frac{1}{2} (\mu y + z)^2 + \frac{1}{2} \mu (a - \mu) y^2 + \gamma \int_{-r}^0 \int_{t+s}^t z^2(\theta) d\theta ds. \end{aligned} \quad (8)$$

Now, by help of the assumptions of Theorem 2, we have

$$a - \mu = a - \frac{ab+c}{2b} = \frac{ab-c}{2b} > 0,$$

$$\mu b - f'(x) = \frac{ab+c}{2} - f'(x) = \frac{ab+c-2f'(x)}{2} = \frac{ab-f'(x)}{2} \geq \frac{ab-c}{2} > 0,$$

and

$$\int_{-r}^0 \int_{t+s}^t z^2(\theta) d\theta ds \geq 0.$$

Hence, one can show from (8) that

$$V_1(x_t, y_t, z_t) \geq D_1 x^2 + D_2 y^2 + D_3 z^2 \geq D_4 (x^2 + y^2 + z^2), \quad (9)$$

where  $D_4 = \min\{D_1, D_2, D_3\}$ .

Next, it is also clear that the function  $V_2$  is continuous and satisfies

$$|V_2| \leq 1. \quad (10)$$

Now, in view of (5), (9) and (10), together, it can be easily seen that the functional  $V(x_t, y_t, z_t)$  satisfies the first part of condition (i) of Theorem 1. Similarly, making use of the assumptions of Theorem 2, it can also be shown that the functional  $V(x_t, y_t, z_t)$  satisfies the second part of condition (i) of Theorem 1.

Let  $\frac{d}{dt}V_1(x_t, y_t, z_t)$  denote the time derivative of functional  $V_1(x_t, y_t, z_t)$  along the solutions of (3). Then, by a straightforward calculation from (6) and (3), we observe that

$$\begin{aligned} \frac{d}{dt}V_1(x_t, y_t, z_t) &= f'(x)y^2 + \mu z^2 - \mu y g(y) - \psi(y)z^2 + (\mu y + z) \int_{t-r}^t g'(y(s))z(s)ds \\ &\quad + \gamma z^2 r - \gamma \int_{t-r}^t z^2(\theta)ds + (\mu y + z) p(t, x, x(t-r), y, y(t-r), z). \end{aligned} \quad (11)$$

Making use of the assumption  $|g'(y)| \leq L$  and the inequality  $2|u||v| \leq u^2 + v^2$ , we have the following:

$$\mu y \int_{t-r}^t g'(y(s))z(s)ds \leq \frac{\mu L}{2} r y^2 + \frac{\mu L}{2} \int_{t-r}^t z^2(s)ds, \quad z \int_{t-r}^t g'(y(s))z(s)ds \leq \frac{L}{2} r z^2 + \frac{L}{2} \int_{t-r}^t z^2(s)ds.$$

By using the afore inequalities in (11), we get

$$\begin{aligned} \frac{d}{dt}V_1(x_t, y_t, z_t) &= f'(x)y^2 + \mu z^2 - \mu y g(y) - \psi(y)z^2 + \gamma z^2 r + \frac{\mu L}{2} r y^2 + \frac{L}{2} r z^2 \\ &\quad + \left( \frac{L}{2} + \frac{L\mu}{2} - \gamma \right) \int_{t-r}^t z^2(s)ds + (\mu y + z) p(t, x, x(t-r), y, y(t-r), z). \end{aligned} \quad (12)$$

Now, suppose that  $\sup\{f'(x)\} = c > 0$ ,  $\psi(y) \geq a$  and  $ab - c > 0$ . Then, (12) implies that

$$\begin{aligned} \frac{d}{dt}V_1(x_t, y_t, z_t) \leq & -\left[\mu \frac{g(y)}{y} - c - \frac{\mu L}{2}r\right]y^2 - \left(a - \mu - \frac{L}{2}r - \gamma\right)z^2 \\ & + \left(\frac{L}{2} + \frac{L\mu}{2} - \gamma\right) \int_{t-r}^t z^2(\theta) d\theta + |\mu y + z| |p(t, x, x(t-r), y, y(t-r), z)|. \end{aligned}$$

By use of  $\mu = \frac{ab+c}{2b}$ , it follows that

$$\begin{aligned} \frac{d}{dt}V_1(x_t, y_t, z_t) \leq & -\left[\mu \frac{g(y)}{y} - c - \frac{\mu L}{2}r\right]y^2 - \left[\frac{ab-c}{2b} - \left(\frac{L}{2} + \gamma\right)r\right]z^2 \\ & + \left(\frac{L}{2} + \frac{L\mu}{2} - \gamma\right) \int_{t-r}^t z^2(s) ds + (\mu|y| + |z|) |p(t, x, x(t-r), y, y(t-r), z)|. \end{aligned}$$

If we choose  $\gamma = \frac{L}{2} + \frac{L\mu}{2}$  and use the assumption  $|p(t, x, x(t-r), y, y(t-r), z)| \leq m$ , then we obtain

$$\frac{d}{dt}V_1(x_t, y_t, z_t) \leq -\left[\mu \frac{g(y)}{y} - c - \frac{\mu L}{2}r\right]y^2 - \left[\frac{ab-c}{2b} - \frac{L}{2}(\mu+2)r\right]z^2 + \mu m|y| + m|z|. \quad (13)$$

Now, a direct computation along  $V_2(x, y, z)$  and solution the  $(x(t), y(t), z(t))$  of system (3) yields:

$$\begin{aligned} \frac{d}{dt}V_2(x, y, z) = & \begin{cases} \frac{1}{M} \{-\psi(y)z - g(y) - f(x) + \int_{t-r}^t g'(y(s))z(s)ds + p(t, x, x(t-r), y, y(t-r), z)\} \operatorname{sgn} x, & |x| \geq 1, |z| \leq M \\ 0, & |x| \geq 1, |z| \geq M \\ \frac{yz}{M} + \frac{x}{M} \{-\psi(y)z - g(y) - f(x) + \int_{t-r}^t g'(y(s))z(s)ds + p(t, x, x(t-r), y, y(t-r), z)\}, & |x| \leq 1, |z| \leq M \\ y \operatorname{sgn} z, & |x| \leq 1, |z| \geq M. \end{cases} \end{aligned}$$

With help of the assumptions of Theorem 2, one can also obtain the following:



$$\frac{d}{dt}V_2(x, y, z) \leq \begin{cases} -\frac{1}{M}f(x)\operatorname{sgn} x + \{a + \delta + |g(y)| + m + L \int_{t-r}^t |z(s)|ds\}, & |x| \geq 1, |z| \leq M \\ 0, & |x| \geq 1, |z| \geq M \\ |y| + |g(y)| + a + \delta + |g(y)| + m + L \int_{t-r}^t |z(s)|ds, & |x| \leq 1, |z| \leq M \\ |y|, & |x| \leq 1, |z| \geq M. \end{cases}$$

First, we consider  $V = V(x_t, y_t, z_t)$  in the domain  $\max\{|y| - K, |z| - M\} \geq 0$ , where the constants  $K$  and  $M$  are large enough, which will be determined later. Hence, we have to consider the following two cases:

**Case1<sup>0</sup>**:  $|y| \geq K \geq 1$ , and  $x, z$  are arbitrary. Then, we have

$$\frac{d}{dt}V_2(x, y, z) \leq |y| + |g(y)| + a + \delta + m + L \int_{t-r}^t |z(s)|ds. \quad (14)$$

Now, by using (5), (13) and (14), we have

$$\begin{aligned} \frac{d}{dt}V(x_t, y_t, z_t) &\leq -\left[\mu y g(y) - c - \frac{\mu L}{2}r\right]y^2 - \left[\frac{ab-c}{2b} - \frac{L}{2}(\mu+2)r\right]z^2 \\ &\quad + (\mu m + 1)|y| + m|z| + |g(y)| + a + \delta + m + L \int_{t-r}^t |z(s)|ds. \end{aligned}$$

Therefore, one can obtain the following:

$$\begin{aligned} \frac{d}{dt}V(x_t, y_t, z_t) &\leq -\left[\mu y g(y) - \left(c + \frac{\mu L}{2}r\right)y^2\right] - \left[\frac{ab-c}{2b} - \frac{L}{2}(\mu+2)r\right]z^2 \\ &\quad + |g(y)| + (\mu m + 1)|y| + m|z| + a + \delta + m + Lr|z| - L \int_{t-r}^t |z(s)|ds + L \int_{t-r}^t |z(s)|ds \\ &= -\left[\mu y g(y) - |g(y)| - \left(c + \frac{\mu L}{2}r\right)y^2\right] - \left[\frac{ab-c}{2b} - \frac{L}{2}(\mu+2)r\right]z^2 \\ &\quad + (\mu m + 1)|y| + m|z| + a + \delta + m + Lr|z|. \end{aligned}$$

Now, we consider the term:

$$\mu y g(y) - |g(y)| - \left(c + \frac{\mu L}{2}r\right)y^2.$$

Let us define

$$h = \frac{ab + 3c}{2(ab + c)} < 1.$$

Then, there exists a constant  $K_1$ , ( $K_1 > 1$ ), satisfying  $\left(1 - \frac{1}{\mu|y|}\right) \geq h$  for  $|y| \geq K_1$ . Therefore, when  $|y| \geq K_1$ , it follows that

$$\begin{aligned} \mu y g(y) - |g(y)| - \left(c + \frac{\mu L}{2} r\right) y^2 &= \mu \frac{g(y)}{y} \left(1 - \frac{1}{\mu|y|}\right) y^2 - \left(c + \frac{\mu L}{2} r\right) y^2 \\ &\geq \frac{ab+c}{2b} b \frac{ab+3c}{2(ab+c)} y^2 - \left(c + \frac{\mu L}{2} r\right) y^2 = \left(\frac{ab-c}{4} - \frac{L\mu}{2} r\right) y^2. \end{aligned}$$

Thus, one can arrive:

$$\begin{aligned} \frac{d}{dt} V(x_t, y_t, z_t) &\leq -\left(\frac{ab-c}{4} - \frac{L\mu}{2} r\right) y^2 - \left[\frac{ab-c}{2b} - \frac{L}{2}(\mu+2)r\right] z^2 \\ &\quad + (\mu m + 1)|y| + (m + Lr)|z| + (a + \delta + m)|y|. \end{aligned}$$

The above inequality implies

$$\frac{d}{dt} V(x_t, y_t, z_t) \leq -\delta_1 (y^2 + z^2) + (\mu m + 1)|y| + (m + Lr)|z| + (a + \delta + m)|y|$$

for a positive constant  $\delta_1$  provided that

$$r < \min\left\{\frac{ab-c}{bL(\mu+2)}, \frac{ab-c}{2\mu L}\right\}.$$

Now, let

$$\tau_1 = \max\{\mu m + \delta + a + m + 1, m + Lr\}.$$

Hence, we have

$$\begin{aligned} \frac{d}{dt} V(x_t, y_t, z_t) &\leq -\delta_1 (y^2 + z^2) + \tau_1(|y| + |z|) \\ &= -\frac{\delta_1}{2} (y^2 + z^2) - \frac{\delta_1}{2} \left[ \left(|y| - \frac{\tau_1}{\delta_1}\right)^2 + \left(|z| - \frac{\tau_1}{\delta_1}\right)^2 - 2\frac{\tau_1^2}{\delta_1^2} \right] \leq -\frac{\delta_1}{2} (y^2 + z^2) \end{aligned}$$

provided that  $|y| \geq (\sqrt{2} + 1)\tau_1\delta_1^{-1}$ .

Let  $K = \max\{(\sqrt{2} + 1)\tau_1\delta_1^{-1}, K_1\}$ . If  $|y| \geq K$ , then we have

$$\frac{d}{dt} V(x_t, y_t, z_t) \leq -\frac{\delta_1}{2} (y^2 + z^2).$$

**Case 2<sup>0</sup>**):  $|z| \geq M$ , and  $x, y$  are arbitrary. Then, clearly, it follows that

$$\frac{d}{dt}V_2(x, y, z) \leq |y|.$$

Next, by following a similar way as afore mentioned, choosing  $\gamma = \frac{L}{2} + \frac{L\mu}{2}$  and taking

$$r < \min \left\{ \frac{ab-c}{bL(\mu+2)}, \frac{ab-c}{2\mu L} \right\},$$

one can easily obtain that

$$\begin{aligned} \frac{d}{dt}V(x_t, y_t, z_t) &\leq -\delta_2 (y^2 + z^2) + (\mu m + 1)|y| + (m + Lr)|z| \\ &\leq -\delta_2 (y^2 + z^2) + \tau_2(|y| + |z|) \leq -\frac{\delta_2}{2} (y^2 + z^2) \end{aligned}$$

for some positive constants  $\delta_2$  and  $\tau_2$  provided that  $|z| \geq M = K$ .

Finally, we consider  $V$  in  $\max\{|y| - K, |z| - M\} \leq 0$ . Now, we assume that  $|x| \geq H > 1$ , where the constant  $H$  will be determined later. Then, we obtain

$$\frac{d}{dt}V_2(x, y, z) \leq -\frac{1}{M} f(x) \operatorname{sgn} x + \max_{|y| \leq K} |g(y)| + \{a + \delta + m + L \int_{t-r}^t |z(s)| ds\}. \quad (15)$$

Hence, we can conclude from (5), (13) and (15) that

$$\begin{aligned} \frac{d}{dt}V(x_t, y_t, z_t) &\leq -\left[ \frac{ab-c}{2} - \frac{\mu L}{2} r \right] y^2 - \left[ \frac{ab-c}{2b} - \frac{L}{2} (\mu+2)r \right] z^2 + \mu m |y| + m |z| \\ &\quad - \frac{1}{M} f(x) \operatorname{sgn} x + \max_{|y| \leq K} |g(y)| + \{a + \delta + m + L \int_{t-r}^t |z(s)| ds\} + Lr|z| - L \int_{t-r}^t |z(s)| ds \\ &\leq -\left[ \frac{ab-c}{2} - \frac{\mu L}{2} r \right] y^2 - \left[ \frac{ab-c}{2b} - \frac{L}{2} (\mu+2)r \right] z^2 \\ &\quad - \frac{1}{M} f(x) \operatorname{sgn} x + \max_{|y| \leq K} |g(y)| + \mu m |y| + m |z| + (a + \delta + m + LMr) \\ &\leq -\left[ \frac{ab-c}{2} - \frac{\mu L}{2} r \right] y^2 - \left[ \frac{ab-c}{2b} - \frac{L}{2} (\mu+2)r \right] z^2 \end{aligned}$$

$$-\frac{1}{M} f(x) \operatorname{sgn} x + \max_{|y| \leq K} |g(y)| + (a + \delta + m + LMr + \mu mK + mM).$$

Now, since  $f(x) \operatorname{sgn} x \rightarrow \infty$  as  $|x| \rightarrow \infty$  and  $|x| \geq H > 1$ , we can write that

$$f(x) \operatorname{sgn} x \geq 2M \left\{ \max_{|y| \leq K} |g(y)| + a + \delta + m + LMr + \mu mK + mM \right\}.$$

Therefore

$$-\frac{f(x) \operatorname{sgn} x}{2M} + \max_{|y| \leq K} |g(y)| + (a + \delta + m + LMr + \mu mK + mM) \leq 0.$$

Now, in view of the above discussion, we have

$$\frac{d}{dt} V(x_t, y_t, z_t) \leq -\left[ \frac{ab-c}{2} - \frac{\mu L}{2} r \right] y^2 - \left[ \frac{ab-c}{2b} - \frac{L}{2} (\mu + 2)r \right] z^2 - \frac{1}{2M} f(x) \operatorname{sgn} x.$$

Subject to the above discussion, one can that there exists a positive constant  $R$ , which is large enough, such that

$$\frac{d}{dt} V(x_t, y_t, z_t) \leq -w(u) \text{ for } u^2 \geq R^2,$$

where  $u = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ . Therefore,  $V(x_t, y_t, z_t)$  satisfies all conditions of Theorem 1.

Thus, the proof of Theorem 2 is now complete.

**Example .** Now, we consider nonlinear third order delay differential equation:

$$\begin{aligned} x'''(t) + \left( 4 + \frac{1}{1 + (x'(t))^2} \right) x''(t) + 4x'(t-r) + \sin x'(t-r) + 11x(t-r) \\ = \frac{1}{2 + \cos t + x^2(t) + x^2(t-r) + (x'(t))^2 + (x'(t-r))^2 + (x''(t))^2} \end{aligned} \quad (16)$$

with the associated system

$$\begin{aligned} x'(t) &= y(t), \quad y'(t) = z(t), \\ z'(t) &= -\left( 4 + \frac{1}{1 + y^2(t)} \right) z(t) - (4y(t) + \sin y(t)) \\ &\quad - 11x(t) + 11 \int_{t-r}^t y(s) ds + \int_{t-r}^t (4 + \cos y(s)) z(s) ds \\ &\quad + \frac{1}{2 + \cos t + x^2(t) + x^2(t-r) + y^2(t) + y^2(t-r) + z^2(t)}. \end{aligned}$$

Now, one can easily observe the following, respectively:

$$\psi(y) = 4 + \frac{1}{1+y^2}, \quad 0 \leq \psi(y) - 4 \leq 1,$$

$$a = 4, \quad \delta = 1, \quad g(y) = 4y + \sin y, \quad g(0) = 0,$$

$$\frac{g(y)}{y} = 4 + \frac{\sin y}{y} \geq 3, \quad b = 3, \quad g'(y) = 4 + \cos y,$$

$$|g'(y)| = |4 + \cos y| \leq 5 = L,$$

$$f(x) = 11x, \quad f(0) = 0, \quad f(x) \operatorname{sgn} x = 11x \operatorname{sgn} x > 0,$$

$$f(x) \operatorname{sgn} x = 11x \operatorname{sgn} x \rightarrow \infty \quad \text{as } |x| \rightarrow \infty,$$

$$f'(x) = 11, \quad c = 11, \quad ab - c = 1 > 0,$$

$$h = \frac{ab + 3c}{2(ab + c)} = \frac{45}{46} < 1,$$

$$p(t, x, x(t-r), y, y(t-r), z)$$

$$= \frac{1}{2 + \cos t + x^2 + x^2(t-r) + y^2 + y^2(t-r) + z^2} \leq 1 = m,$$

$$p(t + 2\pi, x, x(t-r), y, y(t-r), z)$$

$$= \frac{1}{2 + \cos(t + 2\pi) + x^2 + x^2(t-r) + y^2 + y^2(t-r) + z^2}$$

$$= \frac{1}{2 + \cos t + x^2 + x^2(t-r) + y^2 + y^2(t-r) + z^2}$$

$$= p(t, x, x(t-r), y, y(t-r), z), \quad T = 2\pi.$$

Thus, all the assumptions of Theorem 2 hold. This fact shows that equation (16) has a periodic solution of period  $T$ ,  $T = 2\pi$ .

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Inverse limits on  $[0,1]$  using sequence of  $N$ -type maps \*Xianjiu Huang<sup>1†</sup>; Chuanxi Zhu<sup>1</sup>; Xi Wen <sup>2</sup>

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**Abstract**

We examine the inverse limits generated by inverse sequences on  $[0,1]$  with  $N$ -type bonding maps chose from a four-parameter family of piecewise linear continuous functions. We analyze the continua generated by these sequences and obtain sufficient conditions for these sequences to give rise to indecomposable inverse limits.

**Keyword:** Inverse limits;  $N$ -type; Continua; Indecomposable; Two-pass condition**AMS(2000) Subject Classification:** 54F15; 37E05; 37B45**§1 Introduction and some basic terminologies**

Inverse limits, besides being of intrinsic interest to topologists, can often be used to represent attractors of dynamical systems. For example, the inverse limit space with a single full unimodal bonding map is homeomorphic to the attracting set of Smales horseshoe. In a sequence of papers [1,2,3], Ingram conducted an extensive investigation of inverse limit spaces generated by single bonding maps chosen from the two-parameter family  $\mathcal{G}=\{g_{b,c} \mid 0 \leq b \leq 1, 0 < c < 1\}$  where  $g_{b,c}$  is given by

$$g_{b,c}(x) = \begin{cases} \frac{1-b}{c}x + b & \text{if } x \leq c \\ \frac{1-x}{1-c} & \text{if } x \geq c \end{cases}.$$

Brian Raines [4] considered inverse limits of sequences of functions  $g_{b_i,c_i}$  from the family  $\mathcal{G}$ . He gave several sufficient conditions such that the inverse limit generated by  $g_{b_i,c_i}$  is indecomposable.

In this paper, we extended the two-parameter family  $\mathcal{G}=\{g_{b,c} \mid 0 \leq b \leq 1, 0 < c < 1\}$  to four-parameter family  $\mathcal{G}'=\{g_{a,b,c,d} \mid 0 \leq a, d \leq 1, 0 < b < c < 1\}$  where  $g_{a,b,c,d}$  is given by

$$g_{a,b,c,d}(x) = \begin{cases} \frac{1-a}{b}x + a & \text{if } x \in [0, b] \\ \frac{x-c}{b-c} & \text{if } x \in [b, c] \\ \frac{d}{1-c}(x - c) & \text{if } x \in [c, 1] \end{cases}.$$

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We consider inverse limit of sequences of function  $g_{a_i, b_i, c_i, d_i}$ . By looking at the behavior of certain iterates of these complicated  $N$ -type maps on particular subintervals of  $[0, 1]$ , we can study the inverse limit generated by these sequences of function and obtain several sufficient conditions such that the inverse limit generated by  $g_{a_i, b_i, c_i, d_i}$  is indecomposable.

Let  $X_0, X_1, \dots$  be a sequence of metric spaces and  $f_0, f_1, \dots$  be a sequence of maps (continuous functions) such that  $f_i : X_{i+1} \rightarrow X_i$  for each  $i \in \mathbb{N}$ . Define the inverse limit of the inverse sequence  $(X_i, f_i)$  by the following:

$$\varprojlim (X_i, f_i) = \{x = (x_0, x_1, \dots) \in \prod_{i=0}^{\infty} X_i \mid f_i(x_{i+1}) = x_i, \text{ for } i = 0, 1, \dots\}$$

with metric given by  $d(x, y) = \sum_{i=0}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$ , where  $d_i$  is a metric for  $X_i$  bounded by one. Each map  $f_i$  is called a bonding map. We denote the projection mapping from  $\prod_{i \in \mathbb{N}} X_i$  to  $X_i$  by  $\pi_i$  where  $\pi_i(x_1, x_2, x_3, \dots) = x_i$ . Often it will be convenient to consider, for  $j > 1$ , the map  $f_i^j = f_i \circ f_{i+1} \circ \dots \circ f_{j-1} : X_j \rightarrow X_i$ . Throughout this paper, a continuum is a nonempty, compact, connected metric space.

A continuum is decomposable if it is the union of two of its proper subcontinua, otherwise, it is indecomposable. An inverse sequence,  $\{X_i, f_i\}$  is said to satisfy the *two-pass condition* provided that for each positive integer  $i$  whenever  $A_{i+1}$  and  $B_{i+1}$  are subcontinua of  $X_{i+1}$  such that  $X_{i+1} = A_{i+1} \cup B_{i+1}$ , then  $f_i[A_{i+1}] = X_i$  or  $f_i[B_{i+1}] = X_i$ . The following theorems are well-known (see [5]).

**Theorem 1.1** Suppose  $\{X_i, f_i\}$  is an inverse sequence that satisfies the *two-pass condition*. Then  $\varprojlim \{X_i, f_i\}$  is indecomposable.

**Theorem 1.2** (The Subsequence Theorem). Suppose that  $n_1, n_2, n_3, \dots$  is an increasing sequence of positive integers. Then  $\varprojlim \{X_i, f_i\}$  is homeomorphic to  $\varprojlim \{X_{n_j}, f_{n_j}^{n_j}\}$ .

Later we will use the subsequence theorem to show an inverse limit is indecomposable by showing that  $f_i^j$  meets the *two-pass condition*, for some  $j > i$ .

## §2 Main results and their proofs

Firstly we assume that  $f_i = g_{a_i, b_i, c_i, d_i}$  is a sequence of maps with  $b_i = b, c_i = c \in (0, 1)$  for every  $i \in \mathbb{N}$ . This implies that if  $p = \frac{c}{1-b+c}$  then  $f_i(p) = p$  for every  $i \in \mathbb{N}$ . The fact that each of these maps share the same fixed point makes determining when they give rise to an indecomposable inverse limit easier.

**Theorem 2.1** Suppose that for every  $i \in \mathbb{N}, a_i > p > d_i$ . Then  $\varprojlim \{[0, 1], f_i\}$  is decomposable.

**Proof** Notice that  $f_i([0, p]) = [p, 1]$  and that  $f_i([p, 1]) = [0, p]$  for every  $i \in \mathbb{N}$ . For  $n \in \mathbb{N}$  let  $X_{2n-1} = Y_{2n} = [0, p]$  and  $Y_{2n-1} = X_{2n} = [p, 1]$ . Let  $g_i = f_i|_{X_{i+1}}$  and  $h_i = f_i|_{Y_{i+1}}$ . Then

$g_i(X_{i+1}) = X_i$  and  $h_i(Y_{i+1}) = Y_i$ . So  $\varprojlim\{X_i, g_i\}$  and  $\varprojlim\{Y_i, h_i\}$  are proper subcontinua of  $\varprojlim\{[0, 1], f_i\}$  and  $\varprojlim\{[0, 1], f_i\} = \varprojlim\{X_i, g_i\} \cup \varprojlim\{Y_i, h_i\}$ . Thus  $\varprojlim\{[0, 1], f_i\}$  is decomposable.

In [6], Morton Brown shows that if  $\lim_{i \rightarrow \infty} f_i = F$  at or above a certain rate, then  $\varprojlim\{X_i, f_i\}$  is homeomorphic to  $\varprojlim\{X_i, F\}$ . In [1], Ingram shows that  $\varprojlim\{[0, 1], g_{b,c}\}$  is decomposable whenever  $b = p = \frac{1}{2-c}$  and in [4] Brian Raines shows that if  $b'_i$ s are converging from below to  $\frac{1}{2-c}$  fast enough then  $\varprojlim\{[0, 1], g_{b_i, c_i}\}$  is decomposable, even though each  $b_i$  is less than  $\frac{1}{2-c}$ . As a result, he give a sufficient conditions for these inverse limits to be indecomposable. Now we will obtain a similar result for  $f_i$ .

**Theorem 2.2** If one of the following two holds, then  $\varprojlim\{[0, 1], f_i\}$  is indecomposable.

- (1) there is an  $\alpha < p = \frac{c}{1-b+c}$  with  $a_i \leq \alpha$  for infinitely many  $i \in N$ ;
- (2) there is an  $\beta > p = \frac{c}{1-b+c}$  with  $d_i \geq \beta$  for infinitely many  $i \in N$ .

**Proof** We only prove the first case, the second case is similar.

First notice that  $f_i^{-1}(x) \cap [b, c] = \{x(b-c) + c\}$ , for all  $i \in N$  and  $x \in [0, 1]$ . Since this is singleton we will abuse the notation in the future and just write  $f_i^{-1}(x) \cap [b, c] = x(b-c) + c$ . So if  $x, y \in [0, 1]$ , then  $|f_i^{-1}(x) \cap [b, c] - f_j^{-1}(y) \cap [b, c]| = |b-c| \cdot |y-x|$ . For a fixed  $j \in N$ , let  $u_i^j = f_i^{-1}(u_{i-1}^j) \cap [b, c]$ ,  $u_{i-1}^j = b$  and  $v_i^j = f_i^{-1}(v_{i-1}^j) \cap [b, c]$ ,  $v_{i-1}^j = c$ , for all  $i \in N, i \geq j$ . Then  $|u_{j+n-1}^j - p| = |b-c|^n |b-p|$  and  $|v_{j+n-1}^j - p| = |b-c|^n |c-p|$ . Since  $|b-c| < 1$ , the sequence  $u_j^j, u_{j+1}^j, \dots$  and the sequence  $v_j^j, v_{j+1}^j, \dots$  both converges to  $p$ .

Let  $n \in N$  such that  $|u_m^j - p| < |p - \alpha|$  and  $|v_m^j - p| < |p - \alpha|$  for all  $m > n > j$ . Then  $u_m^j \in (\alpha, 1)$  and  $v_m^j \in (\alpha, 1)$  for  $m > n$ . Notice that since for infinitely many  $i \in N, a_i = f_i(0) < \alpha$ , every point,  $x$ , of  $(\alpha, 1) \subseteq (f_i(0), f_i(b))$  has an inverse image,  ${}^i\hat{x}$ . Choose  $i \in N$  large enough so that  $i > n + 1$ , and  $a_i \leq \alpha$ . Then  ${}^i\hat{u}_i^j$  and  ${}^i\hat{v}_i^j$  are both in  $(0, c)$ , with the interval between them mapped across  $[0, 1]$  under the map  $f_j^{i+1}$ . Since  $f_i([b, 1]) = [0, 1]$  for all  $i \in N$ ,  $f_j^{i+1}$  is a *two-pass* map, for infinitely many  $i \in N, i > j$ .

Let  $A, B$  be subcontinua of  $X = \varprojlim\{[0, 1]\}$  such that  $A \cup B = X$ . Since  $A$  and  $B$  are both connected,  $\pi_i[A]$  and  $\pi_i[B]$  are subintervals of  $[0, 1]$  for every  $i \in N$ . Since  $A \cup B = X$ ,  $[0, 1] = \pi_i[A] \cup \pi_i[B]$ . If either  $\pi_i[A]$  or  $\pi_i[B]$  contains  $[b, 1]$  for infinitely many  $i \in N$ , then, since  $f_i([b, 1]) = [0, 1]$  for all  $i \in N$ , either  $\pi_i[A]$  or  $\pi_i[B]$  is  $[0, 1]$ . This would imply that either  $A = X$  or  $B = X$ . So suppose that for some  $n \in N$ , if  $m > n$  then neither  $\pi_m[A]$  nor  $\pi_m[B]$  contains  $[b, 1]$ . Hence we have that for infinitely many  $i \in N$ , either  $\pi_i[A]$  or  $\pi_i[B]$  contains  $[0, b]$ , without loss of generality, assume  $[0, b] \subseteq \pi_i[A]$  for infinitely many  $i \in N$ . Let  $j \in N$ , and define the sequence of points  $\{{}^i\hat{u}_i^j\}_{i \in N}$  and  $\{{}^i\hat{v}_i^j\}_{i \in N}$  as above. Choose  $n, i \in N$  as was done above, such that  $\pi_i[A]$  contains  $[0, b]$ . Then  ${}^i\hat{u}_i^j$  and  ${}^i\hat{v}_i^j$  are both in  $\pi_i[A]$ , and  $\pi_j[A] = f_j^i(\pi_i[A]) = [0, 1]$ . Since this is true for any  $j \in N$ ,  $A = X$ . Either  $A$  or  $B$  cannot be a proper subcontinuum of  $X$ , so  $X$  is indecomposable.



However the converse to Theorem 2.2 is not true.

**Theorem 2.3** If  $\lim_{i \rightarrow \infty} a_i = \lim_{i \rightarrow \infty} d_i = p$  but for infinitely many  $n \in N$ , one of the following two holds, then  $\varprojlim \{[0, 1], f_i\}$  is indecomposable.

- (1)  $a_{2n} \leq p - (c - b)^n(p - b)$ ;
- (2)  $d_{2n} \geq p + (c - b)^n(c - p)$ .

**Proof** We only prove the first case, the second case is similar.

Let  $A$  and  $B$  be proper subcontinua of  $X = \varprojlim \{[0, 1], f_i\}$  with  $X = A \cup B$ ,  $X \neq A$  and  $X \neq B$ . Since  $A$  and  $B$  are proper subcontinua of  $X$  there must be some  $n \in N$  with  $\pi_m[A] \neq [0, 1]$  and  $\pi_m[B] \neq [0, 1]$  for all  $m > n$ . Otherwise either  $A = X$  or  $B = X$ . In fact since each  $f_i$  maps  $[b, 1]$  across  $[0, 1]$ , there must be some  $m \in N$  with  $\pi_k[A] \not\supseteq [b, 1]$  and  $\pi_k[B] \not\supseteq [b, 1]$  for all  $k \geq m$ . Choose  $q \in N, q > m$  such that  $a_{2q} \leq p - (c - b)^q(p - b)$ . Since neither  $A$  nor  $B$  contain  $[b, 1]$  in their  $2q$ th projection, one must contain  $[0, b]$  in its  $2q$ th projection, without loss of generality, assure that  $A$  is this subcontinuum. Then  $\pi_q[A] = f_q^{2q+1}(\pi_{2q}[A]) \supseteq f_q^{2q+1}([0, b])$ . Now building two sequences of pre-images of the critical points treating the  $q$ th factor space as the first. Let  $u_i^q = f_i^{-1}(u_{i-1}^q) \cap [b, c]$ ,  $u_{q-1}^q = b$  and  $v_i^q = f_i^{-1}(v_{i-1}^q) \cap [b, c]$ ,  $v_{q-1}^q = c$ . Using the notation of the proof to the previous theorem, it is easy to see that both  ${}^{2q}\hat{u}_{2q}^q$  and  ${}^{2q}\hat{v}_{2q}^q$  are in  $[0, c] \subseteq \pi_{2q}[A]$ . So  $\pi_q[A] = f_q^{2q+1}(\pi_{2q}[A]) \supseteq f_q^{2q+1}([0, c]) = [0, 1]$ . This contradicts our observation that,  $\pi_k[A] \not\supseteq [b, 1]$  and  $\pi_k[B] \not\supseteq [b, 1]$  for all  $k \geq m$ , since  $A$  and  $B$  are proper subcontinua of  $X$  and  $q$  was chosen to be greater than  $n$ . Hence  $X$  is indecomposable.

So if  $a_i > p$  or  $d_i < p$  for co-finitely many  $i \in N$ , or if  $a_i \rightarrow p$  or  $d_i \rightarrow p$  fast enough, then the inverse limit is decomposable. But if  $a_i < \alpha < p$  or  $d_i > \beta > p$  for infinitely many  $i \in N$  or if  $a_i \rightarrow p$  or  $d_i \rightarrow p$  slowly, then the inverse limit is indecomposable.

Finally, we present a sufficient condition for sequences of  $g_{a_i, b_i, c_i, d_i}$  maps with varying critical points to give rise to indecomposable inverse limits.

**Theorem 2.4** If for infinitely many  $i \in N$ , either  $b_i > a_i$  or  $d_i > c_i$ . Then  $\varprojlim \{[0, 1], f_i\}$  is indecomposable.

**Proof** Suppose  $b_i > a_i$  for infinitely many  $i \in N$ . Let  $\{n_i\}_{i \in N}$  be a sequence in  $N$  such that  $a_{n_i} \leq b_{n_i}$  for all  $i \in N$ . Define  $A_{n_i+1} = [0, b_{n_i+1}]$  and  $B_{n_i+1} = [b_{n_i+1}, 1]$ . Then  $f_{n_i}(B_{n_i+1}) = [0, 1]$  for all  $i \in N$ , and  $f_{n_i}(A_{n_i+1}) = [a_{n_i}, 1] \supseteq [b_{n_i}, 1]$ . Thus  $f_{n_i-1}^{n_i+1}(A_{n_i+1}) \supseteq f_{n_i-1}([b_{n_i}, 1]) = [0, 1]$ , and  $f_{n_i-1}^{n_i+1}$  is a *two-pass* map. By Theorem 1.1 and the Subsequence Theorem,  $\varprojlim \{[0, 1], f_i\}$  is indecomposable.

The case of  $d_i > c_i$  for infinitely many  $i \in N$  is similar, we omit it.

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# Weighted composition operators between $\mu$ -Bloch type spaces on the polydisc of $\mathbf{C}^n$

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**Abstract:** In this paper, we study the weighted composition operators  $T_{\psi,\varphi}f = \psi f \circ \varphi$  between  $\mu$ -Bloch type spaces on the polydisc of  $\mathbf{C}^n$ . For normal functions  $\mu$  and  $\omega$  on  $[0, 1)$ , we characterize the boundedness and the compactness of  $T_{\psi,\varphi} : \mathcal{B}_\mu(\mathbf{D}^n) \longrightarrow \mathcal{B}_\omega(\mathbf{D}^n)$  (respectively,  $\mathcal{B}_{\mu,0}(\mathbf{D}^n) \longrightarrow \mathcal{B}_{\omega,0}(\mathbf{D}^n)$ ).

**Keywords:** Weighted composition operator,  $\mu$ -Bloch type space, Polydisc

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## 1 Introduction

Let  $\mathbf{D}$  be the unit disc in the complex plane  $\mathbf{C}$ , and let  $\mathbf{D}^n = \{z = (z_1, z_2, \dots, z_n) \in \mathbf{C}^n : z_j \in \mathbf{D}, j = 1, 2, \dots, n\}$  be the polydisc of  $\mathbf{C}^n$ .  $H(\mathbf{D}^n)$  and  $H(\mathbf{D}^n, \mathbf{D}^n)$  denote the family of all holomorphic functions and holomorphic self-mappings on  $\mathbf{D}^n$  respectively.

A positive continuous function  $\mu$  on  $[0, 1)$  is called normal if there are three constants  $0 \leq \delta < 1$  and  $0 < a < b < \infty$  such that

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r \rightarrow 1^-} \frac{\mu(r)}{(1-r)^a} = 0,$$

$$\frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r \rightarrow 1^-} \frac{\mu(r)}{(1-r)^b} = \infty.$$

The normal function  $\mu$ , as a weight, has been usually used to defined the weighted Bergman space or mixed norm space. For example,  $\mu(r) = (1-r^2)^p$  with  $0 < p < \infty$ ,  $\mu(r) = (1-r^2) \log \frac{2}{1-r^2}$  and  $\mu(r) = 1/\{\log \log e^2(1-r^2)^{-1}\}$  are all normal weights. Given  $\mu$  we will extend it to  $\mathbf{D}$  by  $\mu(z) = \mu(|z|)$ . A function  $f \in H(\mathbf{D}^n)$  is said to belong to the  $\mu$ -Bloch type space  $\mathcal{B}_\mu(\mathbf{D}^n)$  if

$$\|f\|_{\mathcal{B}_\mu} = \sup_{z \in \mathbf{D}^n} \sum_{j=1}^n \mu(z_j) \left| \frac{\partial f}{\partial z_j}(z) \right| < \infty.$$

It is easy to check that  $\mathcal{B}_\mu(\mathbf{D}^n)$  is a Banach space under the norm  $\|f\|_\mu = |f(0)| + \|f\|_{\mathcal{B}_\mu}$ . When  $\mu(r) = 1-r^2$  and  $\omega(r) = (1-r^2)^{1-p}$  with  $p \in (0, 1)$ , two typical normal weights, the induced spaces  $\mathcal{B}_\mu(\mathbf{D}^n)$  are the Bloch space and Lipschitz type space, respectively.

Notice that  $\lim_{z \rightarrow \partial \mathbf{D}^n} \sum_{j=1}^n \mu(z_j) \left| \frac{\partial f}{\partial z_j}(z) \right| = 0$  will imply  $f$  is a constant when  $n \geq 2$  as pointed out by Timoney in [1]. Meanwhile,  $\mathcal{B}_\mu(\mathbf{D}^n)$  contains all polynomials in  $n$  complex variables. Hence, instead of using the limit in the above expression we define the little  $\mu$ -Bloch type space as the following. The little  $\mu$ -Bloch type space  $\mathcal{B}_{\mu,0}(\mathbf{D}^n)$  is the closure of all polynomials in  $n$  complex variables in the Banach space  $\mathcal{B}_\mu(\mathbf{D}^n)$ . Obvious,  $\mathcal{B}_{\mu,0}(\mathbf{D}^n)$  is a closed subspace of  $\mathcal{B}_\mu(\mathbf{D}^n)$ , and  $\mathcal{B}_{\mu,0}(\mathbf{D}^n)$  is a Banach space also.

Let  $\psi \in H(\mathbf{D}^n)$ ,  $\varphi(z) = (\varphi_1(z), \varphi_2(z), \dots, \varphi_n(z)) \in H(\mathbf{D}^n, \mathbf{D}^n)$ . The weighted composition operator  $T_{\psi,\varphi}$  is defined by

$$T_{\psi,\varphi}f(z) = \psi(z)f(\varphi(z)), \quad f \in H(\mathbf{D}^n), \quad z \in \mathbf{D}^n.$$

It is easy to see that an operator defined in this manner is linear. We can regard this operator as a generalization of a multiplication operator  $M_\psi$  and a composition operator  $C_\varphi$ . In one complex variable case, the behavior of these operators have been studied extensively in [2-8]. In several complex variables case, Zhang and Xiao [9] got the characterization on  $\varphi$  and  $\psi$  for which the induced weighted composition operator is bounded or compact from  $\mathcal{B}_\mu(\mathbf{B})$  to  $\mathcal{B}_\omega(\mathbf{B})$  in the unit ball. Luo and Ueki [10] discussed the same problems on the Bergman and Hardy spaces. And in the polydisc, Zhou and Shi [11] and Zhou [12] obtained the sufficient and necessary condition on  $\varphi$  such that  $C_\varphi$  is bounded or compact on  $\mathcal{B}_{(1-r^2)}(\mathbf{D}^n)$  and  $\mathcal{B}_{(1-r^2)^p}(\mathbf{D}^n)$  ( $0 < p \leq 1$ ) respectively. Hu [13] investigated the same problems between different  $\mu$ -Bloch type spaces. And also, Xu and Liu [14] characterized the boundedness and the compactness of  $T_{\psi,\varphi}$  between  $\mathcal{B}_{(1-r^2)^p}(\mathbf{D}^n)$  and  $\mathcal{B}_{(1-r^2)^q}(\mathbf{D}^n)$  for  $0 \leq p, q \leq \infty$ .

In what follows we always suppose that both of  $\mu$  and  $\omega$  are normal functions on  $[0, 1)$ . The purpose of this work is to obtain the sufficient and necessary conditions on  $\psi \in H(\mathbf{D}^n)$  and  $\varphi(z) \in H(\mathbf{D}^n, \mathbf{D}^n)$ , for which the operator  $T_{\psi,\varphi} : \mathcal{B}_\mu(\mathbf{D}^n) \longrightarrow \mathcal{B}_\omega(\mathbf{D}^n)$  (respectively,  $\mathcal{B}_{\mu,0}(\mathbf{D}^n) \longrightarrow \mathcal{B}_{\omega,0}(\mathbf{D}^n)$ ) is bounded or compact. Our work will generalize [5, 7, 11-14], and main results are the following:

**Theorem A** Let  $\psi \in H(\mathbf{D}^n)$  and  $\varphi \in H(\mathbf{D}^n, \mathbf{D}^n)$ . Then

(1)  $T_{\psi,\varphi} : \mathcal{B}_\mu(\mathbf{D}^n) \longrightarrow \mathcal{B}_\omega(\mathbf{D}^n)$  is a bounded operator if and only if the following are all satisfied:

$$(i) \sup_{z \in \mathbf{D}^n} \sum_{j=1}^n \omega(z_j) \left| \frac{\partial \psi}{\partial z_j}(z) \right| \left( 1 + \sum_{k=1}^n \int_0^{|\varphi_k(z)|} \frac{dt}{\mu(t)} \right) < \infty, \quad (1.1)$$

$$(ii) \sup_{z \in \mathbf{D}^n} \sum_{j,k=1}^n \frac{\omega(z_j)}{\mu(\varphi_k(z))} \left| \psi(z) \frac{\partial \varphi_k}{\partial z_j}(z) \right| < \infty. \quad (1.2)$$

(2)  $T_{\psi,\varphi} : \mathcal{B}_{\mu,0}(\mathbf{D}^n) \longrightarrow \mathcal{B}_{\omega,0}(\mathbf{D}^n)$  is a bounded operator if and only if (1.1) and (1.2) hold, and for each multi-index  $\alpha$ ,  $\psi \varphi^\alpha \in \mathcal{B}_{\omega,0}(\mathbf{D}^n)$ .

**Theorem B** Let  $\psi \in H(\mathbf{D}^n)$ ,  $\varphi \in H(\mathbf{D}^n, \mathbf{D}^n)$  and  $\int_0^1 \frac{dt}{\mu(t)} = \infty$ . Then

(1)  $T_{\psi,\varphi} : \mathcal{B}_\mu(\mathbf{D}^n) \longrightarrow \mathcal{B}_\omega(\mathbf{D}^n)$  is a compact operator if and only if the following are all satisfied:

$$(i) \lim_{\varphi(z) \rightarrow \partial \mathbf{D}^n} \sum_{j=1}^n \omega(z_j) \left| \frac{\partial \psi}{\partial z_j}(z) \right| \left( 1 + \sum_{k=1}^n \int_0^{|\varphi_k(z)|} \frac{dt}{\mu(t)} \right) = 0, \quad (1.3)$$

$$(ii) \lim_{\varphi(z) \rightarrow \partial \mathbf{D}^n} \sum_{j,k=1}^n \frac{\omega(z_j)}{\mu(\varphi_k(z))} \left| \psi(z) \frac{\partial \varphi_k}{\partial z_j}(z) \right| = 0, \quad (1.4)$$

(iii)  $\psi \in \mathcal{B}_\omega(\mathbf{D}^n)$  and  $\psi\varphi_k \in \mathcal{B}_\omega(\mathbf{D}^n)$  for all  $k = 1, 2, \dots, n$ .

(2)  $T_{\psi,\varphi} : \mathcal{B}_{\mu,0}(\mathbf{D}^n) \longrightarrow \mathcal{B}_{\omega,0}(\mathbf{D}^n)$  is a compact operator if and only if (1.3) and (1.4) hold, and for each multi-index  $\alpha$ ,  $\psi\varphi^\alpha \in \mathcal{B}_{\omega,0}(\mathbf{D}^n)$ .

**Theorem C** Let  $\psi \in H(\mathbf{D}^n)$ ,  $\varphi \in H(\mathbf{D}^n, \mathbf{D}^n)$  and  $\int_0^1 \frac{dt}{\mu(t)} < \infty$ . Then

(1)  $T_{\psi,\varphi} : \mathcal{B}_\mu(\mathbf{D}^n) \longrightarrow \mathcal{B}_\omega(\mathbf{D}^n)$  is a compact operator if and only if the following are all satisfied:

$$(i) \lim_{|\varphi_k(z)| \rightarrow 1} \sum_{j=1}^n \frac{\omega(z_j)}{\mu(\varphi_k(z))} \left| \psi(z) \frac{\partial \varphi_k}{\partial z_j}(z) \right| = 0 \text{ for all } k = 1, 2, \dots, n, \quad (1.5)$$

(ii)  $\psi \in \mathcal{B}_\omega(\mathbf{D}^n)$  and  $\psi\varphi_k \in \mathcal{B}_\omega(\mathbf{D}^n)$  for all  $k = 1, 2, \dots, n$ .

(2)  $T_{\psi,\varphi} : \mathcal{B}_{\mu,0}(\mathbf{D}^n) \longrightarrow \mathcal{B}_{\omega,0}(\mathbf{D}^n)$  is a compact operator if and only if (1.5) holds, and for each multi-index  $\alpha$ ,  $\psi\varphi^\alpha \in \mathcal{B}_{\omega,0}(\mathbf{D}^n)$ .

Throughout this paper,  $C, C_1, C_2, \dots$  will stand for positive constants whose value may change from line to line but not depend on the functions in  $H(\mathbf{D}^n)$ .

## 2 The boundedness of $T_{\psi,\varphi}$

**Lemma 2.1** Let  $f \in \mathcal{B}_\mu(\mathbf{D}^n)$ . Then for each  $z \in \mathbf{D}^n$ ,

$$|f(z)| \leq \left( 1 + \sum_{j=1}^n \int_0^{|z_j|} \frac{dt}{\mu(t)} \right) \|f\|_\mu.$$

**Proof.** Since

$$\begin{aligned} f(z) - f(0) &= \sum_{j=1}^n [f(0, \dots, 0, z_j, \dots, z_n) - f(0, \dots, 0, z_{j+1}, \dots, z_n)] \\ &= \sum_{j=1}^n \int_0^1 \frac{d}{dt} f(0, \dots, 0, tz_j, z_{j+1}, \dots, z_n) dt \\ &= \sum_{j=1}^n z_j \int_0^1 \frac{\partial}{\partial w_j} f(0, \dots, 0, tz_j, z_{j+1}, \dots, z_n) dt, \end{aligned}$$

$|f(0)| \leq \|f\|_\mu$  and  $\left| \frac{\partial f}{\partial z_j}(z) \right| \leq \frac{\|f\|_\mu}{\mu(z_j)}$ , we can obtain the result by direct calculation.

Given a normal function  $\mu$ , denote  $k_0 = \max(0, [\log_2 \frac{1}{\mu(\delta)}])$ ,  $r_k = \mu^{-1}(\frac{1}{2^k})$  and  $n_k = [\frac{1}{1-r_k}]$  for  $k > k_0$ , where the symbol  $[x]$  means the greatest integer not more than  $x$ . In what follows, we set the function  $g$ , as in [15], to be

$$g(z) = 1 + \sum_{k > k_0}^{\infty} 2^k z^{n_k}, \quad z \in \mathbf{D}.$$

**Lemma 2.2** ([15]) Let  $\mu$  be a normal function, then the function  $g(z)$  is holomorphic on  $\mathbf{D}$ ,  $g(r)$  is increasing on  $[0, 1)$  and

$$0 < C_1 = \inf_{r \in [0,1)} \mu(r)g(r) \leq \sup_{r \in [0,1)} \mu(r)g(r) = C_2 < \infty.$$

**Proof of Theorem A.** (1) Suppose that (1.1) and (1.2) hold. Then for any  $f \in \mathcal{B}_\mu(\mathbf{D}^n)$ , by Lemma 2.1 we have

$$\begin{aligned}
& \sum_{j=1}^n \omega(z_j) \left| \frac{\partial(\psi \cdot f \circ \varphi)}{\partial z_j}(z) \right| \\
& \leq \sum_{j=1}^n \omega(z_j) \left| \frac{\partial \psi}{\partial z_j}(z) \right| |f(\varphi(z))| + \sum_{j,k=1}^n \omega(z_j) \left| \psi(z) \frac{\partial f}{\partial w_k}(\varphi(z)) \right| \left| \frac{\partial \varphi_k}{\partial z_j}(z) \right| \\
& \leq \sum_{j=1}^n \omega(z_j) \left| \frac{\partial \psi}{\partial z_j}(z) \right| \left( 1 + \sum_{k=1}^n \int_0^{|\varphi_k(z)|} \frac{dt}{\mu(t)} \right) \|f\|_\mu + \sum_{j,k=1}^n \frac{\omega(z_j)}{\mu(\varphi_k(z))} \left| \psi(z) \frac{\partial \varphi_k}{\partial z_j}(z) \right| \|f\|_\mu \\
& \leq C \|f\|_\mu.
\end{aligned} \tag{2.1}$$

Meanwhile,

$$|(T_{\psi,\varphi}f)(0)| = |\psi(0)| |f(\varphi(0))| \leq |\psi(0)| \left( |f(0)| + \max_{|z_k| \leq |\varphi_k(0)|, j,k=1,2,\dots,n} \left| \frac{\partial f}{\partial z_j} \right| \right) \leq C \|f\|_\mu. \tag{2.2}$$

Hence, (2.1) and (2.2) yield that  $T_{\psi,\varphi} : \mathcal{B}_\mu(\mathbf{D}^n) \rightarrow \mathcal{B}_\omega(\mathbf{D}^n)$  is bounded.

Conversely, suppose that  $T_{\psi,\varphi} : \mathcal{B}_\mu(\mathbf{D}^n) \rightarrow \mathcal{B}_\omega(\mathbf{D}^n)$  is bounded. Then  $\psi \in \mathcal{B}_\omega(\mathbf{D}^n)$  and  $\psi\varphi_k \in \mathcal{B}_\omega(\mathbf{D}^n)$ , since  $1, z_k \in \mathcal{B}_\mu(\mathbf{D}^n)$  for all  $k = 1, 2, \dots, n$ . So, we obtain

$$\sup_{z \in \mathbf{D}^n} \sum_{j=1}^n \omega(z_j) \left| \psi(z) \frac{\partial \varphi_k}{\partial z_j}(z) \right| \leq \|\psi\|_\omega + \|\psi\varphi_k\|_\omega < \infty, \quad k = 1, 2, \dots, n. \tag{2.3}$$

We claim that, there exist two constants  $C_1$  and  $C_2$  such that for  $w \in \mathbf{D}^n$  and  $k = 1, 2, \dots, n$ ,

$$\sum_{j=1}^n \omega(w_j) \left| \frac{\partial \psi}{\partial z_j}(w) \right| \int_0^{|\varphi_k(w)|} \frac{dt}{\mu(t)} \leq C_1 \tag{2.4}$$

and

$$\sum_{j=1}^n \frac{\omega(w_j)}{\mu(\varphi_k(w))} \left| \psi(w) \frac{\partial \varphi_k}{\partial z_j}(w) \right| \leq C_2. \tag{2.5}$$

In fact, if  $|\varphi_k(w)| \leq \frac{1}{2}$ , then  $\psi \in \mathcal{B}_\omega(\mathbf{D}^n)$  and (2.3) imply these estimates (2.4) and (2.5). In the following, we always assume that  $|\varphi_k(w)| > \frac{1}{2}$ . First, we prove that (2.5) holds. Take the test function

$$f(z) = \frac{\int_0^{\overline{\varphi_k(w)} z_k} g(t) dt \int_{|\varphi_k(w)|^2}^{\overline{\varphi_k(w)} z_k} g(t) dt}{\overline{\varphi_k(w)} \int_0^{|\varphi_k(w)|^2} g(t) dt}.$$

By the definition of normal weight and Lemma 2.2, we get

$$\int_{|\varphi_k(w)|^2}^{|\varphi_k(w)|} g(t) dt \leq C g(|\varphi_k(w)|^2) [1 - |\varphi_k(w)|] \leq C \int_{|\varphi_k(w)|^4}^{|\varphi_k(w)|^2} g(t) dt.$$

Then

$$\int_0^{|\varphi_k(w)|^2} g(t) dt \leq \int_0^{|\varphi_k(w)|} g(t) dt \leq C \int_0^{|\varphi_k(w)|^2} g(t) dt. \tag{2.6}$$

Thus,

$$\begin{aligned} \sum_{j=1}^n \mu(z_j) \left| \frac{\partial f}{\partial z_j}(z) \right| &= \mu(z_k) |g(z_k \overline{\varphi_k(w)})| \frac{\left| \int_0^{\overline{\varphi_k(w)} z_k} g(t) dt + \int_{|\varphi_k(w)|^2}^{\overline{\varphi_k(w)} z_k} g(t) dt \right|}{\int_0^{|\varphi_k(w)|^2} g(t) dt} \\ &\leq \frac{C \int_0^{|\varphi_k(w)|} g(t) dt}{\int_0^{|\varphi_k(w)|^2} g(t) dt} \leq C. \end{aligned}$$

This means  $f \in \mathcal{B}_\mu(\mathbf{D}^n)$  and  $\|f\|_\mu \leq C$ , where  $C$  does not depend on  $w$ . By the boundedness of  $T_{\psi, \varphi}$  it is clear that  $C\|T_{\psi, \varphi} f\|_\omega \geq \|T_{\psi, \varphi} f\|_\omega \geq \sum_{j=1}^n \omega(z_j) \left| \frac{\partial(\psi \cdot f \circ \varphi)}{\partial z_j}(z) \right|$ . In particular, when  $z = w$  we obtain

$$\begin{aligned} C\|T_{\psi, \varphi}\| &\geq \sum_{j=1}^n \omega(w_j) \left| \frac{\partial \psi}{\partial z_j}(w) f(\varphi(w)) + \psi(w) g(|\varphi_k(w)|^2) \frac{\partial \varphi_k}{\partial z_j}(w) \right| \\ &= \sum_{j=1}^n \omega(w_j) |\psi(w) g(|\varphi_k(w)|^2)| \left| \frac{\partial \varphi_k}{\partial z_j}(w) \right|. \end{aligned} \quad (2.7)$$

On the other hand, by the definition of normal weight, we always have  $\frac{\mu(\varphi_k^2(w))}{\mu(\varphi_k(w))} \leq C_3$ . This, together with (2.7) and Lemma 2.2, gives

$$\begin{aligned} &\sum_{j=1}^n \frac{\omega(w_j)}{\mu(\varphi_k(w))} \left| \psi(w) \frac{\partial \varphi_k}{\partial z_j}(w) \right| \\ &= \sum_{j=1}^n \frac{\omega(w_j) |\psi(w) g(|\varphi_k(w)|^2)|}{g(|\varphi_k(w)|^2) \mu(\varphi_k^2(w))} \left| \frac{\partial \varphi_k}{\partial z_j}(w) \right| \frac{\mu(\varphi_k^2(w))}{\mu(\varphi_k(w))} \\ &\leq C \sum_{j=1}^n \omega(w_j) |\psi(w) g(|\varphi_k(w)|^2)| \left| \frac{\partial \varphi_k}{\partial z_j}(w) \right| \\ &\leq C\|T_{\psi, \varphi}\| < \infty. \end{aligned}$$

This means the estimate (2.5) holds. Now, we prove that (2.4) holds. Take the test function

$$f(z) = \frac{1}{\varphi_k(w)} \int_0^{\overline{\varphi_k(w)} z_k} g(t) dt.$$

It is easy to check that  $\|f\|_\mu \leq C$ , where  $C$  does not depend on  $w$ . Then

$$\begin{aligned} C\|T_{\psi, \varphi}\| &\geq \|T_{\psi, \varphi} f\|_\omega \geq \left( \sum_{j=1}^n \omega(z_j) \left| \frac{\partial(\psi \cdot f \circ \varphi)}{\partial z_j}(z) \right| \right)_{z=w} \\ &\geq \sum_{j=1}^n \omega(w_j) \left| \frac{\partial \psi}{\partial z_j}(w) \right| \frac{1}{|\varphi_k(w)|} \int_0^{|\varphi_k(w)|^2} g(t) dt - \sum_{j=1}^n \omega(w_j) |\psi(w) g(|\varphi_k(w)|^2)| \left| \frac{\partial \varphi_k}{\partial z_j}(w) \right|. \end{aligned}$$

So, by (2.7) we have

$$\sum_{j=1}^n \omega(w_j) \left| \frac{\partial \psi}{\partial z_j}(w) \right| \int_0^{|\varphi_k(w)|^2} g(t) dt \leq C\|T_{\psi, \varphi}\| < \infty. \quad (2.8)$$

It follows from (2.6), (2.8) and Lemma 2.2 that

$$\sum_{j=1}^n \omega(w_j) \left| \frac{\partial \psi}{\partial z_j}(w) \right| \left| \int_0^{|\varphi_k(w)|} \frac{dt}{\mu(t)} \right| \leq C \|T_{\psi, \varphi}\| < \infty.$$

This means the estimate (2.4) holds. Therefore, combining (2.4), (2.5) and the fact that  $\psi \in \mathcal{B}_\omega(\mathbf{D}^n)$ , (1.1) and (1.2) hold.

(2) Suppose that  $T_{\psi, \varphi} : \mathcal{B}_{\mu, 0}(\mathbf{D}^n) \longrightarrow \mathcal{B}_{\omega, 0}(\mathbf{D}^n)$  is bounded. For each multi-index  $\alpha$ , because  $z^\alpha \in \mathcal{B}_{\mu, 0}(\mathbf{D}^n)$  we have  $\psi \varphi^\alpha = T_{\psi, \varphi} z^\alpha \in \mathcal{B}_{\omega, 0}(\mathbf{D}^n)$ . That the boundedness imply (1.1) and (1.2) can be proved in the same way as that in (1), as all the test functions defined in (1) are holomorphic on  $\overline{\mathbf{D}^n}$  and hence in  $\mathcal{B}_{\mu, 0}(\mathbf{D}^n)$ .

Conversely, suppose that (1.1) and (1.2) hold. From (1), we know that  $T_{\psi, \varphi} : \mathcal{B}_\mu(\mathbf{D}^n) \longrightarrow \mathcal{B}_\omega(\mathbf{D}^n)$  is bounded. So, we only need to prove that  $T_{\psi, \varphi} f \in \mathcal{B}_{\omega, 0}(\mathbf{D}^n)$  whenever  $f \in \mathcal{B}_{\mu, 0}(\mathbf{D}^n)$ . In fact, for  $f \in \mathcal{B}_{\mu, 0}(\mathbf{D}^n)$  and any  $\varepsilon > 0$ , by the definition of  $\mathcal{B}_{\mu, 0}(\mathbf{D}^n)$ , there exists some polynomial  $p$  such that

$$\|f - p\|_\mu < \frac{\varepsilon}{2\|T_{\psi, \varphi}\| + 1}.$$

By the boundedness of  $T_{\psi, \varphi}$ ,

$$\|T_{\psi, \varphi} f - T_{\psi, \varphi} p\|_\omega \leq \|T_{\psi, \varphi}\| \cdot \|f - p\|_\mu < \frac{\varepsilon}{2}. \quad (2.9)$$

Since  $\mathcal{B}_{\mu, 0}$  is a linear space and  $\psi \varphi^\alpha \in \mathcal{B}_{\omega, 0}(\mathbf{D}^n)$ , we have  $T_{\psi, \varphi} p \in \mathcal{B}_{\omega, 0}(\mathbf{D}^n)$ . This means we have another polynomial  $q$  such that

$$\|T_{\psi, \varphi} p - q\|_\omega < \frac{\varepsilon}{2}.$$

This and (2.9) yield

$$\|T_{\psi, \varphi} f - q\|_\omega \leq \|T_{\psi, \varphi} f - T_{\psi, \varphi} p\|_\omega + \|T_{\psi, \varphi} p - q\|_\omega < \varepsilon.$$

That is,  $T_{\psi, \varphi} f \in \mathcal{B}_{\omega, 0}(\mathbf{D}^n)$ . The proof is completed.

### 3 The compactness of $T_{\psi, \varphi}$

**Lemma 3.1** Let  $\psi \in H(\mathbf{D}^n)$ ,  $\varphi(z) \in H(\mathbf{D}^n, \mathbf{D}^n)$  and  $T_{\psi, \varphi} : \mathcal{B}_\mu(\mathbf{D}^n) \longrightarrow \mathcal{B}_\omega(\mathbf{D}^n)$  is a bounded operator. Then  $T_{\psi, \varphi} : \mathcal{B}_\mu(\mathbf{D}^n) \longrightarrow \mathcal{B}_\omega(\mathbf{D}^n)$  is a compact operator if and only if for any bounded sequence  $\{f_m\}$  in  $\mathcal{B}_\mu(\mathbf{D}^n)$  which converges to 0 uniformly on any compact subset of  $\mathbf{D}^n$ , we have  $\lim_{m \rightarrow \infty} \|T_{\psi, \varphi} f_m\|_\omega = 0$ .

**Proof.** It can be proved by Montel theorem and Lemma 2.1. The details are omitted here.

**Lemma 3.2** ([13]) Suppose  $\mu$  is normal with  $\int_0^1 \frac{1}{\mu(r)} dr < \infty$  and  $\{f_m\}$  is a bounded sequence in  $\mathcal{B}_\mu(\mathbf{D}^n)$  which converges to 0 uniformly on any compact subset of  $\mathbf{D}^n$ . Then  $\lim_{m \rightarrow \infty} \sup_{z \in \mathbf{D}^n} |f_m(z)| = 0$ . And, for each  $\rho \in [0, 1)$  and  $j = 1, 2, \dots, n$ ,  $\lim_{m \rightarrow \infty} \sup_{z \in \mathbf{D}^n, |z_j| \leq \rho} \left| \frac{\partial f_m}{\partial z_j}(z) \right| = 0$ .

**Proof of Theorem B.** (1) Suppose that  $\psi$  and  $\varphi$  satisfy conditions (i)-(iii). Then by Theorem A,  $T_{\psi, \varphi} : \mathcal{B}_\mu(\mathbf{D}^n) \longrightarrow \mathcal{B}_\omega(\mathbf{D}^n)$  is a bounded operator. Let  $\{f_m\}$  be any a sequence



which converges to 0 uniformly on any compact subset of  $\mathbf{D}^n$  satisfying  $\|f_m\|_\mu \leq 1$ . We claim that

$$\|T_{\psi,\varphi}f_m\|_\omega = |\psi(0)| \cdot |f_m(\varphi(0))| + \sup_{z \in \mathbf{D}^n} \sum_{j=1}^n \omega(z_j) \left| \frac{\partial(\psi \cdot f_m \circ \varphi)}{\partial z_j}(z) \right| \rightarrow 0 \quad (m \rightarrow \infty). \quad (3.1)$$

In fact, for each  $\varepsilon > 0$ , by (i) there exists some compact subset  $K \subset \mathbf{D}^n$  such that

$$\sum_{j=1}^n \omega(z_j) \left| \frac{\partial \psi}{\partial z_j}(z) \right| \left( 1 + \sum_{k=1}^n \int_0^{|\varphi_k(z)|} \frac{dt}{\mu(t)} \right) < \varepsilon$$

whenever  $\varphi(z) \in \mathbf{D}^n \setminus K$ . Thus, if  $\varphi(z) \in \mathbf{D}^n \setminus K$  then by Lemma 2.1,

$$\sum_{j=1}^n \omega(z_j) \left| \frac{\partial \psi}{\partial z_j}(z) \right| |f_m(\varphi(z))| \leq \|f_m\|_\mu \sum_{j=1}^n \omega(z_j) \left| \frac{\partial \psi}{\partial z_j}(z) \right| \left( 1 + \sum_{k=1}^n \int_0^{|\varphi_k(z)|} \frac{dt}{\mu(t)} \right) < \varepsilon. \quad (3.2)$$

If  $\varphi(z) \in K$ , because  $\psi \in \mathcal{B}_\omega(\mathbf{D}^n)$  and  $\{f_m\}$  converges to 0 uniformly on  $K$  we have

$$\sum_{j=1}^n \omega(z_j) \left| \frac{\partial \psi}{\partial z_j}(z) \right| \cdot |f_m(\varphi(z))| \leq \|\psi\|_\omega \cdot \max_{w \in K} |f_m(w)| \rightarrow 0 \quad (m \rightarrow \infty). \quad (3.3)$$

On the other hand, for the above  $\varepsilon > 0$ , by (ii) we have

$$\sum_{j,k=1}^n \omega(z_j) |\psi(z)| \left| \frac{\partial f_m}{\partial w_k}(\varphi(z)) \right| \left| \frac{\partial \varphi_k}{\partial z_j}(z) \right| \leq \|f_m\|_\mu \sum_{j,k=1}^n \frac{\omega(z_j) |\psi(z)|}{\mu(\varphi_k(z))} \left| \frac{\partial \varphi_k}{\partial z_j}(z) \right| < \varepsilon \quad (3.4)$$

whenever  $\varphi(z) \in \mathbf{D}^n \setminus K$ . If  $\varphi(z) \in K$ , because  $\psi, \psi \varphi_k \in \mathcal{B}_\omega(\mathbf{D}^n)$  and  $\{f_m\}$  converges to 0 uniformly on  $K$  we have

$$\begin{aligned} & \sum_{j,k=1}^n \omega(z_j) |\psi(z)| \left| \frac{\partial f_m}{\partial w_k}(\varphi(z)) \right| \left| \frac{\partial \varphi_k}{\partial z_j}(z) \right| \\ & \leq \max_{w \in K, 1 \leq k \leq n} \left| \frac{\partial f_m}{\partial z_k}(w) \right| \sum_{j,k=1}^n \omega(z_j) |\psi(z)| \left| \frac{\partial \varphi_k}{\partial z_j}(z) \right| \rightarrow 0 \quad (m \rightarrow \infty). \end{aligned} \quad (3.5)$$

From (3.2)-(3.5) and  $\lim_{m \rightarrow \infty} |\psi(0)f_m(\varphi(0))| = 0$ , we can obtain (3.1).

Conversely, suppose that  $T_{\psi,\varphi} : \mathcal{B}_\mu(\mathbf{D}^n) \rightarrow \mathcal{B}_\omega(\mathbf{D}^n)$  is a compact operator, then (iii) is trivial. Assume that (1.4) fails. Then there exists some  $\varepsilon_0 > 0$  and some sequence  $\{z^{(m)}\} \subset \mathbf{D}^n$  such that  $\varphi(z^{(m)}) \rightarrow z_0 \in \partial \mathbf{D}^n$  ( $m \rightarrow \infty$ ) and

$$\sum_{j,k=1}^n \frac{\omega(z_j^{(m)})}{\mu(\varphi_k(z^{(m)}))} \left| \psi(z^{(m)}) \frac{\partial \varphi_k}{\partial z_j}(z^{(m)}) \right| \geq n\varepsilon_0.$$

Without loss of generality, we may assume that

$$\sum_{j=1}^n \frac{\omega(z_j^{(m)})}{\mu(\varphi_1(z^{(m)}))} \left| \psi(z^{(m)}) \frac{\partial \varphi_1}{\partial z_j}(z^{(m)}) \right| \geq \varepsilon_0. \quad (3.6)$$

Set  $\lim_{m \rightarrow \infty} |\varphi_1(z^{(m)})| = d \leq 1$ . The construction of the function  $f_m$  will be carried out in two cases.

Case 1. If  $d = 1$ , we may assume that  $|\varphi_1(z^{(m)})| > \frac{1}{2}$ ,  $m = 1, 2, \dots$ . Take

$$f_m(z) = \frac{\int_0^{\overline{\varphi_1(z^{(m)})}z_1} g(t)dt \int_{\frac{\overline{\varphi_1(z^{(m)})}^2 z_1^2}{\varphi_1(z^{(m)})|\varphi_1(z^{(m)})|^2 z_1}} g(t)dt}{\int_0^{|\varphi_1(z^{(m)})|^2} g(t)dt}.$$

Then by Lemma 2.2 and (2.6) we obtain

$$\begin{aligned} & \sum_{j=1}^n \mu(z_j) \left| \frac{\partial f_m}{\partial z_j}(z) \right| \\ &= \frac{\mu(z_1)}{\int_0^{|\varphi_1(z^{(m)})|^2} g(t)dt} \left| g(\overline{\varphi_1(z^{(m)})}z_1) \overline{\varphi_1(z^{(m)})} \int_{\frac{\overline{\varphi_1(z^{(m)})}^2 z_1^2}{\varphi_1(z^{(m)})|\varphi_1(z^{(m)})|^2 z_1}} g(t)dt + \int_0^{\overline{\varphi_1(z^{(m)})}z_1} g(t)dt \right. \\ & \quad \cdot \left( g(\overline{\varphi_1(z^{(m)})}^2 z_1^2) \cdot 2z_1 \overline{\varphi_1(z^{(m)})}^2 - g(z_1 \overline{\varphi_1(z^{(m)})}|\varphi_1(z^{(m)})|^2) \overline{\varphi_1(z^{(m)})}|\varphi_1(z^{(m)})|^2 \right) \Big| \\ &\leq \frac{C\mu(z_1)g(|z_1|)}{\int_0^{|\varphi_1(z^{(m)})|^2} g(t)dt} \left( \left| \int_{\frac{\overline{\varphi_1(z^{(m)})}^2 z_1^2}{\varphi_1(z^{(m)})|\varphi_1(z^{(m)})|^2 z_1}} g(t)dt \right| + \left| \int_0^{\overline{\varphi_1(z^{(m)})}z_1} g(t)dt \right| \right) \\ &\leq \frac{C \int_0^{|\varphi_1(z^{(m)})|} g(t)dt}{\int_0^{|\varphi_1(z^{(m)})|^2} g(t)dt} \leq C. \end{aligned}$$

Because

$$\lim_{m \rightarrow \infty} \int_0^{|\varphi_1(z^{(m)})|} g(t)dt = \lim_{m \rightarrow \infty} \int_0^{|\varphi_1(z^{(m)})|} \frac{dt}{\mu(t)} = \infty,$$

we know that  $\{f_m\}$  converges to 0 uniformly on any compact subset of  $\mathbf{D}^n$ . Hence, from Lemma 3.1,  $\lim_{m \rightarrow \infty} \|T_{\psi, \varphi} f_m\|_\omega = 0$ . But by (3.6), definition of normal weight and Lemma 2.2,

$$\begin{aligned} \|T_{\psi, \varphi} f_m\|_\omega &\geq \sum_{j=1}^n \omega(z_j^{(m)}) \left| \frac{\partial(\psi \cdot f_m \circ \varphi)}{\partial z_j}(z^{(m)}) \right| \\ &= \sum_{j=1}^n \omega(z_j^{(m)}) |\psi(z^{(m)})| g(|\varphi_1(z^{(m)})|^4) |\varphi_1(z^{(m)})|^3 \left| \frac{\partial \varphi_1}{\partial z_j}(z^{(m)}) \right| \\ &= \sum_{j=1}^n \frac{\omega(z_j^{(m)}) |\psi(z^{(m)})| \left| \frac{\partial \varphi_1}{\partial z_j}(z^{(m)}) \right|}{\mu(\varphi_1(z^{(m)}))} \cdot \frac{\mu(\varphi_1(z^{(m)}))}{\mu(\varphi_1^4(z^{(m)}))} g(|\varphi_1(z^{(m)})|^4) \mu(\varphi_1^4(z^{(m)})) |\varphi_1(z^{(m)})|^3 \\ &\geq \frac{C}{4^b} |\varphi_1(z^{(m)})|^3 \sum_{j=1}^n \frac{\omega(z_j^{(m)}) |\psi(z^{(m)})| \left| \frac{\partial \varphi_1}{\partial z_j}(z^{(m)}) \right|}{\mu(\varphi_1(z^{(m)}))} \\ &\geq C\varepsilon_0. \end{aligned}$$

This leads a contradiction.

Case 2. If  $d < 1$ , we have some  $s$  ( $1 < s \leq n$ ) such that  $\lim_{m \rightarrow \infty} |\varphi_s(z^{(m)})| = 1$  since  $\varphi(z^{(m)}) \rightarrow z_0 \in \partial \mathbf{D}^n$  ( $m \rightarrow \infty$ ). Similar to that in Case 1 (replace 1 with  $s$  there), we can get

$$\lim_{m \rightarrow \infty} \sum_{j=1}^n \frac{\omega(z_j^{(m)})}{\mu(\varphi_s(z^{(m)}))} \left| \psi(z^{(m)}) \frac{\partial \varphi_s}{\partial z_j}(z^{(m)}) \right| = 0. \quad (3.7)$$

We may also assume  $|\varphi_s(z^{(m)})| > \frac{1}{2}$ , and take

$$f_m(z) = \frac{z_1}{\int_0^{|\varphi_s(z^{(m)})|} g(t)dt} \int_0^{\overline{\varphi_s(z^{(m)})}z_s} g(t)dt.$$

It is easy to check that  $\|f_m\|_\mu \leq C$  and  $\{f_m\}$  converges to 0 uniformly on any compact subset of  $\mathbf{D}^n$ . So,  $\lim_{m \rightarrow \infty} \|T_{\psi, \varphi} f_m\|_\omega = 0$  by Lemma 3.1. Notice that a compact operator is a bounded operator. Hence, Theorem A and  $\int_0^1 \frac{dt}{\mu(t)} = \infty$  imply

$$\lim_{\varphi(z) \rightarrow \partial \mathbf{D}^n} \sum_{j=1}^n \omega(z_j) \left| \frac{\partial \psi}{\partial z_j}(z) \right| = 0. \quad (3.8)$$

But it follows from (2.6), (3.6), (3.7) and (3.8) that

$$\begin{aligned} & \|T_{\psi, \varphi} f_m\|_\omega \\ & \geq \sum_{j=1}^n \omega(z_j^{(m)}) \left| \psi(z^{(m)}) \sum_{k=0}^n \frac{\partial f_m}{\partial w_k}(\varphi(z^{(m)})) \frac{\partial \varphi_k}{\partial z_j}(z^{(m)}) + \frac{\partial \psi}{\partial z_j}(z^{(m)}) f_m(\varphi(z^{(m)})) \right| \\ & = \sum_{j=1}^n \omega(z_j^{(m)}) \left| \psi(z^{(m)}) \frac{\partial f_m}{\partial w_1}(\varphi(z^{(m)})) \frac{\partial \varphi_1}{\partial z_j}(z^{(m)}) + \psi(z^{(m)}) \frac{\partial f_m}{\partial w_s}(\varphi(z^{(m)})) \frac{\partial \varphi_s}{\partial z_j}(z^{(m)}) \right. \\ & \quad \left. + \frac{\partial \psi}{\partial z_j}(z^{(m)}) f_m(\varphi(z^{(m)})) \right| \\ & \geq \sum_{j=1}^n \mu(\varphi_1(z^{(m)})) \left| \frac{\partial f_m}{\partial w_1}(\varphi(z^{(m)})) \right| \cdot \frac{\omega(z_j^{(m)})}{\mu(\varphi_1(z^{(m)}))} \left| \psi(z^{(m)}) \frac{\partial \varphi_1}{\partial z_j}(z^{(m)}) \right| \\ & \quad - \sum_{j=1}^n \mu(\varphi_s(z^{(m)})) \left| \frac{\partial f_m}{\partial w_s}(\varphi(z^{(m)})) \right| \cdot \frac{\omega(z_j^{(m)})}{\mu(\varphi_s(z^{(m)}))} \left| \psi(z^{(m)}) \frac{\partial \varphi_s}{\partial z_j}(z^{(m)}) \right| \\ & \quad - \sum_{j=1}^n \omega(z_j^{(m)}) \left| \frac{\partial \psi}{\partial z_j}(z^{(m)}) \right| \frac{|\varphi_1(z^{(m)})|}{\int_0^{|\varphi_s(z^{(m)})|} g(t) dt} \int_0^{|\varphi_s(z^{(m)})|^2} g(t) dt \\ & \geq \varepsilon_0 \left[ \min_{t \in [0, \frac{d+1}{2}]} \mu(t) \right] \frac{\int_0^{|\varphi_s(z^{(m)})|^2} g(t) dt}{\int_0^{|\varphi_s(z^{(m)})|} g(t) dt} - \|f_m\|_\mu \sum_{j=1}^n \frac{\omega(z_j^{(m)})}{\mu(\varphi_s(z^{(m)}))} \left| \psi(z^{(m)}) \frac{\partial \varphi_s}{\partial z_j}(z^{(m)}) \right| \\ & \quad - \sum_{j=1}^n \omega(z_j^{(m)}) \left| \frac{\partial \psi}{\partial z_j}(z^{(m)}) \right| \frac{\int_0^{|\varphi_s(z^{(m)})|^2} g(t) dt}{\int_0^{|\varphi_s(z^{(m)})|} g(t) dt} \rightarrow C\varepsilon_0 \quad (m \rightarrow \infty). \end{aligned}$$

This leads a contradiction. This shows that (1.4) holds.

Now we prove (1.3) holds. By (3.8), we only need to claim that

$$\lim_{\varphi(z) \rightarrow \partial \mathbf{D}^n} \sum_{j=1}^n \omega(z_j) \left| \frac{\partial \psi}{\partial z_j}(z) \right| \sum_{k=1}^n \int_0^{|\varphi_k(z)|} \frac{dt}{\mu(t)} = 0.$$

Assume this expression fails. Similarly, there exists some  $\varepsilon_0 > 0$  and some sequence  $\{z^{(m)}\} \subset \mathbf{D}^n$  such that  $\varphi(z^{(m)}) \rightarrow z_0 \in \partial \mathbf{D}^n$  ( $m \rightarrow \infty$ ) and

$$\sum_{j,k=1}^n \omega(z_j^{(m)}) \left| \frac{\partial \psi}{\partial z_j}(z^{(m)}) \right| \int_0^{|\varphi_k(z^{(m)})|} \frac{dt}{\mu(t)} \geq n\varepsilon_0.$$

Without loss of generality, we may assume that

$$\sum_{j=1}^n \omega(z_j^{(m)}) \left| \frac{\partial \psi}{\partial z_j}(z^{(m)}) \right| \int_0^{|\varphi_1(z^{(m)})|} \frac{dt}{\mu(t)} \geq \varepsilon_0.$$

Since  $\varphi(z^{(m)}) \rightarrow z_0 \in \partial \mathbf{D}^n$ , by (3.8) we can get

$$\lim_{m \rightarrow \infty} \sum_{j=1}^n \omega(z_j^{(m)}) \left| \frac{\partial \psi}{\partial z_j}(z^{(m)}) \right| = 0. \quad (3.9)$$

If  $\lim_{m \rightarrow \infty} |\varphi_1(z^{(m)})| = d < 1$ , then by (3.9)

$$\lim_{m \rightarrow \infty} \sum_{j=1}^n \omega(z_j^{(m)}) \left| \frac{\partial \psi}{\partial z_j}(z^{(m)}) \right| \int_0^{|\varphi_1(z^{(m)})|} \frac{dt}{\mu(t)} = 0.$$

This contradicts the assumption. If  $d = 1$ , we may assume  $|\varphi_1(z^{(m)})| > \frac{1}{2}$ . Take

$$f_m(z) = \frac{\left( \int_0^{\overline{\varphi_1(z^{(m)})} z_1} g(t) dt \right)^2}{\int_0^{|\varphi_1(z^{(m)})|^2} g(t) dt},$$

then  $\|f_m\|_\mu \leq C$  and  $\{f_m\}$  converges to 0 uniformly on any compact subset of  $\mathbf{D}^n$ . So  $\lim_{m \rightarrow \infty} \|T_{\psi, \varphi} f_m\|_\omega = 0$ . But (2.6) and (1.4) imply

$$\begin{aligned} \|T_{\psi, \varphi} f_m\|_\omega &\geq \sum_{j=1}^n \omega(z_j^{(m)}) \left| \frac{\partial \psi}{\partial z_j}(z^{(m)}) \int_0^{|\varphi_1(z^{(m)})|^2} g(t) dt \right| \\ &\quad - \sum_{j=1}^n \omega(z_j^{(m)}) \left| \psi(z^{(m)}) \frac{\partial f_m}{\partial w_1}(\varphi(z^{(m)})) \frac{\partial \varphi_1}{\partial z_j}(z^{(m)}) \right| \\ &\geq C \sum_{j=1}^n \omega(z_j^{(m)}) \left| \frac{\partial \psi}{\partial z_j}(z^{(m)}) \int_0^{|\varphi_1(z^{(m)})|} g(t) dt \right| \\ &\quad - \sum_{j=1}^n \mu(\varphi_1(z^{(m)})) \left| \frac{\partial f_m}{\partial w_1}(\varphi(z^{(m)})) \right| \cdot \frac{\omega(z_j^{(m)})}{\mu(\varphi_1(z^{(m)}))} \left| \psi(z^{(m)}) \frac{\partial \varphi_1}{\partial z_j}(z^{(m)}) \right| \\ &\geq C\varepsilon_0 - \|f_m\|_\mu \sum_{j=1}^n \frac{\omega(z_j^{(m)}) \left| \psi(z^{(m)}) \frac{\partial \varphi_1}{\partial z_j}(z^{(m)}) \right|}{\mu(\varphi_1(z^{(m)}))} \rightarrow C\varepsilon_0 \quad (m \rightarrow \infty). \end{aligned}$$

This leads a contradiction. This means that (1.3) holds.

(2) Notice that all the test function sequences defined in (1) are holomorphic on  $\overline{\mathbf{D}^n}$  and hence in  $\mathcal{B}_{\mu, 0}(\mathbf{D}^n)$ . Therefore, by proof of (1) and Theorem A, the result (2) holds. The proof is completed.

**Proof of Theorem C.** Suppose that  $\psi, \psi\varphi_k \in \mathcal{B}_\omega(\mathbf{D}^n)$  and (1.5) holds. Then for each  $\varepsilon > 0$ , there exists some  $\eta \in (0, 1)$  such that for all  $k$ , we have

$$\sup_{|\varphi_k(z)| > \eta} \sum_{j=1}^n \frac{\omega(z_j)}{\mu(\varphi_k(z))} \left| \psi(z) \frac{\partial \varphi_k}{\partial z_j}(z) \right| < \varepsilon. \quad (3.10)$$

Meanwhile,

$$\sup_{|\varphi_k(z)| \leq \eta} \sum_{j=1}^n \frac{\omega(z_j)}{\mu(\varphi_k(z))} \left| \psi(z) \frac{\partial \varphi_k}{\partial z_j}(z) \right| < C(\|\psi\|_\omega + \|\psi\varphi_k\|_\omega). \quad (3.11)$$

From (3.10), (3.11) and Theorem A, we obtain that  $T_{\psi, \varphi} : \mathcal{B}_\mu(\mathbf{D}^n) \rightarrow \mathcal{B}_\omega(\mathbf{D}^n)$  is bounded.

Suppose  $\{f_m\}$  is a bounded sequence in  $\mathcal{B}_\mu(\mathbf{D}^n)$  which converges to 0 uniformly on any compact subset of  $\mathbf{D}^n$ . By Lemma 3.2 and  $\psi \in \mathcal{B}_\omega(\mathbf{D}^n)$ , we can get

$$\lim_{m \rightarrow \infty} \sup_{z \in \mathbf{D}^n} \sum_{j=1}^n \omega(z_j) \left| \frac{\partial \psi}{\partial z_j}(z) f_m(\varphi(z)) \right| = 0. \quad (3.12)$$

On the other hand, for any  $\varepsilon > 0$ , from (1.5), we have some  $\rho \in (0, 1)$  such that for all  $k$

$$\sum_{j=1}^n \frac{\omega(z_j)}{\mu(\varphi_k(z))} \left| \psi(z) \frac{\partial \varphi_k}{\partial z_j}(z) \right| < \frac{\varepsilon}{M+1}, \quad (3.13)$$

whenever  $|\varphi_k(z)| > \rho$ , where  $M = \sup\{\|f_m\|_\mu : m = 1, 2, \dots\}$ . Set

$$I_k = \sum_{j=1}^n \omega(z_j) \left| \psi(z) \frac{\partial f_m}{\partial w_k}(\varphi(z)) \frac{\partial \varphi_k}{\partial z_j}(z) \right|, \quad k = 1, 2, \dots, n.$$

Then by Lemma 3.2 and  $\psi, \psi\varphi_k \in \mathcal{B}_\omega(\mathbf{D}^n)$ , we have

$$\sup_{|\varphi_k(z)| \leq \rho} I_k \leq (\|\psi\|_\omega + \|\psi\varphi_k\|_\omega) \sup_{w \in \mathbf{D}^n, |w_k| < \rho} \left| \frac{\partial f_m}{\partial w_k}(w) \right| \rightarrow 0 \quad (m \rightarrow \infty). \quad (3.14)$$

Thus as  $m$  is sufficiently large, from (3.13) and (3.14),

$$\sup_{z \in \mathbf{D}^n} I_k < \sup_{|\varphi_k(z)| \leq \rho} I_k + \sup_{|\varphi_k(z)| > \rho} I_k < 2\varepsilon. \quad (3.15)$$

Hence, (3.12) and (3.15), together with the fact that  $\lim_{m \rightarrow \infty} |\psi(0)f_m(\varphi(0))| = 0$ , yield  $\lim_{m \rightarrow \infty} \|T_{\psi, \varphi} f_m\|_\omega = 0$ . This means  $T_{\psi, \varphi} : \mathcal{B}_\mu(\mathbf{D}^n) \rightarrow \mathcal{B}_\omega(\mathbf{D}^n)$  is compact.

Conversely, we only need to prove that (1.5) holds. If there exists some  $k$  such that (1.5) fails. Then we would have some constant  $\varepsilon_0 > 0$  and a sequence  $\{z^{(m)}\}$  in  $\mathbf{D}^n$  such  $\lim_{m \rightarrow \infty} |\varphi_k(z^{(m)})| = 1$  and

$$\sum_{j=1}^n \frac{\omega(z_j^{(m)})}{\mu(\varphi_k(z^{(m)}))} \left| \psi(z^{(m)}) \frac{\partial \varphi_k}{\partial z_j}(z^{(m)}) \right| \geq \varepsilon_0.$$

Without loss of generality, we may assume  $k = 1$ , that is

$$\sum_{j=1}^n \frac{\omega(z_j^{(m)})}{\mu(\varphi_1(z^{(m)}))} \left| \psi(z^{(m)}) \frac{\partial \varphi_1}{\partial z_j}(z^{(m)}) \right| \geq \varepsilon_0. \quad (3.16)$$

Writing  $\rho_m = \varphi_1(z^{(m)})$ , we also may assume that

$$1 - \frac{1}{(m+1)^2} < |\rho_m| < 1, \quad \lim_{m \rightarrow \infty} \rho_m = \rho_0 \in \partial \mathbf{D}. \quad (3.17)$$

Take

$$f_m(z) = \frac{1}{\rho_m} \int_0^{\overline{\rho_m} z_1} g(t) dt - \frac{1}{|\rho_m|^m \overline{\rho_m}} \int_0^{|\rho_m|^m \overline{\rho_m} z_1} g(t) dt. \quad (3.18)$$

Then  $\|f_m\|_\mu \leq C$ . From (3.17) we get

$$1 > |\rho_m|^2 > |\rho_m|^{m+2} > \left(1 - \frac{1}{m^2}\right)^{m+2} \rightarrow 1 \quad (m \rightarrow \infty). \quad (3.19)$$

Hence, for fixed  $r \in (0, 1)$  and any  $z \in \overline{\mathbf{D}}_r^n$ ,

$$\begin{aligned} |f_m(z)| &\leq \left| \frac{1}{\overline{\rho_m}} \int_0^{\overline{\rho_m} z_1} g(t) dt - \frac{1}{\rho_m} \int_0^{|\rho_m|^m \overline{\rho_m} z_1} g(t) dt \right| \\ &\quad + \left| \frac{1}{\overline{\rho_m}} \int_0^{|\rho_m|^m \overline{\rho_m} z_1} g(t) dt - \frac{1}{|\rho_m|^m \overline{\rho_m}} \int_0^{|\rho_m|^m \overline{\rho_m} z_1} g(t) dt \right| \\ &\leq |z_1| \left[ (1 - |\rho_m|^m) + \left( \frac{1}{|\rho_m|^m} - 1 \right) \right] \max_{t \in [0, r]} g(t) \\ &\leq C_1 \left[ \frac{1}{|\rho_m|^m} - |\rho_m|^m \right], \end{aligned}$$

where  $\mathbf{D}_r^n = \{z \in \mathbf{C}^n; |z_j| < r, j = 1, 2, \dots, n\}$  and the constant  $C_1$  depends only on  $r$ . So, by (3.19) we obtain

$$\lim_{m \rightarrow \infty} \max_{z \in \overline{\mathbf{D}}_r^n} |f_m(z)| = 0.$$

That is,  $\{f_m\}$  converges to 0 uniformly on any compact subset of  $\mathbf{D}^n$ . By (3.19), we may assume  $|\rho_m|^{m+1} \geq \frac{1}{2}$ , then this and the definition of normal weight imply

$$\begin{aligned} 0 &\leq \frac{\mu(\rho_m)}{\mu(\rho_m^{m+2})} = \frac{\frac{\mu(\rho_m)}{(1-|\rho_m|)^a(1+|\rho_m|+\dots+|\rho_m|^{m+1})^a}}{\frac{\mu(\rho_m^{m+2})}{(1-|\rho_m|^{m+2})^a}} \\ &\leq \frac{1}{(1+|\rho_m|+\dots+|\rho_m|^{m+1})^a} \leq \left( \frac{2}{m+2} \right)^a \rightarrow 0 \quad (m \rightarrow \infty). \end{aligned}$$

This, together the estimate  $\liminf_{m \rightarrow \infty} \frac{\mu(\rho_m)}{\mu(\rho_m^2)} \geq \frac{1}{2^b}$ , yield

$$\begin{aligned} &\mu(\varphi_1(z^{(m)})) \left| \frac{\partial f_m}{\partial w_1}(\varphi(z^{(m)})) \right| \\ &\geq \left( \mu(\rho_m^2) g(|\rho_m|^2) \frac{\mu(\rho_m)}{\mu(\rho_m^2)} - \mu(\rho_m^{m+2}) g(|\rho_m|^{m+2}) \frac{\mu(\rho_m)}{\mu(\rho_m^{m+2})} \right) \\ &\geq \frac{C}{2^b} - 0 \end{aligned} \tag{3.20}$$

as  $m$  is sufficiently large. Because  $\psi \in \mathcal{B}_\omega(\mathbf{D}^n)$  we have

$$0 < \sum_{j=1}^n \omega(z_j^{(m)}) \left| \frac{\partial \psi}{\partial z_j}(z^{(m)}) f_m(\varphi(z^{(m)})) \right| \leq \|\psi\|_\omega |f_m(\varphi(z^{(m)}))| \rightarrow 0 \quad (m \rightarrow \infty). \tag{3.21}$$

Therefore, (3.16), (3.20) and (3.21) imply

$$\begin{aligned} &\|T_{\psi, \varphi} f_m\|_\omega \\ &\geq \sum_{j=1}^n \omega(z_j^{(m)}) \left| \psi(z^{(m)}) \frac{\partial f_m}{\partial w_1}(\varphi(z^{(m)})) \frac{\partial \varphi_1}{\partial z_j}(z^{(m)}) + \frac{\partial \psi}{\partial z_j}(z^{(m)}) f_m(\varphi(z^{(m)})) \right| \\ &\geq \sum_{j=1}^n \frac{\omega(z_j^{(m)})}{\mu(\varphi_1(z^{(m)}))} \left| \psi(z^{(m)}) \frac{\partial \varphi_1}{\partial z_j}(z^{(m)}) \right| \mu(\varphi_1(z^{(m)})) \left| \frac{\partial f_m}{\partial w_1}(\varphi(z^{(m)})) \right| \\ &\quad - \sum_{j=1}^n \omega(z_j^{(m)}) \left| \frac{\partial \psi}{\partial z_j}(z^{(m)}) f_m(\varphi(z^{(m)})) \right| \geq C \varepsilon_0 \end{aligned}$$

as  $m$  is sufficiently large. This contradicts the  $\lim_{m \rightarrow \infty} \|T_{\psi, \varphi} f_m\|_{\omega} = 0$ .

(2) Suppose that  $\psi \varphi^{\alpha} \in \mathcal{B}_{\omega, 0}(\mathbf{D}^n)$  and (1.5) holds. Then by (3.10), (3.11) and Theorem A,  $T_{\psi, \varphi} : \mathcal{B}_{\mu, 0}(\mathbf{D}^n) \rightarrow \mathcal{B}_{\omega, 0}(\mathbf{D}^n)$  is bounded. Hence  $T_{\psi, \varphi}(\mathcal{B}_{\mu, 0}(\mathbf{D}^n)) \subset \mathcal{B}_{\omega, 0}(\mathbf{D}^n)$ . From (1) we know  $T_{\psi, \varphi} : \mathcal{B}_{\mu}(\mathbf{D}^n) \rightarrow \mathcal{B}_{\omega}(\mathbf{D}^n)$  is compact. Because  $\mathcal{B}_{\omega, 0}(\mathbf{D}^n)$  is a closed subspace of the Banach space  $\mathcal{B}_{\omega}(\mathbf{D}^n)$ , we know that  $T_{\psi, \varphi} : \mathcal{B}_{\mu, 0}(\mathbf{D}^n) \rightarrow \mathcal{B}_{\omega, 0}(\mathbf{D}^n)$  is compact. Conversely, it is trivial that  $\psi \varphi^{\alpha} \in \mathcal{B}_{\omega, 0}(\mathbf{D}^n)$  for all multi-index  $\alpha$ . Assume  $\psi$  and  $\varphi$  violate (1.5). Then we only define the test function as (3.18), then  $f_m$  is holomorphic on  $\overline{\mathbf{D}^n}$  and hence in  $\mathcal{B}_{\mu, 0}(\mathbf{D}^n)$ . Similar to that in the proof of (1), we can get a contradiction. The proof is completed.

## 4 Final remarks

Let  $\partial^* \mathbf{D}^n = \{z \in \mathbf{C}^n : |z_j| = 1, j = 1, 2, \dots, n\}$  be the Shilov boundary of  $\mathbf{D}^n$ . Instead of using the closure of all polynomials in  $n$  complex variables under  $\|\cdot\|_{\mu}$ , we also define the other little  $\mu$ -Bloch type space  $\mathcal{B}_{\mu, *}( \mathbf{D}^n )$  as

$$\mathcal{B}_{\mu, *}( \mathbf{D}^n ) = \left\{ f \in \mathcal{B}_{\mu}( \mathbf{D}^n ) : \lim_{z \rightarrow \partial^* \mathbf{D}^n} \sum_{j=1}^n \mu(z_j) \left| \frac{\partial f}{\partial z_j}(z) \right| = 0 \right\}.$$

It is clear that  $\mathcal{B}_{\mu, *}( \mathbf{D}^n )$  is a closed subspace of  $\mathcal{B}_{\mu}( \mathbf{D}^n )$ . Similarly, we have the following theorems.

**Theorem 4.1** Let  $\psi \in H(\mathbf{D}^n)$  and  $\varphi \in H(\mathbf{D}^n, \mathbf{D}^n)$ . Then  $T_{\psi, \varphi} : \mathcal{B}_{\mu, *}( \mathbf{D}^n ) \rightarrow \mathcal{B}_{\omega, *}( \mathbf{D}^n )$  is bounded if and only if the following conditions are all satisfied:

- (1)  $\sup_{z \in \mathbf{D}^n} \sum_{j=1}^n \omega(z_j) \left| \frac{\partial \psi}{\partial z_j}(z) \right| \left( 1 + \sum_{k=1}^n \int_0^1 |\varphi_k(z)| \frac{dt}{\mu(t)} \right) < \infty$ ,
- (2)  $\sup_{z \in \mathbf{D}^n} \sum_{j,k=1}^n \frac{\omega(z_j)}{\mu(\varphi_k(z))} \left| \psi(z) \frac{\partial \varphi_k}{\partial z_j}(z) \right| < \infty$ ,
- (3)  $\psi \in \mathcal{B}_{\omega, *}( \mathbf{D}^n )$  and  $\psi \varphi_k \in \mathcal{B}_{\omega, *}( \mathbf{D}^n )$  for all  $k = 1, 2, \dots, n$ .

**Theorem 4.2** Let  $\psi \in H(\mathbf{D}^n)$ ,  $\varphi \in H(\mathbf{D}^n, \mathbf{D}^n)$  and  $\int_0^1 \frac{dt}{\mu(t)} = \infty$ . Then  $T_{\psi, \varphi} : \mathcal{B}_{\mu, *}( \mathbf{D}^n ) \rightarrow \mathcal{B}_{\omega, *}( \mathbf{D}^n )$  is compact if and only if the following conditions are all satisfied:

- (1)  $\lim_{\varphi(z) \rightarrow \partial \mathbf{D}^n} \sum_{j=1}^n \omega(z_j) \left| \frac{\partial \psi}{\partial z_j}(z) \right| \left( 1 + \sum_{k=1}^n \int_0^1 |\varphi_k(z)| \frac{dt}{\mu(t)} \right) = 0$ ,
- (2)  $\lim_{\varphi(z) \rightarrow \partial \mathbf{D}^n} \sum_{j,k=1}^n \frac{\omega(z_j)}{\mu(\varphi_k(z))} \left| \psi(z) \frac{\partial \varphi_k}{\partial z_j}(z) \right| = 0$ ,
- (3)  $\psi \in \mathcal{B}_{\omega, *}( \mathbf{D}^n )$  and  $\psi \varphi_k \in \mathcal{B}_{\omega, *}( \mathbf{D}^n )$  for all  $k = 1, 2, \dots, n$ .

**Theorem 4.3** Let  $\psi \in H(\mathbf{D}^n)$ ,  $\varphi \in H(\mathbf{D}^n, \mathbf{D}^n)$  and  $\int_0^1 \frac{dt}{\mu(t)} < \infty$ . Then  $T_{\psi, \varphi} : \mathcal{B}_{\mu, *}( \mathbf{D}^n ) \rightarrow \mathcal{B}_{\omega, *}( \mathbf{D}^n )$  is compact if and only if the following conditions are all satisfied:

- (1)  $\lim_{|\varphi_k(z)| \rightarrow 1} \sum_{j=1}^n \frac{\omega(z_j)}{\mu(\varphi_k(z))} \left| \psi(z) \frac{\partial \varphi_k}{\partial z_j}(z) \right| = 0, k = 1, 2, \dots, n$ ,
- (2)  $\psi \in \mathcal{B}_{\omega, *}( \mathbf{D}^n )$  and  $\psi \varphi_k \in \mathcal{B}_{\omega, *}( \mathbf{D}^n )$  for all  $k = 1, 2, \dots, n$ .

The proof of these theorems goes as the proof of Theorem A, Theorem B and Theorem C in Sect. 2 and Sect. 3 respectively. We only need to notice that if conditions (1)-(3) hold in Theorem 4.1, then similar to that in the proof of Theorem 2.3 in [14], we also can obtain  $T_{\psi, \varphi} f \in \mathcal{B}_{\omega, *}( \mathbf{D}^n )$  for any  $f \in \mathcal{B}_{\mu, *}( \mathbf{D}^n )$ .

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# COMPACTNESS OF COMPOSITION OPERATOR IN THE LIPSCHITZ SPACE OF THE POLYDISC

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**ABSTRACT.** In 1987, Shapiro showed that composition operator induced by symbol  $\varphi$  is compact on the Lipschitz space if and only if the infinity norm of  $\varphi$  is less than 1 by a spectral-theoretic argument, where  $\varphi$  is a holomorphic self-map of the unit disk. In this note, we shall generalize Shapiro's result to the  $n$ -dimensional case.

## 1. INTRODUCTION

Let  $U^n$  be the unit polydisc of  $n$ -dimensional complex spaces  $C^n$  with boundary  $\partial U^n$ , the class of all holomorphic functions on domain  $U^n$  will be denoted by  $H(U^n)$ . Let  $\varphi(z) = (\varphi_1(z), \dots, \varphi_n(z))$  be a holomorphic self-map of  $U^n$ , composition operator is defined by

$$C_\varphi(f)(z) = f(\varphi(z))$$

for any  $f \in H(U^n)$  and  $z \in U^n$ .

During the past years much effort has been devoted to the research of such operators on a variety of Banach spaces of holomorphic functions with the goal of explaining the operator-theoretic behavior of  $C_\varphi$ , such as boundedness and compactness, in terms of the function-theoretic properties of the symbol  $\varphi$ . We recommend the interested readers refer to the books by J. H. Shapiro [9] and Cowen and MacCluer [2], which are good sources for information on much of the developments in the theory of composition operators up to the middle of last decade, as well as some papers for  $n$ -dimensional case by Zhou et al [4, 11, 12, 13, 14, 15, 16] on the Bloch space in polydiscs or classical symmetric domains, Gorkin and MacCluer [4] between Hardy spaces in the unit ball.

To our surprise, by a spectral-theoretic argument, Shapiro [10] obtained the following fact:  $C_\varphi$  is compact on the Lipschitz space  $L_{1-\alpha}(D)$  if and only if  $\|\varphi\|_\infty < 1$ . In this paper, we shall generalize Shapiro's result to the unit polydisc.

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## 2. NOTATION AND BACKGROUND

Throughout the paper,  $D$  is the unit disk in one dimensional complex plane, and  $|||z||| = \max_{1 \leq j \leq n} \{|z_j|\}$  stands for the supremum norm on the unit polydisc. Define  $Rf(z) = \langle \nabla f(z), \bar{z} \rangle$  where  $z = (z_1, \dots, z_n) \in U^n$ , and  $H(U^n, D)$  for the class of the holomorphic mappings from  $U^n$  to  $D$ . For  $0 < \alpha < 1$ , it is well known that the Lipschitz space  $L_{1-\alpha}(U^n)$  is equivalent to  $\alpha$ -Bloch space, which is defined to be the space of holomorphic functions  $f \in H(U^n)$  such that

$$||f||_{1-\alpha} = \sup_{z \in U^n} \sum_{j=1}^n (1 - |z_j|^2)^\alpha \left| \frac{\partial f}{\partial z_j}(z) \right| < \infty.$$

Here, Lipschitz space  $L_{1-\alpha}(U^n)$  is a Banach space with the equivalent norm

$$||f|| = |f(0)| + ||f||_{1-\alpha}.$$

Moreover, one should note that every holomorphic function in Lipschitz space extends continuously to the closed unit polydisc.

The Kobayashi distance  $k_{U^n}$  of  $U^n$  is given by

$$k_{U^n}(z, w) = \frac{1}{2} \log \frac{1 + |||\phi_z(w)|||}{1 - |||\phi_z(w)|||},$$

where  $\phi_z : U^n \rightarrow U^n$  is the automorphism of  $U^n$  given by

$$\phi_z(w) = \left( \frac{w_1 - z_1}{1 - \bar{z}_1 w_1}, \dots, \frac{w_n - z_n}{1 - \bar{z}_n w_n} \right)$$

Since the map  $t \rightarrow \log \frac{1+t}{1-t}$  is strictly increasing on  $[0, 1)$ , it follows that

$$k_{U^n}(z, w) = \max_{1 \leq j \leq n} \left\{ \frac{1}{2} \log \frac{1 + \left| \frac{w_j - z_j}{1 - \bar{z}_j w_j} \right|}{1 - \left| \frac{w_j - z_j}{1 - \bar{z}_j w_j} \right|} \right\} = \max_{1 \leq j \leq n} \{\rho(z_j, w_j)\},$$

where  $\rho$  is the Poincaré distance on the unit disk  $D \subset C$ .

Following [1], the horosphere  $E(x, R)$  of center  $x \in \partial U^n$  and radius  $R$  and the Korányi region  $H(x, M)$  of vertex  $x$  and amplitude  $M$  are defined by

$$E(x, R) = \{z \in U^n : \limsup_{w \rightarrow x} [k_{U^n}(z, w) - k_{U^n}(0, w)] < \frac{1}{2} \log R\}$$

and

$$H(x, M) = \{z \in U^n : \limsup_{w \rightarrow x} [k_{U^n}(z, w) - k_{U^n}(0, w)] + k_{U^n}(0, z) < \log M\}.$$

We say that  $f$  has  $K$ -limit  $L \in C$  at  $x$  if  $f(z) \rightarrow L$  as  $z \rightarrow x$  inside any Korányi region  $H(x, M)$ , we shall write  $\tilde{K} - \lim_{z \rightarrow x} f(z) = L$ .

Let  $f \in H(U^n, D)$  and  $x \in \partial U^n$ . If there is  $\delta$  such that

$$\liminf_{w \rightarrow x} \frac{1 - |f(w)|}{1 - |||w|||} = \delta < \infty,$$

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we call  $f$  is  $\delta - Julia$  at  $x$ . If there exists  $\tau \in \partial U^n$  such that

$$f(E(x, R)) \subseteq E(\tau, \delta R)$$

for all  $R$ , we call this  $\tau$  is the restricted  $E$ -limit of  $f$  at  $x$ .

It should be noticed that  $\delta > 0$ . In fact,

$$\rho(0, f(w)) \leq \rho(0, f(0)) + \rho(f(0), f(w)) \leq \rho(0, f(0)) + k_{U^n}(0, w);$$

therefore  $\frac{1-|f(w)|}{1-||w||} \geq \frac{1-|f(0)|}{2(1+|f(0)|)} > 0$ .

## 3. SOME LEMMAS

**Lemma 1.** (*Julia-Wolff-Carathéodory Theorem, Theorem 4.1 in [1]*) Let  $f \in H(U^n, D)$  be  $\delta - Julia$  at  $x \in \partial U^n$ , and  $\tau \in \partial U$  be the restricted  $E$ -limit of  $f$  at  $x$ , then

$$\tilde{K} - \lim_{z \rightarrow x} \frac{\partial f}{\partial x}(z) = \delta \tau.$$

**Lemma 2.** (*Theorem 1 in [11] or Corollary 4.1 in [14]*) Composition operator  $C_\varphi$  is bounded on the Lipschitz space  $L_{1-\alpha}(U^n)$  if and only if there is a constant  $M > 0$  such that

$$\sum_{k,l=1}^n \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \left( \frac{1 - |z_k|^2}{1 - |\varphi_l(z)|^2} \right)^\alpha \leq M$$

for  $z \in U^n$ .

**Lemma 3.** (*Theorem 2 in [11] or Corollary 4.2 in [14]*) Composition operator  $C_\varphi$  is compact on the Lipschitz space  $L_{1-\alpha}(U^n)$  if and only if

$$\lim_{\delta \rightarrow 0} \sup_{\text{dist}(\varphi(z), \partial U^n) < \delta} \sum_{k,l=1}^n \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1 - |z_k|^2)^\alpha}{(1 - |\varphi_l(z)|^2)^\alpha} = 0.$$

**Lemma 4.** (*Lemma 3.2 in [1]*) Let  $f \in H(U^n, D)$  and  $x \in \partial U^n$ . Then

$$\liminf_{w \rightarrow x} \frac{1 - |f(w)|}{1 - ||w||} = \liminf_{t \rightarrow 1^-} \frac{1 - |f(\varphi_x(t))|}{1 - t},$$

where  $\varphi_x(t) = tx$  for any  $t \in [0, 1)$ .

## 4. MAIN THEOREM

**Theorem 1.** Suppose  $C_\varphi$  is bounded on  $L_{1-\alpha}(U^n)$ , then for every  $1 \leq l \leq n$  and  $\xi \in \partial U^n$  with  $|\varphi_l(\xi)| = 1$ ,  $\varphi_l$  is  $\delta - Julia$  at  $\xi$ .

*Proof.* For every  $1 \leq l \leq n$  and  $\xi \in \partial U^n$  with  $\varphi_l(\xi) = \eta$  and  $\eta = e^{\theta_0}$ , we will show that  $\varphi_l$  is  $\delta - Julia$  at  $\xi$  according to the following cases.

**Case 1:**  $\xi = (\xi_1, \xi')$ ,  $\xi_1 = e^{\theta_1}$  and  $||\xi'||| < 1$ .

First we consider the special case for  $\xi = e_1 = (1, 0, \dots, 0)$  and  $\eta = 1$ .

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For  $r \in (1/2, 1)$ , define  $\sigma(r) = (r, 0, \dots, 0) = re_1$  and set  $g(r) = \varphi_l(re_1)$ , then  $g'(r) = \frac{\partial \varphi_l}{\partial z_1}(re_1)$ . It follows from Lemma 2 that the boundedness of  $C_\varphi$  implies that

$$h(r) = R\varphi_l(re_1)\left(\frac{1-r}{1-\varphi_l(re_1)}\right)^\alpha = rg'(r)\left(\frac{1-r}{1-g(r)}\right)^\alpha$$

is bounded.

Putting  $u(r) = \frac{1-g(r)}{1-r}$ , it is easy to see that  $g'(r) = -(1-r)u'(r) + u(r)$  and

$$h(r) = ru(r)^{-\alpha}[-(1-r)u'(r) + u(r)].$$

If we write  $v(r) = u(r)^{1-\alpha}$ , then

$$-\frac{1}{1-\alpha}(1-r)v'(r) + v(r) = \frac{h(r)}{r}$$

the general solution of this differential equation is

$$v(r) = -\frac{1-\alpha}{(1-r)^{1-\alpha}} \int_1^r \frac{h(s)}{s(1-s)^\alpha} ds + \frac{C}{(1-r)^{1-\alpha}}.$$

Since  $h$  is bounded, the first term in the right above is a bounded function of  $r$ , and moreover  $v(r)$  is of the order  $o(\frac{1}{(1-r)^{1-\alpha}})$  as  $r \rightarrow 1^-$ , so we have  $C = 0$ . Hence  $v$ , and moreover  $u$  is also bounded, according to Lemma 4, for some  $\delta$ ,  $\varphi_l$  is  $\delta$ -Julia at  $e_1$ .

Now we return to the proof in case 1. Considering the mapping  $\tilde{\varphi}_l : U^n \rightarrow U^n$ , where

$$\tilde{\varphi}_l(z_1, z') = e^{-i\theta_0} \cdot \varphi_l(e^{i\theta_1} z_1, \phi_{\xi'}(z'))$$

for  $z = (z_1, z') \in U^n$ . It is easy to check that  $C_{\tilde{\varphi}_l}$  is bounded on  $L_{1-\alpha}(U^n)$  and  $\tilde{\varphi}_l(e_1) = 1$ .

By the above argument, we get  $\liminf_{t \rightarrow 1^-} \frac{1-|\tilde{\varphi}_l(te_1)|}{1-t} = \delta < +\infty$ , that is

$$\begin{aligned} \liminf_{t \rightarrow 1^-} \frac{1-|\varphi_l(t\xi_1, \xi')|}{1-t} &= \liminf_{t \rightarrow 1^-} \lim_{r \rightarrow 1^-} \frac{1-|\varphi_l(t\xi_1, r\xi')|}{1-t} \\ &\geq \liminf_{t \rightarrow 1^-} \frac{1-|\varphi_l(t\xi_1, t\xi')|}{1-t}. \end{aligned}$$

It follows from Lemma 4 that

$$\liminf_{w \rightarrow \xi} \frac{1-|\varphi_l(\xi)|}{1-|||\xi|||} = \delta < +\infty.$$

**Case2:**  $\xi = (\xi_1, \xi_2, \xi')$ ,  $\xi_1 = e^{\theta_1}$ ,  $\xi_2 = e^{\theta_2}$  and  $|||\xi'||| < 1$ .

Now assume  $\varphi_l(1, 1, 0, \dots, 0) = 1$ , and set  $g(r) = \varphi_l(r, r, 0, \dots, 0)$  for  $r \in (1/2, 1)$ . then  $g'(r) = \frac{\partial \varphi_l}{\partial z_1}(r, r, 0, \dots, 0) + \frac{\partial \varphi_l}{\partial z_2}(r, r, 0, \dots, 0)$ , and so  $R\varphi_l(r, r, 0, \dots, 0) = rg'(r)$ , we can deal with it as in the case 1, and we can get  $u$  is bounded, furthermore

$$\liminf_{w \rightarrow \xi} \frac{1-|\varphi_l(\xi)|}{1-|||\xi|||} = \delta < +\infty.$$

**Case 3:** For the case  $\varphi_l(\xi) = 1$  with  $\xi = \sum_{k=1}^n \beta_k e_k$ , where  $\beta_k = 0$  or  $1$ , and  $e_k = (0, \dots, 0, 1, 0, \dots, 0)$  with the  $k$ -th component is  $1$ , otherwise  $0$ ; and even more general case, in a similar argument with the cases 1 and 2, we can also show

$$\liminf_{w \rightarrow \xi} \frac{1 - |\varphi_l(\xi)|}{1 - |||\xi|||} = \delta < +\infty.$$

This completes the proof of this theorem.  $\square$

**Remark.** Here  $|\varphi_l(\xi)| = 1$  means the continuous extension of  $\varphi_l$  on the closed unit polydisc, since for each  $1 \leq l \leq n$ ,  $\varphi_l \in L_{1-\alpha}(U^n)$ . And by Theorem 3.1 in [1], if  $\varphi_l$  is  $\delta$ -Julia at  $\xi$ , then  $\varphi_l$  has restricted  $E$ -limit  $\tau \in \partial U^n$  at  $\xi$ . Moreover,  $\varphi_l(\xi) = \tau$  as a non-tangential limit.

**Theorem 2.**  $C_\varphi$  is compact on  $L_{1-\alpha}(U^n)$  if and only if  $\varphi_l \in L_{1-\alpha}(U^n)$  and  $||\varphi_l||_\infty < 1$  for each  $l = 1, 2, \dots, n$ .

*Proof.* Sufficiency is obvious. Now we just turn to the necessity. Suppose to the contrary that there exists  $l$  ( $1 \leq l \leq n$ ) satisfying  $|\varphi_l(\xi)| = 1$  for some  $\xi \in \partial U^n$ . It follows from Theorem 1 that  $\varphi_l$  is  $\delta$ -Julia at  $\xi$ , therefore by Lemma 1, we have  $R\varphi_l(z)$  has  $K$ -limit at  $\xi$ . Hence

$$\begin{aligned} & \sum_{k,l=1}^n \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1 - |z_k|^2)^\alpha}{(1 - |\varphi_l(z)|^2)^\alpha} \\ & \geq \sum_{k,l=1}^n \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1 - |||z|||^2)^\alpha}{(1 - |\varphi_l(z)|^2)^\alpha} \\ & \geq \sum_{k,l=1}^n |z_k| \cdot \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1 - |||z|||^2)^\alpha}{(1 - |\varphi_l(z)|^2)^\alpha} \\ & \geq C \sum_{l=1}^n |R\varphi_l(z)| \frac{(1 - |||z|||^2)^\alpha}{(1 - |\varphi_l(z)|^2)^\alpha} \geq C\delta^{1-\alpha}. \end{aligned}$$

as  $z \rightarrow \xi$  inside any Korányi region, where we can take  $C = \frac{1}{2^\alpha}$ . It is a contradiction to the compactness of  $C_\varphi$  by Lemma 3. Now the proof of Theorem 2 is completed.  $\square$

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# INTERPOLATION FUNCTIONS OF THE $q$ -GENOCCHI AND THE $q$ -EULER POLYNOMIALS OF HIGHER ORDER

YOUNG-HEE KIM, KYUNG-WON HWANG, AND TAEKYUN KIM

**Abstract** Cangul-Ozden-Simsek [1] constructed the  $q$ -Genocchi numbers of high order using a fermionic  $p$ -adic integral on  $\mathbb{Z}_p$ , and gave Witt's formula and the interpolation functions of these numbers. In this paper, we present the generalization of the higher order  $q$ -Euler numbers and  $q$ -Genocchi numbers of Cangul-Ozden-Simsek. We define  $q$ -extensions of  $w$ -Euler numbers and polynomials, and  $w$ -Genocchi numbers and polynomials of high order using the multivariate fermionic  $p$ -adic integral on  $\mathbb{Z}_p$ . We have the interpolation functions of these numbers and polynomials. We obtain the distribution relations for  $q$ -extensions of  $w$ -Euler and  $w$ -Genocchi polynomials. We also have the interesting relation for  $q$ -extensions of these polynomials. We define  $(h, q)$ -extensions of  $w$ -Euler and  $w$ -Genocchi polynomials of high order. We have the interpolation functions for  $(h, q)$ -extensions of these polynomials. Moreover, we obtain some meaningful results of  $(h, q)$ -extensions of  $w$ -Euler and  $w$ -Genocchi polynomials.

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**Key words and phrases** : Genocchi numbers and polynomials, Euler numbers and polynomials,  $q$ -Genocchi numbers,  $q$ -Euler numbers, fermionic  $p$ -adic integral

## 1. INTRODUCTION, DEFINITIONS AND NOTATIONS

Many authors have been studied on the multiple Genocchi and Euler numbers, and multiple zeta functions (cf. [1-2], [4-6], [9-10], [14], [17], [19], [22], [24]). In [10], Kim, the first author of this paper, presented a systematic study of some families of multiple  $q$ -Euler numbers and polynomials. By using the  $q$ -Volkenborn integration on  $\mathbb{Z}_p$ , Kim constructed the  $p$ -adic  $q$ -Euler numbers and polynomials of higher order, and gave the generating function of these numbers and the Euler  $q$ - $\zeta$ -function. In [14], Kim studied some families of multiple  $q$ -Genocchi and  $q$ -Euler numbers by using the multivariate  $p$ -adic  $q$ -Volkenborn integral on  $\mathbb{Z}_p$ , and gave interesting identities related to these numbers.

Recently, Cangul-Ozden-Simsek [1] constructed the  $q$ -Genocchi numbers of high order by using a fermionic  $p$ -adic integral on  $\mathbb{Z}_p$ , and gave Witt's formula and the interpolation functions of these numbers. In [17], Kim gave another constructions of the  $q$ -Euler and  $q$ -Genocchi numbers, which were different from those of Cangul-Ozden-Simsek. Kim obtained the interesting relationship between the  $q$ - $w$ -Euler numbers and  $q$ - $w$ -Genocchi numbers, and gave the interpolation functions of these numbers. In this paper, we will present the generalization of the higher order  $q$ -Euler numbers and  $q$ -Genocchi numbers of Cangul-Ozden-Simsek approaching as Kim did in [17].

Let  $p$  be a fixed odd prime number. Throughout this paper, the symbols  $\mathbb{Z}_p$ ,  $\mathbb{Q}_p$ ,  $\mathbb{C}$  and  $\mathbb{C}_p$  denote the ring of  $p$ -adic rational integers, the field of  $p$ -adic



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rational numbers, the complex number field and the completion of algebraic closure of  $\mathbb{Q}_p$ , respectively. Let  $\mathbb{N}$  be the set of natural numbers and  $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ . Let  $v_p$  be the normalized exponential valuation of  $\mathbb{C}_p$  with  $|p|_p = p^{-v_p(p)} = \frac{1}{p}$ .

The symbol  $q$  can be treated as a complex number,  $q \in \mathbb{C}$ , or as a  $p$ -adic number,  $q \in \mathbb{C}_p$ . If  $q \in \mathbb{C}$ , then we always assume that  $|q| < 1$ . If  $q \in \mathbb{C}_p$ , then we usually assume that  $|1 - q|_p < 1$ .

Now we will recall some  $q$ -notations. The  $q$ -basic natural numbers are defined by  $[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$  ( $n \in \mathbb{N}$ ),  $[n]_{-q} = \frac{1-(-q)^n}{1+q}$  and the  $q$ -factorial by  $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$ . In this paper, we use the notation  $[x]_q = \frac{1-q^x}{1-q}$  and  $[x]_{-q} = \frac{1-(-q)^x}{1+q}$ . Hence  $\lim_{q \rightarrow 1} [x]_q = x$  for any  $x$  with  $|x|_p \leq 1$  in the present  $p$ -adic case (cf. [1-25]).

The  $q$ -shift factorial is given by

$$(a; q)_0 = 1, \quad (a; q)_k = (1-a)(1-aq) \cdots (1-aq^{k-1}).$$

We note that  $\lim_{q \rightarrow 1} (a; q)_k = (1-a)^k$ . It is known that

$$(a; q)_\infty = (1-a)(1-aq)(1-aq^2) \cdots = \prod_{i=1}^{\infty} (1-aq^{i-1}) \quad (\text{see [8]}).$$

From the definition of the  $q$ -shift factorial, we note that

$$(a; q)_k = \frac{(a; q)_\infty}{(aq^k; q)_\infty}.$$

Since  $\binom{-\alpha}{l} = (-1)^l \binom{\alpha+l-1}{l}$ , it follows that

$$\frac{1}{(1-z)^\alpha} = (1-z)^{-\alpha} = \sum_{l=0}^{\infty} \binom{-\alpha}{l} (-z)^l = \sum_{l=0}^{\infty} \binom{\alpha+l-1}{l} z^l.$$

The  $q$ -binomial theorem is given by

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty},$$

where  $z, q \in \mathbb{C}$  with  $|z| < 1$ ,  $|q| < 1$ . For the special case, when  $a = q^\alpha$  ( $\alpha \in \mathbb{C}$ ), we can write as follows:

$$\begin{aligned} \frac{1}{(z; q)_\alpha} &= \frac{(zq^\alpha; q)_\infty}{(z; q)_\infty} = \sum_{n=0}^{\infty} \frac{(q^\alpha; q)_n}{(q; q)_n} z^n \\ &= \sum_{n=0}^{\infty} \frac{(1-q^\alpha)(1-q^{\alpha+1}) \cdots (1-q^{\alpha+n-1})}{(1-q)(1-q^2) \cdots (1-q^n)} z^n \\ &= \sum_{n=0}^{\infty} \frac{[\alpha]_q [\alpha+1]_q \cdots [\alpha+n-1]_q}{[1]_q [2]_q \cdots [n]_q} z^n \\ &= \sum_{n=0}^{\infty} \frac{[\alpha]_q [\alpha+1]_q \cdots [\alpha+n-1]_q}{[n]_q!} z^n. \end{aligned}$$

The  $q$ -binomial coefficients are defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{[n]_q [n-1]_q \cdots [n-k+1]_q}{[k]_q!} \quad (\text{see [14], [16]}).$$

Hence it follows that

$$\frac{1}{(z; q)_\alpha} = \sum_{n=0}^{\infty} \binom{n+\alpha-1}{n}_q z^n,$$

which converges to  $\frac{1}{(1-z)^\alpha} = \sum_{n=0}^{\infty} \binom{n+\alpha-1}{n} z^n$  as  $q \rightarrow 1$ .

We say that  $f$  is a uniformly differentiable function at a point  $a \in \mathbb{Z}_p$ , and write  $f \in UD(\mathbb{Z}_p)$ , the set of uniformly differentiable function, if the difference quotients  $F_g(x, y) = \frac{f(x)-f(y)}{x-y}$  have a limit  $l = f'(a)$  as  $(x, y) \rightarrow (a, a)$ . For  $f \in UD(\mathbb{Z}_p)$ , the  $q$ -deformed bosonic  $p$ -adic integral is defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) \frac{q^x}{[p^N]_q},$$

and the  $q$ -deformed fermionic  $p$ -adic integral is defined by

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) \frac{(-q)^x}{[p^N]_{-q}}.$$

The fermionic  $p$ -adic integral on  $\mathbb{Z}_p$  is defined as

$$I_{-1}(f) = \lim_{q \rightarrow 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x).$$

It follows that  $I_{-1}(f_1) = -I_{-1}(f) + 2f(0)$ , where  $f_1(x) = f(x+1)$  (cf. [4-17]).

The classical Euler polynomials  $E_n(x)$  are defined as

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!},$$

and the Euler numbers  $E_n$  are defined as  $E_n = E_n(0)$  (cf. [1-25]). The Genocchi numbers are defined as

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} \quad \text{for } |t| < \pi,$$

and the Genocchi polynomials  $G_n(x)$  are defined as

$$\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} \quad (\text{see [12], [14], [21]}).$$

It is known that the  $w$ -Euler polynomials  $E_{n,w}(x)$  are defined as

$$\frac{2}{we^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,w}(x) \frac{t^n}{n!},$$

and  $E_{n,w} = E_{n,w}(0)$  are called the  $w$ -Euler numbers. The  $w$ -Genocchi polynomials  $G_{n,w}(x)$  are defined as

$$\frac{2t}{we^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_{n,w}(x) \frac{t^n}{n!},$$

and  $G_{n,w} = G_{n,w}(0)$  are called the  $w$ -Genocchi numbers (see [1]).

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The  $w$ -Euler polynomials  $E_{n,w}^{(r)}(x)$  of order  $r$  are defined as

$$\left(\frac{2}{we^t + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_{n,w}^{(r)}(x) \frac{t^n}{n!} \quad (\text{see [1]}),$$

and  $E_{n,w}^{(r)} = E_{n,w}^{(r)}(0)$  are called the  $w$ -Euler numbers of order  $r$ . The  $w$ -Genocchi polynomials  $G_{n,w}^{(r)}(x)$  of order  $r$  are defined as

$$\frac{2t}{we^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_{n,w}^{(r)}(x) \frac{t^n}{n!} \quad (\text{see [1]}),$$

and  $G_{n,w}^{(r)} = G_{n,w}^{(r)}(0)$  are called the  $w$ -Euler numbers of order  $r$ . When  $r = 1$  and  $w = 1$ ,  $E_{n,w}^{(r)}(x)$  and  $E_{n,w}^{(r)}$  are the ordinary Euler polynomials and numbers, and  $G_{n,w}^{(r)}(x)$  and  $G_{n,w}^{(r)}$  are the ordinary Genocchi polynomials and numbers, respectively.

In Section 2, we define  $q$ -extensions of  $w$ -Euler numbers and polynomials of order  $r$  and  $w$ -Genocchi numbers and polynomials of order  $r$ , respectively, using the multivariate fermionic  $p$ -adic integral on  $\mathbb{Z}_p$ . We obtain the interpolation functions of these numbers and polynomials. We have the distribution relations for  $q$ -extensions of  $w$ -Euler polynomials and those of  $w$ -Genocchi polynomials. We obtain the interesting relation for  $q$ -extensions of these polynomials. We also define  $(h, q)$ -extensions of  $w$ -Euler and  $w$ -Genocchi polynomials of order  $r$ . We have the interpolation functions for  $(h, q)$ -extensions of these polynomials. Moreover, we obtain some meaningful results of  $(h, q)$ -extensions of  $w$ -Euler and  $w$ -Genocchi polynomials when  $h = r - 1$ .

## 2. ON THE EXTENSION OF THE HIGHER ORDER $q$ -GENOCCHI NUMBERS AND $q$ -EULER NUMBERS OF CANGUL-OZDEN-SİMSEK

In this section, we assume that  $w \in \mathbb{C}_p$  with  $|1 - w|_p < 1$  and  $q \in \mathbb{C}_p$  with  $|1 - q|_p < 1$ . Recently, Cangul-Ozden-Simsek [1] constructed  $w$ -Genocchi numbers of order  $r$ ,  $G_{n,w}^{(r)}$ , as follows:

$$\begin{aligned} (1) \quad & t^r \int_{\mathbb{Z}_p^r} w^{x_1 + \dots + x_r} e^{t(x_1 + \dots + x_r)} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= 2^r \left(\frac{t}{we^t + 1}\right)^r = \sum_{n=0}^{\infty} G_{n,w}^{(r)} \frac{t^n}{n!}, \end{aligned}$$

where  $\int_{\mathbb{Z}_p^r} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}$  ( $r$ -times) and  $r \in \mathbb{N}$ . They also consider the  $q$ -extension of  $G_{n,w}^{(r)}$  as follows:

$$\begin{aligned} (2) \quad & t^r \int_{\mathbb{Z}_p^r} q^{\sum_{i=1}^r (h-i+1)x_i} e^{t(\sum_{i=1}^r x_i)} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= \frac{2^r t^r}{(q^h e^t + 1) \cdots (q^{h-r+1} e^t + 1)} = \sum_{n=0}^{\infty} G_{n,q}^{(h,r)} \frac{t^n}{n!}. \end{aligned}$$

From (2), they obtained the following interesting formula:

$$(3) \quad G_{n+r,q}^{(r-1,r)} = 2^r r! \binom{n+r}{r} \sum_{v=0}^{\infty} \binom{r+v-1}{v}_q (-1)^v v^n.$$

The following similar formula obtains from the above formula. We are very interested to study the following formula using  $[v]_q$  instead of  $v$  in the (3).

$$2^r r! \binom{n+r}{r} \sum_{v=0}^{\infty} \binom{r+v-1}{v}_q (-1)^v [v]_q^n.$$

In the viewpoint of the  $q$ -extension of (1) using the multivariate  $p$ -adic integral on  $\mathbb{Z}_p$ , we define the  $q$ -analogue of  $w$ -Euler numbers of order  $r$ ,  $E_{n,w,q}^{(r)}$ , as follows:

$$(4) \quad E_{n,w,q}^{(r)} = \int_{\mathbb{Z}_p^r} w^{x_1+\dots+x_r} [x_1+\dots+x_r]_q^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r).$$

From (4), we derive that

$$\begin{aligned} E_{n,w,q}^{(r)} &= \frac{2^r}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \left( \frac{1}{1+q^l w} \right)^r \\ &= \frac{2^r}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m q^{lm} w^m \\ &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m [m]_q^n. \end{aligned}$$

Therefore, we obtain the following theorem.

**Theorem 1.** Let  $r \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ . Then we have

$$(5) \quad E_{n,w,q}^{(r)} = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m [m]_q^n.$$

Let  $F^{(r)}(t, w|q) = \sum_{n=0}^{\infty} E_{n,w,q}^{(r)} \frac{t^n}{n!}$ . By (4) and (5), we see that

$$\begin{aligned} F^{(r)}(t, w|q) &= \int_{\mathbb{Z}_p^r} w^{x_1+\dots+x_r} e^{t[x_1+\dots+x_r]_q} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m e^{t[m]_q}. \end{aligned}$$

Thus we obtain the following corollary.

**Corollary 2.** Let  $F^{(r)}(t, w|q) = \sum_{n=0}^{\infty} E_{n,w,q}^{(r)} \frac{t^n}{n!}$ . Then we have

$$F^{(r)}(t, w|q) = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m e^{t[m]_q}.$$

Let us define the  $q$ -extension of  $w$ -Euler polynomials of order  $r$  as follows:

$$(6) \quad E_{n,w,q}^{(r)}(x) = \int_{\mathbb{Z}_p^r} w^{x_1+\dots+x_r} [x+x_1+\dots+x_r]_q^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r).$$

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By (6), we have that

$$\begin{aligned} E_{n,w,q}^{(r)}(x) &= \frac{2^r}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \left( \frac{1}{1+q^l w} \right)^r \\ &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m [m+x]_q^n \end{aligned}$$

Therefore, we obtain the following theorem.

**Theorem 3.** *Let  $r \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ . Then we have*

$$(7) \quad E_{n,w,q}^{(r)}(x) = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m [m+x]_q^n.$$

Let  $F^{(r)}(t, w, x|q) = \sum_{n=0}^{\infty} E_{n,w,q}^{(r)}(x) \frac{t^n}{n!}$ . By (6) and (7), we have

$$\begin{aligned} F^{(r)}(t, w, x|q) &= \int_{\mathbb{Z}_p^r} w^{x_1+\dots+x_r} e^{t[x+x_1+\dots+x_r]_q} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m e^{t[m+x]_q}. \end{aligned}$$

Therefore we have the following corollary.

**Corollary 4.** *Let  $F^{(r)}(t, w, x|q) = \sum_{n=0}^{\infty} E_{n,w,q}^{(r)}(x) \frac{t^n}{n!}$ . Then we have*

$$(8) \quad F^{(r)}(t, w, x|q) = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m e^{t[m+x]_q}.$$

Now we define the  $q$ -extension of  $w$ -Genocchi polynomials of order  $r$ ,  $G_{n,w,q}^{(r)}(x)$ , as follows:

$$(9) \quad 2^r t^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m e^{t[m+x]_q} = \sum_{n=0}^{\infty} G_{n,w,q}^{(r)}(x) \frac{t^n}{n!}.$$

Then we have

$$\begin{aligned} (10) \quad & \sum_{n=0}^{\infty} G_{n,w,q}^{(r)}(x) \frac{t^n}{n!} \\ &= t^r \int_{\mathbb{Z}_p^r} w^{x_1+\dots+x_r} e^{t[x+x_1+\dots+x_r]_q} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p^r} w^{x_1+\dots+x_r} [x+x_1+\dots+x_r]_q^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) r! \binom{n+r}{r} \frac{t^{n+r}}{(n+r)!}. \end{aligned}$$

By comparing the coefficients on the both sides of (10), we see that

$$G_{0,w,q}^{(r)}(x) = G_{1,w,q}^{(r)}(x) = \cdots = G_{r-1,w,q}^{(r)}(x) = 0,$$

and

$$\begin{aligned}
 (11) \quad & G_{n+r,w,q}^{(r)}(x) \\
 &= r! \binom{n+r}{r} \int_{\mathbb{Z}_p^r} w^{x_1+x_2+\dots+x_r} [x+x_1+\dots+x_r]_q^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
 &= r! \binom{n+r}{r} E_{n,w,q}^{(r)}(x).
 \end{aligned}$$

In the special case of  $x = 0$ ,  $G_{n,w,q}^{(r)}(0) = G_{n,w,q}^{(r)}$  are called the  $q$ -extension of  $w$ -Genocchi numbers of order  $r$ . By (11), we have the following theorem.

**Theorem 5.** *Let  $r \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ . Then we have*

$$\begin{aligned}
 \frac{G_{n+r,w,q}^{(r)}(x)}{r! \binom{n+r}{r}} &= \int_{\mathbb{Z}_p^r} w^{x_1+\dots+x_r} [x+x_1+\dots+x_r]_q^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
 &= E_{n,w,q}^{(r)}(x),
 \end{aligned}$$

and  $G_{0,w,q}^{(r)}(x) = G_{1,w,q}^{(r)}(x) = \cdots = G_{r-1,w,q}^{(r)}(x) = 0$ .

Now we consider the distribution relation for the  $q$ -extension of  $w$ -Euler polynomials of order  $r$ . For  $d \in \mathbb{N}$  with  $d \equiv 1 \pmod{2}$ , by (8), we see that

$$\begin{aligned}
 (12) \quad & F^{(r)}(t, w, x|q) \\
 &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m e^{t[m+x]_q} \\
 &= \sum_{a_1, \dots, a_r=0}^{d-1} \left( \prod_{i=1}^r w^{a_i} \right) (-1)^{a_1+\dots+a_r} 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^{md} e^{t[d]_q [m + \frac{a_1+\dots+a_r+x}{d}]_{q^d}} \\
 &= \sum_{a_1, \dots, a_r=0}^{d-1} \left( \prod_{i=1}^r w^{a_i} \right) (-1)^{a_1+\dots+a_r} F^{(r)}([d]_q t, w^d, \frac{a_1+\dots+a_r+x}{d} | q^d).
 \end{aligned}$$

By (12), we obtain the following distribution relations for  $E_{n,w,q}^{(r)}(x)$  and  $G_{n+r,w,q}^{(r)}(x)$ , respectively.

**Theorem 6.** *Let  $r \in \mathbb{N}$ ,  $n \in \mathbb{Z}_+$  and  $d \in \mathbb{N}$  with  $d \equiv 1 \pmod{2}$ . Then we have*

$$E_{n,w,q}^{(r)}(x) = [d]_q^n \sum_{a_1, \dots, a_r=0}^{d-1} \left( \prod_{i=1}^r w^{a_i} \right) (-1)^{a_1+\dots+a_r} E_{n,w^d,q^d}^{(r)}\left(\frac{a_1+\dots+a_r+x}{d}\right).$$

Furthermore,

$$G_{n+r,w,q}^{(r)}(x) = [d]_q^n \sum_{a_1, \dots, a_r=0}^{d-1} \left( \prod_{i=1}^r w^{a_i} \right) (-1)^{a_1+\dots+a_r} G_{n+r,w^d,q^d}^{(r)}\left(\frac{a_1+\dots+a_r+x}{d}\right).$$

For the extension of (2), we consider the  $(h, q)$ -extension of  $w$ -Euler polynomials of order  $r$ . For  $h \in \mathbb{Z}$ ,  $r \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ , let us define the  $(h, q)$ -extension of  $w$ -Euler polynomial of order  $r$  as follows:

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$$\begin{aligned}
 (13) \quad E_{n,w,q}^{(h,r)}(x) &= \int_{\mathbb{Z}_p^r} w^{x_1+\dots+x_r} [x+x_1+\dots+x_r]_q^n q^{\sum_{i=1}^r (h-i+1)x_i} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r).
 \end{aligned}$$

From (13), we obtain that

$$\begin{aligned}
 (14) \quad E_{n,w,q}^{(h,r)}(x) &= \frac{2^r}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{lx}}{(1+q^{l+h}w)(1+q^{l+h-1}w) \cdots (1+q^{l+h-r+1}w)} \\
 &= \frac{2^r}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{lx}}{(-q^{l+h}w : q^{-1})_r} \\
 &= \frac{2^r}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \sum_{m=0}^{\infty} \binom{m+r-1}{m}_{q^{-1}} (-1)^m q^{(l+h)m} w^m \\
 &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_{q^{-1}} (-1)^m q^{hm} w^m [m+x]_q^n.
 \end{aligned}$$

Therefore, we have the following theorem.

**Theorem 7.** Let  $h \in \mathbb{Z}$ ,  $r \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ . Then we have

$$\begin{aligned}
 (15) \quad E_{n,w,q}^{(h,r)}(x) &= \frac{2^r}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{lx}}{(-q^{l+h}w : q^{-1})_r} \\
 &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_{q^{-1}} (-1)^m q^{hm} w^m [m+x]_q^n.
 \end{aligned}$$

We also have the following result.

**Corollary 8.** Let  $F^{(h,r)}(t, w, x|q) = \sum_{n=0}^{\infty} E_{n,w,q}^{(h,r)}(x) \frac{t^n}{n!}$ . Then we have

$$(16) \quad F^{(h,r)}(t, w, x|q) = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_{q^{-1}} (-1)^m q^{hm} w^m e^{t[m+x]_q}.$$

**Remark 1.** In the special case  $x = 0$ ,  $E_{n,w,q}^{(h,r)}(0) = E_{n,w,q}^{(h,r)}$  are called the  $(h, q)$ -extension of  $w$ -Euler numbers of order  $r$ .

If we take  $h = r - 1$  in (14), then we have

$$\begin{aligned}
 (17) \quad E_{n,w,q}^{(r-1,r)}(x) &= \frac{2^r}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{lx}}{(1+q^{l+r-1}w)(1+q^{l+r-2}w) \cdots (1+q^l w)} \\
 &= \frac{2^r}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{lx}}{(-q^l w : q)_r} \\
 &= \frac{2^r}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-1)^m q^{lm} w^m \\
 &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-1)^m w^m [m+x]_q^n.
 \end{aligned}$$

Then we have the following theorem.

**Theorem 9.** Let  $r \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ . Then we have

$$\begin{aligned} E_{n,w,q}^{(r-1,r)}(x) &= \frac{2^r}{(1-q)^n} \sum_{l=0}^n \frac{\binom{n}{l} (-1)^l q^{lx}}{(-q^l w : q)_r} \\ &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-1)^m w^m [m+x]_q^n. \end{aligned}$$

We also the following corollary.

**Corollary 10.** Let  $F^{(r-1,r)}(t, w, x|q) = \sum_{n=0}^{\infty} E_{n,w,q}^{(r-1,r)}(x) \frac{t^n}{n!}$ . Then we have

$$(18) \quad F^{(r-1,r)}(t, w, x|q) = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-1)^m w^m e^{t[m+x]_q}.$$

From (18), we note that

$$\begin{aligned} F^{(r-1,r)}(t, w, x|q) &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-1)^m w^m e^{t[m+x]_q} \\ (19) \quad &= \sum_{a_1, \dots, a_r=0}^{d-1} q^{\sum_{i=0}^r (r-i)a_i} (-1)^{a_1+\dots+a_r} w^{a_1+\dots+a_r} \\ &\quad \times 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_{q^d} (-1)^m w^{md} e^{t[d]_q [m + \frac{a_1+\dots+a_r+x}{d}]_{q^d}} \\ &= \sum_{a_1, \dots, a_r=0}^{d-1} q^{\sum_{i=0}^r (r-i)a_i} (-1)^{a_1+\dots+a_r} w^{a_1+\dots+a_r} \\ &\quad \times F^{(r-1,r)}([d]_q t, w^d, \frac{a_1+\dots+a_r+x}{d} | q^d), \end{aligned}$$

where  $d \in \mathbb{N}$  with  $d \equiv 1 \pmod{2}$ . By (19), we obtain the following the distribution relation for  $E_{n,w,q}^{(r-1,r)}(x)$ .

**Theorem 11.** For  $r \in \mathbb{N}$ ,  $n \in \mathbb{Z}_+$  and  $d \in \mathbb{N}$  with  $d \equiv 1 \pmod{2}$ . Then we have

$$\begin{aligned} E_{n,w,q}^{(r-1,r)}(x) &= [d]_q^n \sum_{a_1, \dots, a_r=0}^{d-1} q^{\sum_{i=0}^r (r-i)a_i} (-1)^{a_1+\dots+a_r} w^{a_1+\dots+a_r} E_{n,w^d,q^d}^{(r-1,r)}\left(\frac{a_1+\dots+a_r+x}{d}\right). \end{aligned}$$

Now we define the  $(h, q)$ -extension of  $w$ -Genocchi polynomials  $G_{n,w,q}^{(h,r)}(x)$  of order  $r$  as follows:

$$(20) \quad 2^r t^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_{q^{-1}} (-1)^m q^{hm} w^m e^{t[m+x]_q} = \sum_{n=0}^{\infty} G_{n,w,q}^{(h,r)}(x) \frac{t^n}{n!}.$$



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Then we have

$$\begin{aligned}
 & \sum_{n=0}^{\infty} G_{n,w,q}^{(h,r)}(x) \frac{t^n}{n!} \\
 (21) \quad &= t^r \int_{\mathbb{Z}_p^r} q^{\sum_{i=0}^{\infty} (h-i+1)x_i} w^{x_1+\dots+x_r} e^{t[x+x_1+\dots+x_r]_q} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
 &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p^r} q^{\sum_{i=0}^{\infty} (h-i+1)x_i} w^{x_1+\dots+x_r} [x+x_1+\dots+x_r]_q^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
 &\quad \times r! \binom{n+r}{r} \frac{t^{n+r}}{(n+r)!}.
 \end{aligned}$$

From (13) and (21), we derive the following result.

**Theorem 12.** *Let  $r \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ . Then we have*

$$\begin{aligned}
 \frac{G_{n+r,w,q}^{(h,r)}(x)}{r! \binom{n+r}{r}} &= \int_{\mathbb{Z}_p^r} q^{\sum_{i=0}^{\infty} (h-i+1)x_i} w^{x_1+\dots+x_r} [x+x_1+\dots+x_r]_q^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
 &= E_{n,w,q}^{(h,r)}(x),
 \end{aligned}$$

and  $G_{0,w,q}^{(h,r)}(x) = G_{1,w,q}^{(h,r)}(x) = \cdots = G_{r-1,w,q}^{(h,r)}(x) = 0$ .

When  $h = r - 1$  in Theorem 12, we have

$$\begin{aligned}
 \frac{G_{n+r,w,q}^{(r-1,r)}(x)}{r! \binom{n+r}{r}} &= \int_{\mathbb{Z}_p^r} q^{\sum_{i=0}^{\infty} (r-i)x_i} w^{x_1+\dots+x_r} [x+x_1+\dots+x_r]_q^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
 &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-1)^m w^m [m+x]_q^n \\
 &= \frac{2^r}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \\
 &= E_{n,w,q}^{(r-1,r)}(x).
 \end{aligned}$$

**Remark 2.** *In the special case  $x = 0$ ,  $G_{n,w,q}^{(h,r)}(0) = G_{n,w,q}^{(h,r)}$  are called the  $(h, q)$ -extension of  $w$ -Genocchi numbers of order  $r$ .*

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# The improved preconditioned AOR method for irreducible $L$ -matrices \*

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## Abstract

In this paper, we first improve the preconditioned AOR method for irreducible  $L$ -matrices considered by Yun and Kim [Convergence of the preconditioned AOR method for irreducible  $L$ -matrices, *Appl. Math. Comput.*, (2008), doi:10.1016/j.amc.2007.11.045], and then we prove the convergence of our method. Lastly, numerical experiments to illustrate the theoretical results are provided. When choosing the approximately optimal parameters, our method has smaller spectral radii of the iterative matrices than the method provided in Yun and Kim's, which is shown through numerical examples.

*Keywords:* AOR method; Linear system;  $L$ -matrix; Preconditioned AOR method;

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## 1 Introduction

For solving the linear system

$$Ax = b, \quad x, b \in \mathbb{R}^n, \quad (1)$$

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where  $A = (a_{i,j}) \in \mathbb{R}^{n \times n}$  is a nonsingular matrix. An iteration method is often considered. For any splitting,  $A = M - N$  with  $\det(M) \neq 0$ , the basic iterative method for system (1) is

$$x^{i+1} = M^{-1}Nx^k + M^{-1}b, \quad k = 0, 1, \dots$$

For simplicity, without loss of generality, we assume that  $A$  has a splitting of the form  $A = I - L - U$ , where  $I$  denotes  $n \times n$  identity matrix,  $-L$  and  $-U$  are the strictly lower, and upper triangular parts of  $A$ , respectively. In [2], the AOR iterative method is defined

$$x^{(i+1)} = (I - rL)^{-1}[(1 - \omega)I + (\omega - r)L + \omega U]x^{(i)} + (I - rL)^{-1}\omega b. \quad (2)$$

Then the iteration matrix of the AOR iterative method is

$$T_{r\omega} = (I - rL)^{-1}[(1 - \omega)I + (\omega - r)L + \omega U], \quad (3)$$

where  $\omega$  and  $r$  are real parameters with  $\omega \neq 0$ .

We now transform the original linear system (1) into the preconditioned linear system

$$PAx = Pb, \quad (4)$$

where  $P$  is called a preconditioner. Then the basic iterative method for solving the linear system (1) is

$$x_{k+1} = M_p^{-1}N_px_k + M_p^{-1}Pb, \quad k = 0, 1, \dots, \quad (5)$$

where  $x_0$  is an initial vector and  $PA = M_p - N_p$  is a splitting of  $PA$ .

In this paper, we consider the following two cases where

$$P = P_{S_1} \quad \text{or} \quad P = P_{S_2}.$$

The preconditioner  $P_{S_1}$  is of the form  $P_{S_1} = I + S_1$ , where

$$S_1 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ -\alpha_2 a_{21} & 0 & \dots & 0 \\ -\alpha_3 a_{31} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -\alpha_n a_{n1} & 0 & \dots & 0 \end{pmatrix}.$$

The preconditioner  $P_{S_2}$  is of the form  $P_{S_2} = I + S_2$ , where

$$S_2 = \begin{pmatrix} 0 & -\alpha_2 a_{12} & 0 & \dots & 0 \\ 0 & 0 & -\alpha_3 a_{23} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\alpha_n a_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Here,  $\alpha_i$  ( $i = 2, 3, \dots, n$ ) are real parameters and  $\alpha_i > 0$  for  $i = 2, 3, \dots, n$ . Especially, if  $\alpha_i = 1$  ( $i = 2, 3, \dots, n$ ), then the preconditioners are the case in Yun and Kim [1]. Let

$$\tilde{A} = P_{S_1}A, \text{ and } S_1U = D' + L' + U',$$

where  $D'$  is a diagonal matrix,  $L'$  is a strictly lower triangular matrix, and  $U'$  is a strictly upper triangular matrix. Then, from  $S_1L = 0$  we can obtain

$$\tilde{A} = (I + S_1)(I - L - U) = I - L - U + S_1 - S_1U = \tilde{D} - \tilde{L} - \tilde{U}, \quad (6)$$

where  $\tilde{D} = I - D'$ ,  $\tilde{L} = L - S_1 + L'$ , and  $\tilde{U} = U + U'$ .

Let  $A = P_{S_2}A$  and  $S_2L = D^* + L^*$ , where  $D^*$  is a diagonal matrix and  $L^*$  is a strictly lower triangular matrix. Then, we obtains

$$\bar{A} = (I + S_2)(I - L - U) = I - L - U + S_2 - S_2L - S_2U = \bar{D} - \bar{L} - \bar{U}, \quad (7)$$

where  $\bar{D} = I - D^*$ ,  $\bar{L} = L + L^*$ , and  $\bar{U} = U - S_2 + S_2U$ .

If we apply the AOR iterative method to the preconditioned linear system (4), then we get the *preconditioned AOR iterative method* whose iteration matrix is

$$\tilde{T}_{r\omega} = (\tilde{D} - r\tilde{L})^{-1}((1 - \omega)\tilde{D} + (\omega - r)\tilde{L} + \omega\tilde{U}) \quad \text{if } P = P_{S_1}, \quad (8)$$

or

$$\bar{T}_{r\omega} = (\bar{D} - r\bar{L})^{-1}((1 - \omega)\bar{D} + (\omega - r)\bar{L} + \omega\bar{U}) \quad \text{if } P = P_{S_2}, \quad (9)$$

When  $\omega = r$ , the (preconditioned) AOR iterative method reduces to the (preconditioned) SOR iterative method [5]. For  $\omega = r$ ,  $T_{r\omega}$ ,  $\tilde{T}_{r\omega}$  and  $\bar{T}_{r\omega}$  defined by (3), (8) and (9) are denoted by  $T_\omega$ ,  $\tilde{T}_\omega$  and  $\bar{T}_\omega$ , respectively. That is,

$$T_\omega = (I - \omega L)^{-1}((1 - \omega)I + \omega U), \quad (10)$$

$$\tilde{T}_\omega = (\tilde{D} - \omega\tilde{L})^{-1}((1 - \omega)\tilde{D} + \omega\tilde{U}), \quad (11)$$

$$\bar{T}_\omega = (\bar{D} - \omega\bar{L})^{-1}((1 - \omega)\bar{D} + \omega\bar{U}), \quad (12)$$

In this paper, first in section 2, we present some notation, definitions and preliminary results. Next, we discuss the convergence of the preconditioned AOR iterative method which uses

$$P = I + S_1 \text{ or } P = I + S_2$$

as a preconditioner in section 3. Furthermore, in section 4, we provide numerical experiments to illustrate the theoretical results obtained in Section 3, and we find if we choose the set of parameters then our method has smaller spectral radii of the iterative matrices than the method provided in [1], which is shown through numerical examples. Lastly, in section 5, we obtain some conclusions.

## 2 Preliminaries

We shall use the following notations and lemmas. A matrix  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  is called a *Z-matrix* if  $a_{ij} \leq 0$  for  $i \neq j$ , and it is called an *L-matrix* if  $A$  is a Z-matrix and  $a_{ii} > 0$  for  $i = 1, 2, \dots$ . For a vector  $x \in \mathbb{R}^n$ ,  $x \geq 0$  ( $x > 0$ ) denotes that all components of  $x$  are nonnegative (positive). For two vectors  $x, y \in \mathbb{R}^n$ ,  $x \geq y$  ( $x > y$ ) means that  $x - y \geq 0$ , ( $x - y > 0$ ). These definitions carry immediately over to matrices.  $A(2 : n, 2 : n)$  denotes the submatrix of  $A \in \mathbb{R}^{n \times n}$  whose rows are indexed by  $2, 3, \dots, n$  and columns by  $2, 3, \dots, n$ . Let  $\rho(A)$  denotes the spectral radius of  $A$ , and  $A$  is called *irreducible* if the directed graph of  $A$  is strongly connected [3].

We first refer to the following result which is essentially.

**Theorem 2.1 ([3]).** *Let  $A \geq 0$  be an irreducible matrix. Then*

- (a)  *$A$  has a positive eigenvalue equal to  $\rho(A)$ ;*
- (b)  *$A$  has an eigenvector  $x > 0$  corresponding to  $\rho(A)$ ;*
- (c)  *$\rho(A)$  is a simple eigenvalue of  $A$ .*

**Theorem 2.2 ([4]).** *Let  $A \geq 0$  be a matrix. Then the following hold.*

- (a) *If  $Ax \geq \beta x$  for a vector  $x \geq 0$  and  $x \neq 0$ , then  $\rho(A) \geq \beta$ ;*
- (b) *If  $Ax \leq \gamma x$  for a vector  $x > 0$ , then  $\rho(A) \leq \gamma$ . Moreover, if  $A$  is irreducible and if  $\beta x \leq Ax \leq \gamma x$ , equality excluded, for a vector  $x \geq 0$  and  $x \neq 0$  and  $x \neq 0$ , then  $\beta < \rho(A) < \gamma$  and  $x > 0$ .*

## 3 Main results

**Theorem 3.1.** *Let  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  be an L-matrix and  $A(2 : n, 2 : n)$  be an irreducible submatrix of  $A$ . Suppose that there exists a nonempty set  $\beta \subset N_1 = \{2, 3, \dots, n\}$  and real parameters  $\alpha_i > 0$  for  $i = 2, 3, \dots, n$  such that*

$$\begin{cases} 0 < \alpha_i a_{1i} a_{i1} < 1, & \text{if } i \in \beta, \\ a_{1i} a_{i1} = 0, & \text{if } i \in N_1 - \beta. \end{cases}$$

*Let  $T_{r\omega}$  and  $\tilde{T}_{r\omega}$  be defined by (3) and (8). If  $0 \leq r \leq \omega \leq 1$  ( $\omega \neq 0, r \neq 1$ ), then*

- (a)  *$\rho(\tilde{T}_{r\omega}) < \rho(T_{r\omega})$  if  $\rho(T_{r\omega}) < 1$ ;*
- (b)  *$\rho(\tilde{T}_{r\omega}) = \rho(T_{r\omega})$  if  $\rho(T_{r\omega}) = 1$ ;*
- (c)  *$\rho(\tilde{T}_{r\omega}) > \rho(T_{r\omega})$  if  $\rho(T_{r\omega}) > 1$ .*

**Proof.** By (3),  $T_{r\omega}$  can be expressed as

$$T_{r\omega} = (1 - \omega)I + \omega(1 - r)L + \omega U + H, \quad (13)$$

where  $H$  is a nonnegative matrix. Since  $A$  is an  $L$ -matrix,  $L$  and  $U$  are nonnegative. From (13),  $T_{r\omega} \geq 0$ . Since  $A(2 : n, 2 : n)$  is irreducible and  $a_{1i}a_{i1} \neq 0$  for  $i \in \beta$ , it is easy to show that  $A$  is irreducible. Since  $\omega \neq 0$ ,  $r \neq 1$  and  $A$  is irreducible,  $\omega(1-r)L + \omega U$  is irreducible. Hence,  $T_{r\omega}$  is irreducible from (13). From Theorem 2.1, there exists a vector  $x > 0$  such that  $T_{r\omega}x = \lambda x$ , where  $\lambda = \rho(T_{r\omega})$ . Since  $L$  is a strictly lower triangular matrix,  $S_1L = 0$ . From  $T_{r\omega}x = \lambda x$ , one easily obtains

$$\begin{aligned} ((1-\omega)I + (\omega-r)L + \omega U)x &= \lambda(I-rL)x, \\ \omega S_1 Ux &= (\lambda + \omega - 1)S_1x. \end{aligned} \quad (14)$$

Using (8) and (14),

$$\begin{aligned} \tilde{T}_{r\omega}x - \lambda x &= (\tilde{D} - r\tilde{L})^{-1}((1-\omega)\tilde{D} + (\omega-r)\tilde{L} + \omega\tilde{U} - \lambda(\tilde{D} - r\tilde{L}))x \\ &= (\tilde{D} - r\tilde{L})^{-1}((1-\omega-\lambda)\tilde{D} + (\omega-r+\lambda r)\tilde{L} + \omega\tilde{U})x \\ &= (\tilde{D} - r\tilde{L})^{-1}((\omega+\lambda-1)D' + (\omega-r+\lambda r)(L' - S_1) + \omega U')x \\ &= (\tilde{D} - r\tilde{L})^{-1}((\lambda-1)D' + (\lambda-1)rL' + (r-\omega-\lambda r)S_1 + \omega S_1 U)x \\ &= (\tilde{D} - r\tilde{L})^{-1}((\lambda-1)(D' + rL') + (-\lambda r + \lambda + r - 1)S_1)x \\ &= (\lambda-1)(\tilde{D} - r\tilde{L})^{-1}(D' + rL' + (1-r)S_1)x. \end{aligned} \quad (15)$$

Since  $0 < \alpha_i a_{1i} a_{i1} < 1$ ,  $D'$ ,  $L'$  and  $S_1$  are nonnegative. Since  $\tilde{A} = \tilde{L} - \tilde{U}$  is also an  $L$ -matrix,  $\tilde{D}$ ,  $\tilde{L}$  and  $\tilde{U}$  are all nonnegative. By simple calculation,  $\tilde{T}_{r\omega}$  can be expressed as

$$\tilde{T}_{r\omega} = (1-\omega)I + \omega(1-r)\tilde{D}^{-1}\tilde{L} + \omega\tilde{D}^{-1}\tilde{U} + \tilde{H} = \begin{pmatrix} 1-\omega & \tilde{T}_{12} \\ 0 & \tilde{T}_{22} \end{pmatrix}, \quad (16)$$

where  $\tilde{H}$  is a nonnegative matrix,  $\tilde{T}_{12} \geq 0$  is an  $1 \times (n-1)$  matrix and  $\tilde{T}_{22} \geq 0$  is an  $(n-1) \times (n-1)$  matrix. Since  $a_{1i} \neq 0$  for  $i \in \beta$ ,  $\tilde{T}_{12}$  is a nonzero matrix. Since  $A(2 : n, 2 : n)$  is irreducible, it is easy to show that  $\tilde{A}(2 : n, 2 : n)$  is irreducible. Since  $\omega \neq 0$  and  $r \neq 1$ , from (16)  $\tilde{T}_{r\omega}(2 : n, 2 : n) = \tilde{T}_{22}$  is irreducible. Let

$$y = (D' + rL' + (1-r)S_1)x \text{ and } z = (\tilde{D} - r\tilde{L})^{-1}y. \quad (17)$$

Since  $a_{i1} \neq 0$  for  $i \in \beta$ ,  $r \neq 1$  and  $x > 0$ ,  $y \geq 0$  is a nonzero vector and the first component of  $y$  is zero. Since  $(\tilde{D} - r\tilde{L})^{-1}$  is a nonnegative lower triangular matrix,  $z \geq 0$  is also a nonzero vector and the first component of  $z$  is zero. Thus, we can set

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } z = \begin{pmatrix} 0 \\ z_2 \end{pmatrix}, \quad (18)$$

where  $x_1 \in \mathbb{R}^1 > 0$ ,  $x_2 \in \mathbb{R}^{n-1} > 0$ , and  $z_2 \in \mathbb{R}^{n-1} \geq 0$  is a nonzero vector. From (15)-(18),  $\tilde{T}_{r\omega}x - \lambda x = (\lambda-1)z$  and hence

$$(1-\omega)x_1 + \tilde{T}_{12}x_2 = \lambda_1, \quad (19)$$

$$\tilde{T}_{22}x_2 - \lambda x_2 = (\lambda - 1)z_2. \quad (20)$$

If  $\lambda > 1$ , then from (20) one obtains

$$\tilde{T}_{22}x_2 \geq \lambda x_2 \quad \text{and} \quad \tilde{T}_{22}x_2 \neq \lambda x_2. \quad (21)$$

From (20) and Theorem 2.2,  $\rho(\tilde{T}_{22}) > \lambda$ . Since  $0 \leq (1-\omega) < 1$ ,  $\rho(\tilde{T}_{r\omega}) = \rho(\tilde{T}_{22}) > \lambda = \rho(T_{r\omega})$ .

If  $\lambda < 1$ , then from (20) one obtains

$$\tilde{T}_{22}x_2 \leq \lambda x_2, \quad \tilde{T}_{22}x_2 \neq \lambda x_2. \quad (22)$$

Since  $\tilde{T}_{22} \geq 0$  is irreducible and  $x_2 > 0$ , from (22) and Theorem 2.2

$$\rho(\tilde{T}_{22}) < \lambda. \quad (23)$$

Since  $\tilde{T}_{12} \geq 0$  is nonzero and  $x_2 > 0$ ,  $\tilde{T}_{12}x_2 > 0$ . From (19)  $(1-\omega)x_1 < \lambda x_1$  and thus

$$(1-\omega) < \lambda. \quad (24)$$

Since  $\rho(\tilde{T}_{r\omega}) = \max\{(1-\omega), \rho(\tilde{T}_{22})\}$ , from (23) and (24)  $\rho(\tilde{T}_{r\omega}) < \lambda = \rho(T_{r\omega})$ .

If  $\lambda = 1$ , from (15)  $\tilde{T}_{r\omega}x = \lambda x$ . Hence, from Theorem 2.2  $\rho(\tilde{T}_{r\omega}) = \lambda = \rho(T_{r\omega})$  is obtained.  $\square$

**Corollary 3.2.** *Let  $A = (a_{ij}) \in R^{n \times n}$  be an L-matrix and  $A(2 : n, 2 : n)$  be an irreducible submatrix of  $A$ . Suppose that there exists a nonempty set  $\beta \subset N_1 = \{2, 3, \dots, n\}$  and real parameters  $\alpha_i > 0$  for  $i = 2, 3, \dots, n$  such that*

$$\begin{cases} 0 < \alpha_i a_{1i} a_{i1} < 1, & \text{if } i \in \beta, \\ a_{1i} a_{i1} = 0, & \text{if } i \in N_1 - \beta. \end{cases}$$

Let  $T_\omega$  and  $\tilde{T}_\omega$  be defined by (10) and (11). If  $0 < \omega < 1$ , then

- (a)  $\rho(\tilde{T}_\omega) < \rho(T_\omega)$  if  $\rho(T_\omega) < 1$ .
- (b)  $\rho(\tilde{T}_\omega) = \rho(T_\omega)$  if  $\rho(T_\omega) = 1$ .
- (c)  $\rho(\tilde{T}_\omega) > \rho(T_\omega)$  if  $\rho(T_\omega) > 1$ .

**Remark 3.3.** If  $r = \omega = 1$ , then the (preconditioned) AOR method reduces to the (preconditioned) Gauss-Seidel method. For  $r = \omega = 1$ , since  $T_{r\omega}$  and  $\tilde{T}_{22}$  used in the proof of Theorem 3.1 are not necessarily irreducible, the proof of Theorem 3.1 does not make sense. Hence, it can not be guaranteed that Corollary 3.2 holds for  $r = \omega = 1$ . Further work will discuss the case of  $r = \omega = 1$  under the similar assumptions used in Corollary 3.2.



**Lemma 3.4.** *Let  $A = (a_{ij}) \in R^{n \times n}$  be an  $L$ -matrix. Suppose that there exist a nonempty set  $\gamma \subset N_2 = \{1, 2, \dots, n-1\}$  and real parameters  $\alpha_i > 0$  for  $i = 2, 3, \dots, n$  such that  $\alpha_{i+1}a_{i,i+1}a_{i+1,i} < 1$  for all  $i \in N_2$ . Let  $T_{r\omega}$  and  $\bar{T}_{r\omega}$  be defined by (3) and (9). If  $0 \leq r \leq \omega \leq 1$  ( $\omega \neq 0, r \neq 1$ ) and  $A$  is irreducible even for  $a_{i,i+1}$  set to 0 for every  $i \in \gamma$ , then  $T_{r\omega}$  and  $\bar{T}_{r\omega}$  are nonnegative and irreducible.*

**Proof.** Since  $A$  is irreducible even for  $a_{i,i+1}$  set to zero for every  $i \in \gamma$ ,  $A = I - L - U$  is irreducible. Hence,  $T_{r\omega}$  is irreducible and nonnegative from (13). Let

$$\bar{A} = P_2 A = (I + S_2)A = \bar{D} - \bar{L} - \bar{U},$$

where  $\bar{D}$ ,  $\bar{L}$  and  $\bar{U}$  are defined as in (7). Since  $A$  is an  $L$ -matrix and  $\alpha_{i+1}a_{i,i+1}a_{i+1,i} < 1$  for all  $i \in N_2$ ,  $\bar{A}$  is also an  $L$ -matrix and thus  $\bar{D}$ ,  $\bar{L}$  and  $\bar{U}$  are all nonnegative. Since the nonzero structure of  $\bar{A}$  is the same as that of  $A$  with  $a_{i,i+1}$  set to zero for every  $i \in \gamma$ ,  $\bar{A}$  is also irreducible by assumption. Note that  $\bar{T}_{r\omega}$  can be expressed as

$$\bar{T}_{r\omega} = (1 - \omega)I + \omega(1 - r)\bar{D}^{-1}\bar{L}^{-1} + \omega\bar{D}^{-1}\bar{U}^{-1} + \bar{H}, \quad (25)$$

where  $\bar{H}$  is a nonnegative matrix. From (25),  $\bar{T}_{r\omega}$  is nonnegative. Since  $\omega \neq 0$ ,  $r \neq 1$  and  $\bar{A}$  is irreducible,  $\omega(1 - r)\bar{D}^{-1}\bar{L}^{-1} + \omega\bar{D}^{-1}\bar{U}^{-1}$  is irreducible. Hence,  $\bar{T}_{r\omega}$  is irreducible from (25).  $\square$

**Theorem 3.5.** *Let  $A = (a_{ij}) \in R^{n \times n}$  be an  $L$ -matrix. Suppose that there exists a nonempty set  $\gamma \subset N_2 = \{1, 2, \dots, n-1\}$  such that  $a_{i,i+1} \neq 0$  for all  $i \in \gamma$  and real parameters  $\alpha_i > 0$  for  $i = 2, 3, \dots, n$  such that  $\alpha_{i+1}a_{i,i+1}a_{i+1,i} < 1$  for all  $i \in N_2$ . Let  $T_{r\omega}$  and  $\bar{T}_{r\omega}$  be defined by (3) and (9). If  $0 \leq r \leq \omega \leq 1$  ( $\omega \neq 0, r \neq 1$ ) and  $A$  is irreducible even for  $a_{i,i+1}$  set to 0 for every  $i \in \gamma$ , then*

- (a)  $\rho(\bar{T}_{r\omega}) < \rho(T_{r\omega})$  if  $\rho(T_{r\omega}) < 1$ .
- (b)  $\rho(\bar{T}_{r\omega}) = \rho(T_{r\omega})$  if  $\rho(T_{r\omega}) = 1$ .
- (c)  $\rho(\bar{T}_{r\omega}) > \rho(T_{r\omega})$  if  $\rho(T_{r\omega}) > 1$ .

**Proof.** From Lemma 3.4,  $T_{r\omega}$  is nonnegative and irreducible. By Theorem 2.1, there exists a vector  $x > 0$  such that  $T_{r\omega}x = \lambda x$ , where  $\lambda = \rho(T_{r\omega})$ . From  $T_{r\omega}x = \lambda x$ , one easily obtains

$$\begin{aligned} ((1 - \omega)I + (\omega - r)L + \omega U)x &= \lambda(I - rL)x, \\ ((\lambda + \omega - I)S_2 + (r - \omega - \lambda r)S_2L)x &= \omega S_2 Ux. \end{aligned} \quad (26)$$

Using (9) and (26),

$$\begin{aligned}
\bar{T}_{r\omega}x - \lambda x &= (\bar{D} - r\bar{L})^{-1}((1 - \omega)\bar{D} + (\omega - r)\bar{L} + \omega\bar{U} - \lambda(\bar{D} - r\bar{L}))x \\
&= (\bar{D} - r\bar{L})^{-1}((1 - \omega - \lambda)\bar{D} + (\omega - r + \lambda r)\bar{L} + \omega\bar{U})x \\
&= (\bar{D} - r\bar{L})^{-1}((\omega + \lambda - 1)D^* + (\omega - r + \lambda r)L^* + \omega(S_2U - S_2))x \\
&= (\bar{D} - r\bar{L})^{-1}((\omega + \lambda - 1)(D^* + S_2) + (\omega - r + \lambda r)(L^* - S_2L) - \omega S_2)x \\
&= (\bar{D} - r\bar{L})^{-1}((\omega + \lambda - 1)D^* + (\lambda - 1)S_2 - (\omega - r + \lambda r)D^*)x \\
&= (\bar{D} - r\bar{L})^{-1}((-\lambda r + \lambda + r - 1)D^* + (\lambda - 1)S_2)x \\
&= (\lambda - 1)(\bar{D} - r\bar{L})^{-1}((1 - r)D^* + S_2)x.
\end{aligned} \tag{27}$$

Since  $0 < \alpha_i a_{1i} a_{i1} < 1$ ,  $\bar{D}$ ,  $D^*$ ,  $\bar{L}$  and  $S_2$  are all nonnegative. Let

$$y = ((1 - r)D^* + S_2)x \text{ and } z = (\bar{D} - r\bar{L})^{-1}y. \tag{28}$$

Since  $a_{i,i+1} \neq 0$  for  $i \in \gamma$  and  $x > 0$ ,  $y = (y_i) \geq 0$  and  $y_i$  is nonzero for  $i \in \gamma$ . Since  $(\bar{D} - r\bar{L})^{-1}$  is a nonnegative lower triangular matrix,  $z = (z_i) \geq 0$  and  $z_i$  is also nonzero for  $i \in \gamma$ . From (27) and (28), one obtains

$$\bar{T}_{r\omega}x - \lambda x = (\lambda - 1)z. \tag{29}$$

For the case of  $\lambda = 1$  and  $\lambda > 1$ ,  $\bar{T}_{r\omega}x = \lambda x$  and  $\bar{T}_{r\omega}x \geq \lambda x$  (with  $\bar{T}_{r\omega}x \neq \lambda x$ ) are obtained directly from Theorem 3.1 and 3.3, respectively. If  $\lambda < 1$ , then from (29)  $\bar{T}_{r\omega}x \leq \lambda x$  and  $\bar{T}_{r\omega}x \neq \lambda x$ . Since  $\bar{T}_{r\omega}$  is irreducible from Lemma 3.3, Theorem 2.2 implies that  $\rho(\bar{T}_{r\omega}) < \lambda = \rho(T_{r\omega})$ . Hence, the theorem follows from Theorem 2.2.  $\square$

**Corollary 3.6.** *Let  $A = (a_{ij}) \in R^{n \times n}$  be an L-matrix. Suppose that there exists a nonempty set  $\gamma \subset N_2 = \{1, 2, \dots, n-1\}$  such that  $a_{i,i+1} \neq 0$  for all  $i \in \gamma$  and real parameters  $\alpha_i > 0$  for  $i = 2, 3, \dots, n$  such that  $\alpha_{i+1} a_{i,i+1} a_{i+1,i} < 1$  for all  $i \in N_2$ . Let  $T_\omega$  and  $\bar{T}_\omega$  be defined by (10) and (12). If  $0 < \omega < 1$  and  $A$  is irreducible even for  $a_{i,i+1}$  set to 0 for every  $i \in \gamma$ , then*

- (a)  $\rho(\bar{T}_\omega) < \rho(T_\omega)$  if  $\rho(T_\omega) < 1$ .
- (b)  $\rho(\bar{T}_\omega) = \rho(T_\omega)$  if  $\rho(T_\omega) = 1$ .
- (c)  $\rho(\bar{T}_\omega) > \rho(T_\omega)$  if  $\rho(T_\omega) > 1$ .

**Remark 3.7.** If  $r = \omega = 1$ , then the proof of Theorem 3.4 does not make sense since it is not generally true that  $T_{r\omega}$  and  $\bar{T}_{r\omega}$  are irreducible. Hence, it can not be guaranteed that Corollary 3.6 holds for  $\omega = 1$ . Further work will discuss the case of  $r = \omega = 1$  under the similar assumptions used in Corollary 3.6.

## 4 Numerical experiments

In this section, we provide numerical experiments to illustrate the theoretical results obtained in Section 3. If we choose the set of parameters, then our method has smaller spectral radii of the iterative matrices than the method provided in [1]. All numerical experiments are carried out using MATLAB 7.1. For simplicity of comparison, suppose that all of  $\alpha_i$  are equal and let  $\alpha = \alpha_i$  for  $i = 2, 3, \dots, n$ . Let  $\rho(\tilde{T}_{r\omega})$  and  $\rho(\bar{T}_{r\omega})$  denote the spectral radii of the corresponding iteration matrices for  $\alpha = 1$ . Let  $\rho_{opt}(\tilde{T}_{r\omega})$  and  $\rho_{opt}(\bar{T}_{r\omega})$  denote the spectral radii of the corresponding iteration matrices when using approximately optimal parameters  $\alpha_{(\tilde{T}_{r\omega})}$  and  $\alpha_{(\bar{T}_{r\omega})}$ , respectively.

**Example 4.1.[1]** Consider a  $4 \times 4$  matrix  $A$  of the form

$$A = \begin{pmatrix} 1 & 0 & 0 & -0.3 \\ -0.3 & 1 & -0.3 & -0.3 \\ 0 & -0.3 & 1 & -0.3 \\ -0.3 & 0 & -0.3 & 1 \end{pmatrix}.$$

It is easy to see that the matrix  $A$  satisfies all assumption of Theorems 3.1 and 3.5. Note that  $\beta = \{4\} \subset N_1$  and  $\gamma = \{2, 3\} \subset N_2$ . Since the smaller spectral radius of the iteration matrix is, the faster the convergence rate is, we compare spectral radii of the iterative matrices in tables 1 and 2 for example 4.1.

Table 1 Spectral radii of the iterative matrices  $\rho(T_{r\omega})$ ,  $\rho(\tilde{T}_{r\omega})$  and  $\rho(\bar{T}_{r\omega})$  with various of  $r$  and  $\omega$

$r$	$\omega$	$\rho(T_{r\omega})$	$\rho(\tilde{T}_{r\omega})$	$\rho(\bar{T}_{r\omega})$	$\rho_{opt}(\tilde{T}_{r\omega})$	$\alpha_{(\tilde{T}_{r\omega})}$	$\rho(\bar{T}_{r\omega})$	$\alpha_{(\bar{T}_{r\omega})}$
0.9	1	0.3406	0.2937	0.2331	0.1881	4.5	0.1557	2.3
0.8	1	0.3852	0.3377	0.2844	0.2290	4.0	0.2111	2.1
0.7	1	0.4201	0.3721	0.3240	0.2599	3.7	0.2546	2.0
0.7	0.8	0.5361	0.4977	0.4592	0.3662	7.2	0.3679	4.5
0.6	0.8	0.5593	0.5204	0.4854	0.3850	6.0	0.3952	3.7
0.5	0.8	0.5793	0.5400	0.5079	0.3998	5.4	0.4214	3.3
0.5	0.6	0.6845	0.6550	0.6309	0.5297	8.4	0.5562	5.5
0.4	0.6	0.6976	0.6680	0.6458	0.5372	7.2	0.5706	4.8
0.3	0.6	0.7093	0.6795	0.6590	0.5425	6.5	0.5855	4.2
0.3	0.4	0.8062	0.7863	0.7727	0.6844	9.0	0.7209	5.7
0.2	0.4	0.8133	0.7933	0.7807	0.6855	7.9	0.7284	5.6
0.1	0.4	0.8197	0.7997	0.7879	0.6852	7.1	0.7364	4.9

Table 2 Spectral radii of the iterative matrices  $\rho(T_{r\omega})$ ,  $\rho(\tilde{T}_{r\omega})$  and  $\rho(\bar{T}_{r\omega})$  with various of  $\omega$ , where  $r = \omega$ 

$\omega$	$\rho(T_{r\omega})$	$\rho(\tilde{T}_{r\omega})$	$\rho(\bar{T}_{r\omega})$	$\rho_{opt}(\tilde{T}_{r\omega})$	$\alpha_{(\tilde{T}_{r\omega})}$	$\rho(\bar{T}_{r\omega})$	$\alpha_{(\bar{T}_{r\omega})}$
0.95	0.3465	0.3018	0.2391	0.2082	4.4	0.1645	2.5
0.9	0.4066	0.3643	0.3098	0.2585	5.5	0.2273	3.2
0.8	0.5081	0.4702	0.4275	0.3521	7.3	0.3416	4.6
0.7	0.5941	0.5604	0.5268	0.4396	8.6	0.4460	5.3
0.6	0.6695	0.6403	0.6140	0.5230	9.5	0.5416	5.4
0.5	0.7371	0.7125	0.6924	0.6036	10.1	0.6301	5.6
0.4	0.7984	0.7786	0.7638	0.6824	10.5	0.7129	5.5
0.3	0.8547	0.8398	0.8295	0.7607	10.5	0.7907	5.6
0.2	0.9066	0.8967	0.8903	0.8382	11.0	0.8641	5.9
0.1	0.9549	0.9499	0.9470	0.9175	10.9	0.9338	5.6

**Example 4.2.** Consider a  $n \times n$  matrix  $A$  of the form

$$A = \begin{pmatrix} 1 & c_1 & c_2 & c_3 & c_1 & \dots \\ c_3 & 1 & c_1 & c_2 & \ddots & c_1 \\ c_2 & c_3 & \ddots & \ddots & \ddots & c_3 \\ c_1 & \ddots & \ddots & 1 & c_1 & c_2 \\ c_3 & \ddots & c_2 & c_3 & 1 & c_1 \\ \dots & c_3 & c_1 & c_2 & c_3 & 1 \end{pmatrix},$$

where  $c_1 = -2/n$ ,  $c_2 = 0$ ,  $c_3 = -1/(n+2)$ . Clearly, the matrix  $A$  satisfies all assumptions of Theorems 3.1 and 3.5. Note that  $\beta = \{2, 4, 5, 7, 9, 10, \dots\} \subset N_1$  and  $\gamma = \{1, 2, 3, \dots, n-1\} \subset N_2$ . Numerical results for this matrix  $A$  are provided in Tables 3 and 4 for  $n = 30$  and in Tables 5 and 6 for  $n = 100$ .

Table 3 Spectral radii of the iterative matrices  $\rho(T_{r\omega})$ ,  $\rho(\tilde{T}_{r\omega})$  and  $\rho(\bar{T}_{r\omega})$  with various of  $r$  and  $\omega$ 

$r$	$\omega$	$\rho(T_{r\omega})$	$\rho(\tilde{T}_{r\omega})$	$\rho(\bar{T}_{r\omega})$	$\rho_{opt}(\tilde{T}_{r\omega})$	$\alpha_{(\tilde{T}_{r\omega})}$	$\rho(\bar{T}_{r\omega})$	$\alpha_{(\bar{T}_{r\omega})}$
0.9	1	0.9076	0.9047	0.8966	0.6781	19.0	0.1557	2.3
0.8	1	0.9147	0.9120	0.9054	0.6906	19.3	0.2111	2.1
0.7	1	0.9207	0.9182	0.9128	0.7032	19.6	0.2546	2.0
0.7	0.8	0.9365	0.9345	0.9302	0.7624	19.6	0.3679	4.5
0.6	0.8	0.9407	0.9388	0.9352	0.7712	19.8	0.3952	3.7
0.5	0.8	0.9443	0.9426	0.9395	0.7788	19.9	0.4214	3.3
0.5	0.6	0.9583	0.9569	0.9546	0.8340	19.9	0.5562	5.5
0.4	0.6	0.9607	0.9594	0.9574	0.8395	20.0	0.5706	4.8
0.3	0.6	0.9628	0.9616	0.9599	0.8448	20.1	0.5855	4.2
0.3	0.4	0.9752	0.9744	0.9733	0.8966	20.1	0.7209	5.7
0.2	0.4	0.9765	0.9757	0.9747	0.8999	20.2	0.7284	5.6
0.1	0.4	0.9776	0.9769	0.9761	0.9032	20.3	0.7364	4.9

Table 4 Spectral radii of the iterative matrices  $\rho(T_{r\omega})$ ,  $\rho(\tilde{T}_{r\omega})$  and  $\rho(\bar{T}_{r\omega})$  with various of  $\omega$ , where  $r = \omega$

$\omega$	$\rho(T_{r\omega})$	$\rho(\tilde{T}_{r\omega})$	$\rho(\bar{T}_{r\omega})$	$\rho_{opt}(\tilde{T}_{r\omega})$	$\alpha_{(\tilde{T}_{r\omega})}$	$\rho(\bar{T}_{r\omega})$	$\alpha_{(\bar{T}_{r\omega})}$
0.95	0.9083	0.9055	0.8969	0.6908	19.0	0.6308	9.8
0.9	0.9168	0.9142	0.9069	0.7109	19.0	0.6812	10.4
0.8	0.9317	0.9296	0.9243	0.7524	19.3	0.7588	11.3
0.7	0.9445	0.9427	0.9389	0.7921	19.6	0.8195	12.1
0.6	0.9555	0.9541	0.9514	0.8284	19.8	0.8622	12.6
0.5	0.9652	0.9641	0.9622	0.8617	19.9	0.8980	13.1
0.4	0.9738	0.9729	0.9716	0.8930	20.0	0.9265	13.5
0.3	0.9814	0.9808	0.9800	0.9224	20.1	0.9493	13.8
0.2	0.9882	0.9878	0.9874	0.9500	20.2	0.9690	14.1
0.1	0.9944	0.9942	0.9940	0.9758	20.3	0.9858	14.4

Table 5 Spectral radii of the iterative matrices  $\rho(T_{r\omega})$ ,  $\rho(\tilde{T}_{r\omega})$  and  $\rho(\bar{T}_{r\omega})$  with various of  $r$  and  $\omega$

$r$	$\omega$	$\rho(T_{r\omega})$	$\rho(\tilde{T}_{r\omega})$	$\rho(\bar{T}_{r\omega})$	$\rho_{opt}(\tilde{T}_{r\omega})$	$\alpha_{(\tilde{T}_{r\omega})}$	$\rho(\bar{T}_{r\omega})$	$\alpha_{(\bar{T}_{r\omega})}$
0.9	1	0.9705	0.9702	0.9694	0.8208	75.0	0.8576	38.8
0.8	1	0.9729	0.9726	0.9720	0.8228	75.2	0.8981	39.9
0.7	1	0.9749	0.9747	0.9742	0.8296	76.1	0.9210	40.5
0.7	0.8	0.9799	0.9797	0.9793	0.8637	76.1	0.9259	45.4
0.6	0.8	0.9813	0.9812	0.9808	0.8687	76.8	0.9419	45.8
0.5	0.8	0.9826	0.9824	0.9821	0.8735	77.4	0.9524	45.8
0.5	0.6	0.9869	0.9868	0.9866	0.9051	77.4	0.9584	47.9
0.4	0.6	0.9877	0.9876	0.9874	0.9083	77.8	0.9639	48.5
0.3	0.6	0.9884	0.9883	0.9881	0.9115	78.2	0.9676	48.9
0.3	0.4	0.9923	0.9922	0.9921	0.9410	78.2	0.9784	48.9
0.2	0.4	0.9927	0.9926	0.9925	0.9430	78.5	0.9807	49.3
0.1	0.4	0.9931	0.9930	0.9929	0.9449	78.8	0.9824	49.6

Table 6 Spectral radii of the iterative matrices  $\rho(T_{r\omega})$ ,  $\rho(\tilde{T}_{r\omega})$  and  $\rho(\bar{T}_{r\omega})$  with various of  $\omega$ , where  $r = \omega$

$\omega$	$\rho(T_{r\omega})$	$\rho(\tilde{T}_{r\omega})$	$\rho(\bar{T}_{r\omega})$	$\rho_{opt}(\tilde{T}_{r\omega})$	$\alpha_{(\tilde{T}_{r\omega})}$	$\rho(\bar{T}_{r\omega})$	$\alpha_{(\bar{T}_{r\omega})}$
0.95	0.9707	0.9704	0.9696	0.8218	73.5	0.8185	38.8
0.9	0.9735	0.9732	0.9725	0.8342	74.1	0.8532	40.9
0.8	0.9783	0.9781	0.9776	0.8583	75.2	0.8961	44.3
0.7	0.9825	0.9823	0.9819	0.8807	76.1	0.9296	46.0
0.6	0.9860	0.9859	0.9856	0.9015	76.8	0.9507	47.1
0.5	0.9891	0.9890	0.9888	0.9209	77.4	0.9653	47.9
0.4	0.9918	0.9917	0.9916	0.9389	77.8	0.9759	48.5
0.3	0.9942	0.9942	0.9941	0.9558	78.2	0.9838	48.9
0.2	0.9963	0.9963	0.9963	0.9715	78.5	0.9903	49.3
0.1	0.9983	0.9983	0.9982	0.9862	78.8	0.9956	49.6

**Remark 4.3.** When  $\alpha = 1$ , in this case, the preconditioner is the one in [1]. The above numerical experiments indicate that the spectral radii of iterative matrices with the proposed preconditioners achieve significant improvement over the spectral radii of iterative matrices with the existing preconditioners in [1].

## 5 Conclusions

In this paper, we improve the preconditioned AOR method for irreducible  $L$ -matrices and analyze the convergence of our method. When choosing the various parameters, the spectral radii of the iteration matrices with the proposed preconditioner is smaller than those in [1], which is shown through numerical experiments. Particularly, one may discuss how choose the set of parameters in order to really accelerate the convergence of the considered method. Furthermore, the optimal choice of this set of parameters is valuably studied.

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## SOME RESULTS IN FUZZY COMPACT LINEAR OPERATORS

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ABSTRACT. In this paper we introduce the concept of fuzzy compact operators between fuzzy  $n$ -normed linear spaces in the sense of Narayanan and Vijayabalaji [12].

## 1. INTRODUCTION

The concept of 2-norm and  $n$ -norm on a linear space has been introduced and developed by Gähler in [5, 6]. After that Misiak [11], Kim and Cho [8] and Malčeski [10] developed the theory of  $n$ -normed space. In [7], Gunawan and Mashadi gave a simple way to derive an  $(n-1)$ -norm from the  $n$ -norm and realized that any  $n$ -normed space is an  $(n-1)$ -normed space.

In 2003, Bag and Samanta [1] introduced a definition of a fuzzy norm and proved a decomposition theorem of a fuzzy norm into a family of crisp norms. Also they [2, 3] gave some important properties on fuzzy norms. Lael and Nourouzi [9] introduced the fuzzy compact linear operators between fuzzy normed linear spaces.

Narayanan and Vijayabalaji [12] introduced the concept of fuzzy  $n$ -normed linear space as a generalization of fuzzy normed linear space. In [4], Efe defined various types of continuities of operators and boundedness of linear operators over fuzzy  $n$ -normed linear spaces such as fuzzy continuity, sequential fuzzy continuity, weakly fuzzy continuity, strongly fuzzy continuity, weakly fuzzy boundedness and strongly fuzzy boundedness.

In this paper we study on fuzzy compact operator between fuzzy  $n$ -normed linear spaces in the sense of Narayanan and Vijayabalaji [12]. Some definitions and theorems are generalized in fuzzy  $n$ -normed linear space.

2. FUZZY  $n$ -NORMED LINEAR SPACES

**Definition 1** ([7]). Let  $n \in \mathbb{N}$  and let  $X$  be a real vector space of dimension  $d \geq n$ . (Here we allow  $d$  to be infinite.) A real-valued function  $\|\cdot, \dots, \cdot\|$  on  $\underbrace{X \times \dots \times X}_n$

satisfying the following four properties,

- (1)  $\|x_1, x_2, \dots, x_n\| = 0$  if and only if  $x_1, x_2, \dots, x_n$  are linearly dependent,
- (2)  $\|x_1, x_2, \dots, x_n\|$  is invariant under any permutation,
- (3)  $\|x_1, x_2, \dots, \alpha x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$  for any  $\alpha \in \mathbb{R}$ ,
- (4)  $\|x_1, x_2, \dots, x_{n-1}, y + z\| \leq \|x_1, x_2, \dots, x_{n-1}, y\| + \|x_1, x_2, \dots, x_{n-1}, z\|$ ,

is called an  $n$ -norm on  $X$  and the pair  $(X, \|\cdot, \dots, \cdot\|)$  is called an  $n$ -normed space.

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**Definition 2** ([12]). Let  $X$  be a linear space over a real field  $F$ . A fuzzy subset  $N$  of  $\underbrace{X \times \cdots \times X}_n \times \mathbb{R}$  ( $\mathbb{R}$ , set of real numbers) is called a fuzzy  $n$ -norm on  $X$  if and only if

- (N1) for all  $t \in \mathbb{R}$  with  $t \leq 0$ ,  $N(x_1, x_2, \dots, x_n, t) = 0$ ,
- (N2) for all  $t \in \mathbb{R}$  with  $t > 0$ ,  $N(x_1, x_2, \dots, x_n, t) = 1$  if and only if  $x_1, x_2, \dots, x_n$  are linearly dependent,
- (N3)  $N(x_1, x_2, \dots, x_n, t)$  is invariant under any permutation of  $x_1, \dots, x_n$ ,
- (N4) for all  $t \in \mathbb{R}$  with  $t > 0$ ,  $N(x_1, x_2, \dots, cx_n, t) = N(x_1, x_2, \dots, x_n, t/|c|)$ , if  $c \neq 0$ ,  $c \in F$ ,
- (N5) for all  $s, t \in \mathbb{R}$ ,

$$N(x_1, x_2, \dots, x_n + x'_n, s + t) \geq \min \left\{ \frac{N(x_1, x_2, \dots, x_n, s)}{N(x_1, x_2, \dots, x'_n, t)} \right\},$$

- (N6)  $N(x_1, x_2, \dots, x_n, \cdot)$  is a nondecreasing function of  $\mathbb{R}$  and

$$\lim_{t \rightarrow \infty} N(x_1, x_2, \dots, x_n, t) = 1.$$

Then  $(X, N)$  is called a fuzzy  $n$ -normed linear space or in short  $f$ - $n$ -NLS.

**Remark 1.** From (N3), it follows that in a  $f$ - $n$ -NLS,

- (N4) for all  $t \in \mathbb{R}$  with  $t > 0$ ,

$$N(x_1, x_2, \dots, cx_i, \dots, x_n, t) = N(x_1, x_2, \dots, x_i, \dots, x_n, t/|c|),$$

if  $c \neq 0$ ,

- (N5) for all  $s, t \in \mathbb{R}$ ,

$$N(x_1, x_2, \dots, x_i + x'_i, \dots, x_n, s + t) \geq \min \left\{ \frac{N(x_1, x_2, \dots, x_i, \dots, x_n, s)}{N(x_1, x_2, \dots, x'_i, \dots, x_n, t)} \right\}.$$

**Example 1.** Let  $(X, \|\cdot, \cdot, \dots, \cdot\|)$  be an  $n$ -normed space as in Definition 1. Define,

$$N(x_1, x_2, \dots, x_n, t) = \begin{cases} \frac{t}{t + \|x_1, x_2, \dots, x_n\|} & \text{if } t > 0, t \in \mathbb{R}, \\ 0 & \text{if } t \leq 0 \end{cases}$$

for all  $x_1, x_2, \dots, x_n \in X$ . Then  $(X, N)$  is a  $f$ - $n$ -NLS.

**Theorem 1** ([12]). Let  $(X, N)$  be a  $f$ - $n$ -NLS. Assume further those

- (N7)  $N(x_1, x_2, \dots, x_n, t) > 0$  for all  $t > 0$  implies  $x_1, x_2, \dots, x_n$  are linearly dependent.

Define

$$\|x_1, x_2, \dots, x_n\|_\alpha = \inf\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha\}, \alpha \in (0, 1).$$

Then  $\{\|\cdot, \cdot, \dots, \cdot\|_\alpha : \alpha \in (0, 1)\}$  ascending family of  $n$ -norms on  $X$ . These  $n$ -norms are called  $\alpha$ - $n$ -norms on  $X$  corresponding to the fuzzy  $n$ -norm on  $X$ .

- (N8) We assume that, for  $x_1, x_2, \dots, x_n$  are linearly independent,

$N(x_1, x_2, \dots, x_n, \cdot)$  is a continuous function of  $\mathbb{R}$  and strictly increasing on the subset  $\{t : 0 < N(x_1, x_2, \dots, x_n, t) < 1\}$  of  $\mathbb{R}$ .



**Definition 3** ([4]). Let  $(X, N)$  be a  $f$ - $n$ -NLS and  $\{x(k)\}$  be a sequence in  $X$ . Then  $\{x(k)\}$  is said to be convergent if there exists a  $x \in X$  such that

$$\lim_{k \rightarrow \infty} N(x_1, x_2, \dots, x_{n-1}, x(k) - x, t) = 1$$

for every  $x_1, x_2, \dots, x_{n-1}, x \in X$  and for all  $t > 0$ . Then  $x \in X$  called limit of the sequence  $\{x(k)\}$  and denoted by  $\lim x(k) = x$  or  $x(k) \rightarrow x$ .

**Definition 4** ([4]). A sequence  $\{x(k)\}$  in  $(X, N)$  is called Cauchy sequence, if

$$\lim_{k, l \rightarrow \infty} N(x_1, x_2, \dots, x_{n-1}, x(k) - x(l), t) = 1$$

for every  $x_1, x_2, \dots, x_{n-1} \in X$  and for all  $t > 0, k, l \in \mathbb{N}$ .

**Definition 5** ([4]). Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS. A mapping  $T : X \rightarrow Y$  is said to be fuzzy continuous at  $z \in X$ , if for given  $\varepsilon > 0, \alpha \in (0, 1)$ , there exists  $\delta = \delta(\alpha, \varepsilon) > 0, \beta = \beta(\alpha, \varepsilon) \in (0, 1)$  such that for all  $x_1, x_2, \dots, x_{n-1}, y \in X, y_1, y_2, \dots, y_{n-1} \in Y$ ,

$$N_1(x_1, x_2, \dots, x_{n-1}, y - z, \delta) > \beta \Rightarrow N_2(y_1, y_2, \dots, y_{n-1}, Ty - Tz, \varepsilon) > \alpha.$$

If  $T$  is fuzzy continuous at each point of  $X$ , then  $T$  is said to be fuzzy continuous on  $X$ .

**Definition 6** ([4]). Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS. A mapping  $T : X \rightarrow Y$  is said to be strongly fuzzy continuous at  $z \in X$ , if for each  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $x_1, x_2, \dots, x_{n-1}, y \in X, y_1, y_2, \dots, y_{n-1} \in Y$ ,

$$N_2(y_1, y_2, \dots, y_{n-1}, Ty - Tz, \varepsilon) \geq N_1(x_1, x_2, \dots, x_{n-1}, y - z, \delta).$$

If  $T$  is strongly fuzzy continuous at each point of  $X$ , then  $T$  is said to be strongly fuzzy continuous on  $X$ .

**Definition 7** ([4]). Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS. A mapping  $T : X \rightarrow Y$  is said to be weakly fuzzy continuous at  $z \in X$ , if for a given  $\varepsilon > 0, \alpha \in (0, 1)$ , there exists  $\delta = \delta(\alpha, \varepsilon) > 0$  such that for all  $x_1, x_2, \dots, x_{n-1}, y \in X, y_1, y_2, \dots, y_{n-1} \in Y$ ,

$$\begin{aligned} N_1(x_1, x_2, \dots, x_{n-1}, y - z, \delta) &\geq \alpha \Rightarrow \\ N_2(y_1, y_2, \dots, y_{n-1}, Ty - Tz, \varepsilon) &\geq \alpha. \end{aligned}$$

If  $T$  is weakly fuzzy continuous at each point of  $X$ , then  $T$  is said to be weakly fuzzy continuous on  $X$ .

**Definition 8** ([4]). Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS. A mapping  $T : X \rightarrow Y$  is said to be sequentially fuzzy continuous at  $z \in X$ , if for any sequence  $\{x(k)\}$  in  $X$  with  $x(k) \rightarrow z$  implies  $Tx(k) \rightarrow Tz, k \in \mathbb{N}$ . I.e., for all  $t > 0$ ,

$$\begin{aligned} \lim_{k \rightarrow \infty} N_1(x_1, x_2, \dots, x_{n-1}, x(k) - z, t) &= 1 \Rightarrow \\ \lim_{k \rightarrow \infty} N_2(y_1, y_2, \dots, y_{n-1}, Tx(k) - Tz, t) &= 1. \end{aligned}$$

for all  $x_1, x_2, \dots, x_{n-1}, y \in X, y_1, y_2, \dots, y_{n-1} \in Y$ . If  $T$  is sequentially fuzzy continuous at each point of  $X$ , then  $T$  is said to be sequentially fuzzy continuous on  $X$ .

**Remark 2.** It is easy to see that if a mapping is strongly fuzzy continuous then it is weakly fuzzy continuous.

**Theorem 2** ([4]). *Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ -n-NLS and  $T : X \longrightarrow Y$  be a mapping. If  $T$  is strongly fuzzy continuous then it is sequentially fuzzy continuous.*

**Theorem 3** ([4]). *Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ -n-NLS and  $T : X \longrightarrow Y$  be a mapping. Then  $T$  is fuzzy continuous iff it is sequentially fuzzy continuous.*

**Definition 9** ([4]). *Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ -n-NLS and  $T : X \longrightarrow Y$  be a linear operator.  $T$  is said to be strongly fuzzy bounded on  $X$  iff there exists positive real number  $M$  such that for all  $x_1, x_2, \dots, x_{n-1}, y \in X$ ,  $y_1, y_2, \dots, y_{n-1} \in Y$  and for all  $s \in \mathbb{R}$ ,*

$$N_2(y_1, y_2, \dots, y_{n-1}, Ty, s) \geq N_1\left(x_1, x_2, \dots, x_{n-1}, y, \frac{s}{M}\right).$$

Let us denote the set of all strongly fuzzy bounded linear operators from a  $f$ -n-NLS  $(X, N_1)$  to  $(Y, N_2)$  by  $\mathcal{F}(X, Y)$ .

**Theorem 4.**  $\mathcal{F}(X, Y)$  is a linear space.

*Proof.* First note that for  $T_1, T_2 \in \mathcal{F}(X, Y)$  and  $y \in X$ , we have  $(T_1 + T_2)(y) = T_1(y) + T_2(y)$  and  $(\lambda T_1)(y) = \lambda T_1(y)$ . Since  $T_1$  and  $T_2$  are strongly fuzzy bounded, then there exist positive numbers  $M_1$  and  $M_2$  such that

$$\begin{aligned} N_2(y_1, y_2, \dots, y_{n-1}, T_1 y, t) &\geq N_1\left(x_1, x_2, \dots, x_{n-1}, y, \frac{t}{M_1}\right) \text{ and} \\ N_2(y_1, y_2, \dots, y_{n-1}, T_2 y, t) &\geq N_1\left(x_1, x_2, \dots, x_n, \frac{t}{M_2}\right) \end{aligned}$$

for all  $x_1, x_2, \dots, x_{n-1}, y \in X$  and  $y_1, y_2, \dots, y_{n-1} \in Y$  and for all  $t \in \mathbb{R}$ . Now for any scalars  $\alpha, \beta$  and for all  $x \in X$  we have

$$\begin{aligned} &N_2(y_1, y_2, \dots, y_{n-1}, (\alpha T_1 + \beta T_2)y, t) \\ &= N_2(y_1, y_2, \dots, y_{n-1}, \alpha T_1(y) + \beta T_2(y), t) \\ &\geq \min \left\{ \begin{array}{l} N_2\left(y_1, y_2, \dots, y_{n-1}, T_1(\alpha y), \frac{t}{\alpha}\right), \\ N_2\left(y_1, y_2, \dots, y_{n-1}, T_2(\beta y), \frac{t}{\beta}\right) \end{array} \right\} \\ &\geq \min \left\{ \begin{array}{l} N_1\left(x_1, x_2, \dots, x_{n-1}, \alpha y, \frac{t}{\alpha M_1}\right), \\ N_1\left(x_1, x_2, \dots, x_{n-1}, \beta y, \frac{t}{\beta M_2}\right) \end{array} \right\} \\ &= \min \left\{ \begin{array}{l} N_1\left(x_1, x_2, \dots, x_{n-1}, y, \frac{t}{2|\alpha|M_1}\right), \\ N_1\left(x_1, x_2, \dots, x_{n-1}, y, \frac{t}{2|\beta|M_2}\right) \end{array} \right\}. \end{aligned}$$

Choose  $M = \max\{2|\alpha|M_1, 2|\beta|M_2\} + 1$ . Thus  $M \geq 2|\alpha|M_1$  and  $M \geq 2|\beta|M_2$  and this shows that

$$\frac{t}{2|\alpha|M_1} \geq \frac{t}{M} \text{ and } \frac{t}{2|\beta|M_2} \geq \frac{t}{M}$$

for all  $t \geq 0$ . Hence

$$\begin{aligned} N_1\left(x_1, x_2, \dots, x_{n-1}, y, \frac{t}{2|\alpha|M_1}\right) &\geq N_1\left(x_1, x_2, \dots, x_{n-1}, y, \frac{t}{M}\right) \text{ and} \\ N_1\left(x_1, x_2, \dots, x_{n-1}, y, \frac{t}{2|\beta|M_2}\right) &\geq N_1\left(x_1, x_2, \dots, x_{n-1}, y, \frac{t}{M}\right) \end{aligned}$$

which implies

$$\min \left\{ \begin{array}{l} N_1 \left( x_1, x_2, \dots, x_{n-1}, y, \frac{t}{2|\alpha|M_1} \right), \\ N_1 \left( x_1, x_2, \dots, x_{n-1}, y, \frac{t}{2|\beta|M_2} \right) \end{array} \right\} \geq N_1 \left( x_1, x_2, \dots, x_{n-1}, y, \frac{t}{M} \right).$$

Then we have

$$N_2(y_1, y_2, \dots, y_{n-1}, (\alpha T_1 + \beta T_2)y, t) \geq N_1 \left( x_1, x_2, \dots, x_{n-1}, y, \frac{t}{M} \right)$$

for all  $t \geq 0$ . If  $t < 0$ , then the relation is obvious.

Thus, there exists  $M > 0$  such that

$$N_2(y_1, y_2, \dots, y_{n-1}, (\alpha T_1 + \beta T_2)y, t) \geq N_1 \left( x_1, x_2, \dots, x_{n-1}, y, \frac{t}{M} \right)$$

for all  $x_1, x_2, \dots, x_{n-1}, y \in X$ ,  $y_1, y_2, \dots, y_{n-1} \in Y$  and for all  $t \in \mathbb{R}$ . This implies that  $\alpha T_1 + \beta T_2 \in \mathcal{F}(X, Y)$ . Hence  $\mathcal{F}(X, Y)$  is a linear space.  $\square$

**Definition 10** ([4]). Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS and  $T : X \rightarrow Y$  be a linear operator.  $T$  is said to be weakly fuzzy bounded on  $X$  if for any  $\alpha \in (0, 1)$ , there exists  $M_\alpha > 0$  such that for all  $x_1, x_2, \dots, x_{n-1}, y \in X$ ,  $y_1, y_2, \dots, y_{n-1} \in Y$  and for all  $t \in \mathbb{R}$ ,

$$N_1 \left( x_1, x_2, \dots, x_{n-1}, y, \frac{t}{M_\alpha} \right) \geq \alpha \Rightarrow N_2(y_1, y_2, \dots, y_{n-1}, Ty, t) \geq \alpha.$$

Let us denote the set of all weakly fuzzy bounded linear operators from a  $f$ - $n$ -NLS  $(X, N_1)$  to  $(Y, N_2)$  by  $\mathcal{F}'(X, Y)$ .

**Theorem 5.**  $\mathcal{F}'(X, Y)$  is a linear space.

*Proof.* First note that for  $T_1, T_2 \in \mathcal{F}'(X, Y)$  and  $y \in X$ , we have  $(T_1 + T_2)(y) = T_1(y) + T_2(y)$  and  $(\lambda T_1)(y) = \lambda T_1(y)$ . Since  $T_1$  and  $T_2$  are weakly fuzzy bounded, for all  $\alpha \in (0, 1)$ , there exist positive numbers  $M_\alpha^1, M_\alpha^2$  such that

$$\begin{aligned} N_1 \left( x_1, x_2, \dots, x_{n-1}, y, \frac{t}{M_\alpha^1} \right) &\geq \alpha \Rightarrow N_2(y_1, y_2, \dots, y_{n-1}, T_1(y), t) \geq \alpha, \\ N_1 \left( x_1, x_2, \dots, x_{n-1}, y, \frac{t}{M_\alpha^2} \right) &\geq \alpha \Rightarrow N_2(y_1, y_2, \dots, y_{n-1}, T_2(y), t) \geq \alpha \end{aligned}$$

for all  $x_1, x_2, \dots, x_{n-1}, y \in X$ ,  $y_1, y_2, \dots, y_{n-1} \in Y$  and for all  $t \in \mathbb{R}$ .

Let  $k_1$  and  $k_2$  be any two arbitrary nonzero scalars. Then,

$$\begin{aligned} N_1 \left( x_1, x_2, \dots, x_{n-1}, y, \frac{t}{2|k_1|M_\alpha^1} \right) &= N_1 \left( x_1, x_2, \dots, x_{n-1}, k_1 y, \frac{t}{2M_\alpha^1} \right) \geq \alpha, \\ N_1 \left( x_1, x_2, \dots, x_{n-1}, y, \frac{t}{2|k_2|M_\alpha^2} \right) &= N_1 \left( x_1, x_2, \dots, x_{n-1}, k_2 y, \frac{t}{2M_\alpha^2} \right) \geq \alpha. \end{aligned}$$

Choose  $M_\alpha = 2|k_1|M_\alpha^1 + 2|k_2|M_\alpha^2$ . Then

$$\begin{aligned}
N_1 \left( x_1, x_2, \dots, x_{n-1}, y, \frac{t}{M_\alpha} \right) &= N_1 \left( x_1, x_2, \dots, x_{n-1}, y, \frac{t}{2|k_1|M_\alpha^1 + 2|k_2|M_\alpha^2} \right) \geq \alpha \\
&\implies \begin{cases} N_1 \left( x_1, x_2, \dots, x_{n-1}, y, \frac{t}{2|k_1|M_\alpha^1} \right) \geq \alpha \\ N_1 \left( x_1, x_2, \dots, x_{n-1}, y, \frac{t}{2|k_2|M_\alpha^2} \right) \geq \alpha \end{cases} \\
&\implies \begin{cases} N_2 \left( y_1, y_2, \dots, y_{n-1}, T_1(k_1y), \frac{t}{2} \right) \geq \alpha \\ N_2 \left( y_1, y_2, \dots, y_{n-1}, T_2(k_2y), \frac{t}{2} \right) \geq \alpha \end{cases} \\
&\implies N_2(y_1, y_2, \dots, y_{n-1}, (k_1T_1 + k_2T_2)y, t) \geq \alpha \\
&\implies k_1T_1 + k_2T_2 \in \mathcal{F}'(X, Y).
\end{aligned}$$

If  $k_1 = k_2 = 0$ , obviously  $k_1T_1 + k_2T_2 \in \mathcal{F}'(X, Y)$ . Hence  $\mathcal{F}'(X, Y)$  is a linear space.  $\square$

**Theorem 6** ([4]). *Let  $(X, N_1)$  and  $(Y, N_2)$  be  $f$ - $n$ -NLS and  $T : X \longrightarrow Y$  be a linear operator. If  $T$  is strongly fuzzy bounded then it is weakly fuzzy bounded but not conversely.*

**Definition 11** ([4]). *Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS and  $T : X \longrightarrow Y$  be a linear operator.  $T$  is said to be uniformly bounded if there exists  $M > 0$  such that for all  $\alpha \in (0, 1)$ ,*

$$\|y_1, y_2, \dots, y_{n-1}, Ty\|_\alpha^2 \leq M \|x_1, x_2, \dots, x_{n-1}, y\|_\alpha^1$$

where  $\|\cdot, \cdot, \dots, \cdot\|_\alpha^1$  and  $\|\cdot, \cdot, \dots, \cdot\|_\alpha^2$  are  $\alpha$ - $n$ -norms of  $N_1$  and  $N_2$  respectively.

**Theorem 7** ([4]). *Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS and  $T : X \rightarrow Y$  be a linear operator. Then*

- (i)  $T$  is strongly fuzzy continuous everywhere on  $X$  if  $T$  is strongly fuzzy continuous at a point  $z \in X$ .
- (ii)  $T$  is strongly fuzzy continuous iff  $T$  is strongly fuzzy bounded.

**Theorem 8** ([4]). *Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS satisfying (N7) and (N8). Let  $T : X \longrightarrow Y$  be a linear operator. Then  $T$  is strongly fuzzy bounded iff it is uniformly bounded with respect to  $\alpha$ - $n$ -norms of  $N_1$  and  $N_2$ .*

**Remark 3.** *If  $T$  is strongly fuzzy bounded then it is sequentially fuzzy continuous on  $X$ .*

**Theorem 9** ([4]). *Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS and  $T : X \longrightarrow Y$  be a linear operator. If  $T$  is sequentially fuzzy continuous at a point then it is sequentially fuzzy continuous on  $X$ .*

**Theorem 10** ([4]). *Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS and  $T : X \longrightarrow Y$  be a linear operator. Then*

- (i)  $T$  is weakly fuzzy continuous everywhere on  $X$  if  $T$  is weakly fuzzy continuous at a point  $y_0 \in X$ .
- (ii)  $T$  is weakly fuzzy continuous iff  $T$  is weakly fuzzy bounded.

**Theorem 11** ([4]). *Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS satisfying (N7) and (N8). Let  $T : X \longrightarrow Y$  be a linear operator. Then  $T$  is weakly fuzzy bounded iff  $T$  be bounded w.r.t.  $\alpha$ - $n$ -norms of  $N_1$  and  $N_2$ ,  $\alpha \in (0, 1)$ .*

**Theorem 12** ([4]). Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS satisfying (N7) and (N8). Let  $T : X \longrightarrow Y$  be a linear operator. If  $X$  is of finite dimension then  $T$  is weakly fuzzy bounded.

**Definition 12.** Let  $(X, N)$  be a fuzzy  $n$ -normed linear space. A subset  $B$  of  $X$  is said to be the closure of  $K \subset X$  if for any  $x \in B$ , there exists a sequence  $\{x(k)\}$  in  $K$  such that

$$\lim_{k \rightarrow \infty} N(x_1, x_2, \dots, x_{n-1}, x(k) - x, t) = 1$$

for all  $t > 0$ . We denote the set  $B$  by  $\overline{K}$ .

**Definition 13.** A subset  $A$  of a fuzzy  $n$ -normed linear space  $(X, N)$  is said to be bounded iff there exist  $t > 0$  and  $0 < r < 1$  such that

$$N(x_1, x_2, \dots, x_n, t) > 1 - r$$

for all  $x_1, x_2, \dots, x_n \in A$ .

**Definition 14.** A subset  $A$  of a fuzzy  $n$ -normed linear space  $(X, N)$  is said to be compact if any sequence  $\{x(k)\}$  in  $A$  has a subsequence converging to an element of  $A$ .

### 3. MAIN RESULTS

**Definition 15.** Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS. A linear operator  $T : X \longrightarrow Y$  is called fuzzy compact operator if for every fuzzy bounded subset  $M$  of  $X$  the subset of  $T(M) \subset Y$  is relatively compact, i.e., the fuzzy closure of  $T(M)$  is a fuzzy compact set.

**Example 2.** Let  $(X, \|\cdot, \cdot, \dots, \cdot\|_1)$  and  $(Y, \|\cdot, \cdot, \dots, \cdot\|_2)$  be two ordinary  $n$ -normed linear spaces, and  $T : X \longrightarrow Y$  be a compact operator. Then  $T : (X, N_1) \longrightarrow (Y, N_2)$  is a fuzzy compact operator, where  $N_1$  and  $N_2$  are the standard fuzzy norms induced by ordinary norms  $\|\cdot, \cdot, \dots, \cdot\|_1$  and  $\|\cdot, \cdot, \dots, \cdot\|_2$ , respectively, i.e.,

$$N_1(x_1, x_2, \dots, x_n, t) = \begin{cases} \frac{t}{t + \|\cdot, \cdot, \dots, \cdot\|_1} & \text{if } t > 0, t \in \mathbb{R}, \\ 0 & \text{if } t \leq 0 \end{cases},$$

and

$$N_2(y_1, y_2, \dots, y_n, t) = \begin{cases} \frac{t}{t + \|\cdot, \cdot, \dots, \cdot\|_2} & \text{if } t > 0, t \in \mathbb{R}, \\ 0 & \text{if } t \leq 0 \end{cases}.$$

**Theorem 13.** Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS and  $T : X \longrightarrow Y$  be a linear operator. Then  $T$  is fuzzy compact iff it maps every fuzzy bounded sequence  $\{x(k)\}$  in  $X$  onto a sequence  $\{Tx(k)\}$  in  $Y$  which has a fuzzy convergent subsequence.

*Proof.* Suppose that  $T$  be a fuzzy compact operator and  $\{x(k)\}$  be a fuzzy bounded sequence in  $X$ . The fuzzy closure of  $\{Tx(k) : k \in \mathbb{N}\}$  is a fuzzy compact set. So  $\{Tx(k)\}$  has a fuzzy convergent subsequence by definition.

Conversely, let  $A$  be a fuzzy bounded subset of  $X$ . We show that the fuzzy closure of  $T(A)$  is fuzzy compact. Let  $\{x(k)\}$  be a sequence in the closure of  $T(A)$ . For given  $\varepsilon > 0$ ,  $k \in \mathbb{N}$  and  $t > 0$ , there exists  $\{y(k)\}$  in  $T(A)$  such that  $N_2(y_1, y_2, \dots, y_{n-1}, x(k) - y(k), \frac{t}{2}) > 1 - \varepsilon$ . Let  $y(k) = Tz(k)$ , where  $z(k) \in A$ . Since  $A$  is fuzzy bounded set, so is  $\{z(k)\}$ . On the other hand, since  $T$  is a fuzzy compact operator,  $\{Tz(k)\}$  has a fuzzy convergent subsequence  $y(k_i) = Tz(k_i)$ .

Let  $y(k_i) \longrightarrow y$  for some  $y \in Y$ . Hence  $N_2(y_1, y_2, \dots, y_{n-1}, y(k_i) - y, \frac{t}{2}) > 1 - \varepsilon$  for all  $n_i > n_0$ . Then we get

$$\begin{aligned} & N_2\left(y_1, y_2, \dots, y_{n-1}, x(k_i) - y, \frac{t}{2}\right) \\ & \geq \min \left\{ \begin{array}{l} N_2\left(y_1, y_2, \dots, y_{n-1}, x(k_i) - y(k_i), \frac{t}{2}\right), \\ N_2\left(y_1, y_2, \dots, y_{n-1}, y(k_i) - y, \frac{t}{2}\right) \end{array} \right\} \\ & > 1 - \varepsilon \end{aligned}$$

for all  $n_i > n_0$ . Hence  $\{x(k_i)\}$  is a fuzzy convergent subsequence of  $\{x(k)\}$ . Thus the fuzzy closure of  $T(A)$  is a fuzzy compact set.  $\square$

**Lemma 1.** *Let  $(X, N)$  be a fuzzy  $n$ -normed space satisfying  $(N7)$  and  $\{x(k)\}$  be a sequence in  $X$ . Then*

$$\lim_{k \rightarrow \infty} N(x_1, x_2, \dots, x_{n-1}, x(k) - x, t) = 1 \text{ iff } \lim_{k \rightarrow \infty} \|x_1, x_2, \dots, x_{n-1}, x(k) - x\|_\alpha = 0$$

for all  $\alpha \in (0, 1)$ .

*Proof.* Suppose that  $x(k) \xrightarrow{N} x$ . Choose  $\alpha \in (0, 1)$  and  $t > 0$ . Then there exists  $k_0 \in \mathbb{N}$  such that

$$N(x_1, x_2, \dots, x_{n-1}, x(k) - x, t) > 1 - \alpha,$$

for all  $k \geq k_0$ . It follows that

$$\|x_1, x_2, \dots, x_{n-1}, x_k - x\|_{1-\alpha} \leq t$$

for all  $k \geq k_0$ . Thus

$$\|x_1, x_2, \dots, x_{n-1}, x(k) - x\|_{1-\alpha} \rightarrow 0.$$

Conversely, let

$$\|x_1, x_2, \dots, x_{n-1}, x(k) - x\|_\alpha \rightarrow 0$$

for all  $\alpha \in (0, 1)$ . Fix  $\alpha \in (0, 1)$  and  $t > 0$ . There exists  $k_0 \in \mathbb{N}$  such that

$$\wedge \{r > 0 : N(x_1, x_2, \dots, x_{n-1}, x(k) - x, t) \geq 1 - \alpha\} < t,$$

for all  $k \geq k_0$ . This implies that

$$N(x_1, x_2, \dots, x_{n-1}, x(k) - x, t) \geq 1 - \alpha$$

for all  $k \geq k_0$ , i.e.,  $x(k) \xrightarrow{N} x$ .  $\square$

**Definition 16.** *Let  $(X, N)$  be a fuzzy  $n$ -normed space. We define the following subset of  $X$ :*

$$B_\alpha[x, r] = \{y \in X : N(x_1, x_2, \dots, x_{n-1}, x - y, t) \geq \alpha\}$$

where  $x_1, x_2, \dots, x_{n-1}, x \in X$ ,  $\alpha \in (0, 1)$  and  $r > 0$ .

**Theorem 14.** *Let  $(X, N)$  be a fuzzy  $n$ -normed space satisfying  $(N7)$  and*

*$N(x_1, x_2, \dots, x_n, \cdot)$  is a continuous function on  $\mathbb{R}$ . Then  $X$  is finite dimensional iff  $B_\alpha[x, r]$  is a fuzzy compact set in  $X$ , for each  $\alpha \in (0, 1)$  and  $r > 0$ .*

*Proof.* Let

$$A_\alpha[x, r] = \{y \in X : \|x_1, x_2, \dots, x_{n-1}, x - y\|_\alpha \leq r\},$$

where  $\alpha \in (0, 1)$  and  $r > 0$ .

We first show that  $B_\alpha[x, r] = A_\alpha[x, r]$ . If  $y \in B_\alpha[x, r]$ , then

$$N(x_1, x_2, \dots, x_{n-1}, x - y, t) \geq \alpha.$$

Since  $\|x_1, x_2, \dots, x_{n-1}, x - y\|_\alpha \leq r$ , then  $y \in A_\alpha[x, r]$ .

Now if  $y \in A_\alpha[x, r]$ , then

$$\|x_1, x_2, \dots, x_{n-1}, x - y\|_\alpha \leq r \text{ or } \wedge \{t > 0 : N(x_1, x_2, \dots, x_{n-1}, x - y, t) \geq \alpha\} \leq r.$$

If

$$\wedge \{t > 0 : N(x_1, x_2, \dots, x_{n-1}, x - y, t) \geq \alpha\} < r$$

then

$$N(x_1, x_2, \dots, x_{n-1}, x - y, t) \geq \alpha.$$

Thus  $y \in B_\alpha[x, r]$ .

If

$$\wedge \{t > 0 : N(x_1, x_2, \dots, x_{n-1}, x - y, t) \geq \alpha\} = r,$$

there exists a sequence  $\{t_k\}$  in  $\mathbb{R}$  such that  $t_k \rightarrow r$ , and  $N(x_1, x_2, \dots, x_{n-1}, x - y, t_k) \geq \alpha$ . By continuity of  $N(x_1, x_2, \dots, x_n, \cdot)$  we get

$$N(x_1, x_2, \dots, x_{n-1}, x - y, r) = \lim_{k \rightarrow \infty} N(x_1, x_2, \dots, x_{n-1}, x - y, t_k) \geq \alpha.$$

Hence  $y \in B_\alpha[x, r]$  and therefore  $B_\alpha[x, r] = A_\alpha[x, r]$ .

Now suppose that  $\dim X < \infty$ ,  $x_1, x_2, \dots, x_{n-1}, x \in X$ , and  $r > 0$ . Choose the sequence  $\{x(k)\}$  in  $B_\alpha[x, r]$ . It is clear that  $A_\alpha[x, r]$  is a compact subset of  $(X, \|\cdot, \cdot, \dots, \cdot\|_\alpha)$ . Hence there exists a subsequence  $\{x(k_i)\}$  of  $\{x(k)\}$  and  $v \in A_\alpha[x, r]$  such that  $x(k_i) \xrightarrow{\|\cdot, \cdot, \dots, \cdot\|_\alpha} v$ . Since all norms are equivalent in finite dimensional spaces,  $x(k_i) \xrightarrow{\|\cdot, \cdot, \dots, \cdot\|_\beta} v$ , for all  $\beta \in (0, 1)$ . Thus, we obtain  $x(k_i) \xrightarrow{N} v$  by Lemma 1. Since  $B_\alpha[x, r] = A_\alpha[x, r]$ , we have  $v \in B_\alpha[x, r]$ .

Conversely, let  $B_\alpha[x, r]$  be fuzzy compact. To show that  $X$  is finite dimensional, it suffices to prove that  $A_\alpha[x, r]$  is compact with respect to  $\alpha$ - $n$ -norm. Choose a sequence  $\{x(k)\}$  in  $A_\alpha[x, r]$ . Since  $B_\alpha[x, r]$  is fuzzy compact, it has a fuzzy convergent subsequence  $\{x(k_i)\}$ . Lemma 1 implies that  $\{x(k_i)\}$  is convergent under  $\|\cdot, \cdot, \dots, \cdot\|_\alpha$ . Thus  $A_\alpha[x, r]$  is compact in  $n$ -normed linear space  $(X, \|\cdot, \cdot, \dots, \cdot\|_\alpha)$  which shows that  $X$  is finite dimensional.  $\square$

**Lemma 2.** Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS satisfying (N7) and  $T : X \rightarrow Y$  be a fuzzy compact operator. Then

$$T : (X, \|\cdot, \cdot, \dots, \cdot\|_\alpha^1) \rightarrow (Y, \|\cdot, \cdot, \dots, \cdot\|_\alpha^2)$$

is an ordinary compact operator for all  $\alpha \in (0, 1)$ .

*Proof.* We show that for each bounded sequence  $\{x(k)\}$  in  $(X, \|\cdot, \cdot, \dots, \cdot\|_\alpha^1)$ , the sequence  $\{Tx(k)\}$  has a convergent subsequence in  $(Y, \|\cdot, \cdot, \dots, \cdot\|_\alpha^2)$ . Let  $\{x(k)\}$  be a bounded sequence in  $(X, \|\cdot, \cdot, \dots, \cdot\|_\alpha^1)$ . There exists  $M > 0$  such that

$$\|x_1, x_2, \dots, x_{n-1}, x(k)\|_\alpha^1 < M$$

for all  $k \in \mathbb{N}$ . Hence

$$N_1(x_1, x_2, \dots, x_{n-1}, x(k), M) \geq \alpha$$

for all  $k$ , that is  $\{x(k)\}$  is fuzzy bounded. Thus  $\{Tx(k)\}$  has a fuzzy convergent subsequence  $\{Tx(k_i)\}$ . By Lemma 1,  $\{Tx(k_i)\}$  is convergent under  $\|\cdot, \cdot, \dots, \cdot\|_\alpha^2$ .  $\square$

**Theorem 15.** *Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS satisfying (N7) and (N8). Then*

- (a) Every fuzzy compact operator  $T : (X, N_1) \longrightarrow (Y, N_2)$  is weakly fuzzy continuous.
- (b) If  $\dim X = \infty$  then the identity operator  $I : (X, N_1) \longrightarrow (X, N_1)$  is not a fuzzy compact operator.

*Proof.* (a) Choose  $\alpha \in (0, 1)$ . Let  $\|\cdot, \cdot, \dots, \cdot\|_\alpha^1$  and  $\|\cdot, \cdot, \dots, \cdot\|_\alpha^2$  are  $\alpha$ - $n$ -norms on  $X$  and  $Y$  corresponding to the fuzzy  $n$ -norms  $N_1$  and  $N_2$ , respectively. By Lemma 2,

$$T : (X, \|\cdot, \cdot, \dots, \cdot\|_\alpha^1) \longrightarrow (Y, \|\cdot, \cdot, \dots, \cdot\|_\alpha^2)$$

is a compact operator. Since compact operator is bounded, there exists  $M_\alpha > 0$  such that

$$\|y_1, y_2, \dots, y_{n-1}, Tx\|_\alpha^2 \leq M_\alpha \|x_1, x_2, \dots, x_{n-1}, x\|_\alpha^1.$$

Hence  $T$  is weakly fuzzy bounded by Theorem 11. Now Theorem 10(ii) implies that  $T$  is weakly fuzzy bounded.

(b) The identity operator  $I$  maps  $B_\alpha[0, 1]$  to itself. Suppose on the contrary that  $I$  is a fuzzy compact operator. Then  $\overline{B}_\alpha[0, 1]$  is fuzzy compact for all  $\alpha \in (0, 1)$ . Now  $\overline{B}_\alpha[0, 1] \subseteq A_\alpha[0, 1] = B_\alpha[0, 1]$  implies that  $B_\alpha[0, 1]$  is closed and so fuzzy compact. Thus  $X$  is finite dimensional by Theorem 14, which is a contradiction.  $\square$

**Theorem 16.** *Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS. Then the set of all fuzzy compact linear operators from  $X$  to  $Y$  is a linear subspace of  $\mathcal{F}'(X, Y)$ .*

*Proof.* Suppose that  $T_1$  and  $T_2$  are fuzzy compact linear operators from  $X$  to  $Y$  and  $\{x(k)\}$  be any fuzzy bounded sequence in  $X$ . Then the sequence  $\{T_1x(k)\}$  has a fuzzy convergent subsequence  $\{T_1x(k_i)\}$ . The sequence  $\{T_2x(k_i)\}$  also has a fuzzy convergent subsequence  $\{T_2z(k)\}$ . Hence  $\{T_1z(k)\}$  and  $\{T_2z(k)\}$  are fuzzy convergent sequences. Let  $T_1z(k) \longrightarrow u$ , and  $T_2z(k) \longrightarrow v$ . If  $t > 0$ , we have

$$\begin{aligned} & \lim_{k \rightarrow \infty} N_2(y_1, y_2, \dots, y_{n-1}, (T_1 + T_2)z(k) - u - v, t) \\ & \geq \lim_{k \rightarrow \infty} \min \left\{ \begin{array}{l} N_2(y_1, y_2, \dots, y_{n-1}, T_1z(k) - u, \frac{t}{2}), \\ N_2(y_1, y_2, \dots, y_{n-1}, T_2z(k) - v, \frac{t}{2}) \end{array} \right\}. \end{aligned}$$

for all  $y_1, y_2, \dots, y_{n-1} \in Y$ . Thus,  $\lim_{k \rightarrow \infty} N_2(y_1, y_2, \dots, y_{n-1}, (T_1 + T_2)z(k) - u - v, t) = 1$ , for all  $t > 0$ . This implies  $T_1 + T_2$  is a fuzzy compact operator. Now if  $T_1x(k_i) \longrightarrow y$ , then

$$\begin{aligned} & \lim_{k \rightarrow \infty} N_2(y_1, y_2, \dots, y_{n-1}, \alpha T_1x(k_i) - \alpha y, t) \\ & = \lim_{k \rightarrow \infty} N_2\left(y_1, y_2, \dots, y_{n-1}, T_1x(k_i) - y, \frac{t}{|\alpha|}\right) = 1, \end{aligned}$$

for all  $\alpha \in \mathbb{R} \setminus \{0\}$ , and  $t > 0$ . Hence  $\alpha T_1$  is also a fuzzy compact operator which completes the proof.  $\square$

**Theorem 17.** *Let  $(X, N)$  be a  $f$ - $n$ -NLS,  $T : X \longrightarrow X$  be a fuzzy compact linear operator, and  $S : X \longrightarrow X$  be a strongly fuzzy continuous linear operator. Then  $ST$  and  $TS$  are fuzzy compact operators.*



*Proof.* Let  $\{x(k)\}$  be any fuzzy bounded sequence in  $X$ . Then  $\{Tx(k)\}$  has a fuzzy convergent subsequence  $\{Tx(k_i)\}$ . Let  $\lim_{k \rightarrow \infty} Tx(k_i) = y$ . Since  $S$  is strongly fuzzy continuous, by Theorem 2 we have  $ST(x(k_i)) \rightarrow S(y)$ . Hence  $ST(x(k))$  has a fuzzy convergent subsequence. This proves  $ST$  is fuzzy compact.

Next we show that  $TS$  is fuzzy compact. Let choose any fuzzy bounded sequence  $\{x(k)\}$  in  $X$ . Then there exist  $t_0 > 0$  and  $r_0 \in (0, 1)$  such that

$$N_1(x_1, x_2, \dots, x_{n-1}, x(k), t_0) > 1 - r_0$$

for all  $k \geq 1$ . By Theorem 7(ii) we conclude that the operator  $S$  is a strongly fuzzy bounded linear operator. Thus there exists  $M > 0$  such that

$$N_2(y_1, y_2, \dots, y_{n-1}, Sx(k), t_0 M) > 1 - r_0$$

for all  $k$ . It follows that  $\{Sx(k)\}$  is fuzzy bounded sequence in  $S(X)$ . Because  $T$  is fuzzy compact,  $\{TSx(k)\}$  has a fuzzy convergent subsequence. This completes the proof.  $\square$

**Lemma 3.** Let  $(X, N)$  be a  $f$ - $n$ -NLS satisfying (N7),  $N(x_1, x_2, \dots, x_n, \cdot)$  be a continuous function on  $\mathbb{R}$  and  $\dim X < \infty$ . Then each fuzzy bounded sequence  $\{x(k)\}$  in  $(X, N)$  has a fuzzy convergent subsequence.

*Proof.* Let  $\{x(k)\}$  be a fuzzy bounded sequence in  $(X, N)$ . There exist  $t_0 > 0$  and  $r_0 \in (0, 1)$  such that

$$N(x_1, x_2, \dots, x_{n-1}, x(k), t_0) > 1 - r_0$$

for all  $k \in \mathbb{N}$ . Hence  $x(k) \in B_{1-r_0}[0, t_0]$ , for all  $k \in \mathbb{N}$ . By Theorem 14,  $B_{1-r_0}[0, t_0]$  is a fuzzy compact set, so  $\{x(k)\}$  has a fuzzy convergent subsequence.  $\square$

**Theorem 18.** Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS satisfying (N7) and (N8). If  $T : X \rightarrow Y$  is a linear operator where  $\dim X < \infty$ , then  $T$  is weakly fuzzy continuous.

*Proof.* It is clear from Theorem 12 and Theorem 10(ii).  $\square$

**Theorem 19.** Let  $(X, N_1)$  and  $(Y, N_2)$  be two  $f$ - $n$ -NLS satisfying (N7) and

$N_2(y_1, y_2, \dots, y_n, \cdot)$  is continuous function on  $\mathbb{R}$ , and  $T : (X, N_1) \rightarrow (Y, N_2)$  a linear operator. Then the following hold:

- (a) If  $T$  is weakly fuzzy bounded and  $\dim T(X) < \infty$ , then  $T$  is a fuzzy compact operator.
- (b) In addition if  $(X, N_1)$  and  $(Y, N_2)$  satisfying (N8) and  $\dim T(X) < \infty$ , then  $T$  is a fuzzy compact operator.

*Proof.* (a) Let  $\{x(k)\}$  be a fuzzy bounded sequence of  $(X, N_1)$ . There exist  $t_0 > 0$  and  $r_0 \in (0, 1)$  such that

$$N(x_1, x_2, \dots, x_{n-1}, x(k), t_0) > 1 - r_0$$

for all  $k \in \mathbb{N}$ . Since  $T$  is weakly fuzzy bounded, there exists  $M_{1-r_0} > 0$  such that for all  $k$ ,

$$\begin{aligned} N(x_1, x_2, \dots, x_{n-1}, x(k), t_0) &\geq 1 - r_0 \Rightarrow \\ N_2\left(y_1, y_2, \dots, y_{n-1}, Tx(k), \frac{t_0}{M_{1-r_0}}\right) &\geq 1 - r_0. \end{aligned}$$

It follows that  $\{Tx(k)\}$  is a fuzzy bounded sequence in  $T(X)$ . Since  $\dim T(X) < \infty$ , the sequence  $\{Tx(k)\}$  has a convergent subsequence by Lemma 3. Hence  $T$  is fuzzy compact.

(b)  $T$  is weakly fuzzy continuous by Theorem 18. Furthermore Theorem 10(ii) implies that  $T$  is weakly fuzzy bounded. Since  $\dim T(X) < \infty$ , by the (a) we conclude that  $T$  is a fuzzy compact operator.  $\square$

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# A Note on Shape Preserving Weighted Uniform Approximation \*

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## Abstract

In this paper, new results concerning shape preserving weighted uniform approximation on the real line are presented.

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## 1 Introduction

Shape preserving approximation by real polynomials of real variables on the compact interval  $[a, b]$  in the classical non-weighted  $L^p[a, b]$ -norms with  $0 < p \leq \infty$ , is a well developed topic in mathematics (for a comprehensive treatment of the subject see for example the very recent book [3]).

But papers concerning shape preserving weighted approximation on the real line seem to be almost nonexistent. In fact, the only paper we found on the topic is the very recent paper [5].

The aim of this paper is to show that the so-called  $L$ -positive approximation method developed in [1] is powerful enough to produce new results in shape preserving weighted approximation.

## 2 Shape Preserving Weighted Uniform Approximation

For a continuous weight function  $w : \mathbb{R} \rightarrow (0, 1]$ , define the weighted space

$$C_w(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R}; f \text{ continuous on } \mathbb{R} \text{ and } \lim_{x \rightarrow \pm\infty} f(x)w(x) = 0\}.$$

It is a linear space endowed with the norm  $\|f\|_{C_w(\mathbb{R})} = \sup\{w(x)|f(x)|; x \in \mathbb{R}\}$ .

Also, for any  $r \in \mathbb{N} \cup \{0\}$  define the space

$$C_w^r(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R}; f^{(\gamma)} \in C_w(\mathbb{R}), \text{ for all } \gamma = 0, 1, \dots, r\},$$

endowed with the norm  $\|f\|_{C_w^r} = \max\{\|f^{(\gamma)}\|_{C_w(\mathbb{R})}; \gamma = 0, 1, \dots, r\}$ . Clearly we have  $C_w^0(\mathbb{R}) = C_w(\mathbb{R})$ .

In all what follows we will consider the exponential (Freud) weight

$$w_\alpha(x) = e^{-|x|^\alpha}, \text{ with } \alpha \geq 1.$$

The general results in [1] will allow us to obtain in an easy way shape preserving results in weighted approximation. Thus, first we obtain the following results in simultaneous shape preserving weighted approximation.

**Theorem 2.1.** *Let  $r \geq 0$  be an even number. For any  $f \in C_{w_\alpha}^r(\mathbb{R})$  satisfying  $f^{(j)}(x) \geq 0$ , for all  $x \in \mathbb{R}$  and  $j = 0, 2, 4, \dots, r$ , there exists a sequence of polynomials  $(P_n)_n$  with degree  $(P_n) \leq n$ , such that  $P_n^{(j)}(x) \geq 0$ , for all  $x \in \mathbb{R}$ ,  $n \in \mathbb{N}$  and  $j = 0, 2, 4, \dots, r$  and*

$$\|f - P_n\|_{C_{w_\alpha}^r} \leq CE_n(f; C_{w_\alpha}^r(\mathbb{R})), \text{ for all } n \in \mathbb{N},$$

where  $C > 0$  is independent of  $n$  and  $f$  and

$$E_n(f; C_{w_\alpha}^r(\mathbb{R})) := \inf\{\|f - P\|_{C_{w_\alpha}^r}; P \in \mathcal{P}_n\}.$$

**Proof.** If we fix  $r$  an even number and in Corollary 2.1 in [1] we take  $L_\gamma(f) = f^{(\gamma)}$ ,  $\gamma = 0, 2, 4, \dots, r$ ,  $F = C_{w_\alpha}^r(\mathbb{R})$  and define  $\rho(x) = \sum_{j=0}^r x^{2j} \in C_{w_\alpha}^r(\mathbb{R})$ , then we immediately obtain the conclusion in the theorem.  $\square$

As an immediate consequence we obtain the following result.

**Corollary 2.2.** *Let  $r \geq 0$  be an even number and  $f \in C_{w_\alpha}^r(\mathbb{R})$  satisfying  $f^{(j)}(x) \geq 0$ , for all  $x \in \mathbb{R}$  and  $j = 0, 2, 4, \dots, r$ . There exists a sequence of polynomials  $(P_n)_{n \in \mathbb{N}}$  with degree  $(P_n) \leq n$ , such that for every  $j = 0, 2, 4, \dots, r$  we have*

$$\lim_{n \rightarrow \infty} \|P_n^{(j)} - f^{(j)}\|_{C_{w_\alpha}(\mathbb{R})} = 0 \text{ and } P_n^{(j)}(x) \geq 0, \forall x \in \mathbb{R}.$$

**Proof.** Taking into account Theorem 2.1, clearly that it is sufficient to prove that for any fixed even number  $r$ , we have

$$\lim_{n \rightarrow \infty} E_n(f; C_{w_\alpha}^r(\mathbb{R})) = 0.$$

For this purpose, let us denote by  $Q_n$  a polynomial of degree  $\leq n$  attached to  $f$  such that

$$\|f - Q_n\|_{C_{w_\alpha}(\mathbb{R})} \leq c \inf_{Q \in \mathcal{P}_n} \|f - Q\|_{C_{w_\alpha}(\mathbb{R})},$$

with a constant  $c \geq 1$ . We clearly have  $\lim_{n \rightarrow \infty} \|f - Q_n\|_{C_{w_\alpha}(\mathbb{R})} = 0$ .

But according to a classical result of Freud ([2, Theorem 4.1]) (see also for example [4, p. 90, Theorem 4.1.7]), this immediately will imply that

$$\lim_{n \rightarrow \infty} \|f^{(j)} - Q_n^{(j)}\|_{C_{w_\alpha}(\mathbb{R})} = 0, \text{ for all } 1 \leq j \leq r.$$

Since

$$E_n(f; C_{w_\alpha}^r(\mathbb{R})) \leq \max_{0 \leq j \leq r} \|f^{(j)} - Q_n^{(j)}\|_{C_{w_\alpha}(\mathbb{R})},$$

passing to limit with  $n \rightarrow \infty$  we get the desired conclusion.  $\square$

**Remark.** Given  $r \in \mathbb{N}$  and  $f$  with  $f^{(r)} \geq 0$  on  $\mathbb{R}$  and denoting

$$E_n^r(f, C_{w_\alpha}(\mathbb{R})) := \inf\{\|f - P\|_{C_{w_\alpha}(\mathbb{R})}; P \in \mathcal{P}_n, P^{(r)}(x) \geq 0\},$$

the main result in [5, Theorem 1] is that we have

$$\lim_{n \rightarrow \infty} E_n^r(f, C_{w_\alpha}(\mathbb{R})) = 0,$$

or equivalently, that there exists a sequence of polynomials  $(P_n)_{n \in \mathbb{N}}$  with degree  $(P_n) \leq n$ , such that we have

$$\lim_{n \rightarrow \infty} \|P_n - f\|_{C_{w_\alpha}(\mathbb{R})} = 0 \text{ and } P_n^{(r)}(x) \geq 0, \forall x \in \mathbb{R}.$$

It is clear that for even  $r \in \mathbb{N}$ , Corollary 2.2 is a simultaneous approximation-type result corresponding to Theorem 1 in [5].

Now, if for fixed  $\delta \geq 0$  we define as in [1, p. 483] the set  $M_\delta(\mathbb{R})$  of all  $\delta$ -increasing functions, by the set of functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfying the property

$$\frac{f(x) - f(\gamma)}{x - \gamma} \geq 0, \text{ for all } x, \gamma \in \mathbb{R}, |x - \gamma| \geq \delta, x \neq \gamma,$$

applying Corollary 2.2 in [1] we immediately obtain the following.

**Theorem 2.2.** *For any  $\delta > 0$ ,  $f \in C_{w_\alpha}(\mathbb{R}) \cap M_\delta(\mathbb{R})$ , there exists a sequence of polynomials  $(P_n)_n$  with degree  $(P_n) \leq n$  such that  $P_n \in M_\delta(\mathbb{R})$  for all  $n \in \mathbb{N}$  and*

$$\|f - P_n\|_{w_\alpha} \leq C E_n(f; C_{w_\alpha}(\mathbb{R})), \text{ for all } n \in \mathbb{N},$$

where  $C > 0$  is independent of  $f$  and  $n$ .

**Remarks.** 1). Theorem 2.2 is the weighted correspondent of the non-weighted approximation result in [1, Corollary 3.6].

2) In fact, all the applicative results in the Sections 3 and 4 in [1] can be re-written in the weighted approximation setting, at least for Freud-type weights of one or several variables.

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# STABLE MIXTURE MODEL WITH DEPENDENT STATES FOR FINANCIAL RETURN SERIES EXHIBITING SHORT HISTORIES AND PERIODS OF STRONG PASSIVITY

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WEI SUN, AND IGORIS BELOVAS

**ABSTRACT.** The paper provides some analysis solutions for financial return series exhibiting short histories and periods of strong passivity. The mixed-stable law is used to fit the forex data and the self-similarity analysis is made as well. The power-correlation measure is used to describe the relation between the presented series.

## 1. INTRODUCTION

Adequate distributional fitting of empirical financial series for financial risk factors, such as asset returns, fix income rates, FX-rates, implied volatilities, etc., has a great influence on forecasting and investment decisions. Gaussian models were the first to be applied and became the cornerstone of much of financial economic theory. However, there is no sufficient empirical evidence that the classical Gaussian models adequately describe the behavior of financial series. More specifically, real-world financial time series are often characterized by skewness, kurtosis, heavy tails, self-similarity and multifractality. Alternative distributions have been proposed. One distribution supported by empirical evidence, first observed more than 45 years ago by Mandelbrot (1963), is the stable distribution. The advantages of the stable distribution for modeling financial risk factors are now well documented (see, for example, Rachev and Mitnik, 2000 and *Rachev et al.*, 2005).

A limitation of the wide spread usage of stable distributions in the financial industry is that, with the exception of a few special cases, they do not have analytical distribution and density functions. They are however easily described by their characteristic functions (CF). Today, this limitation is overcome due to the availability of various numerical methods to estimate the parameters of the stable distribution (see Rachev and Mitnik, 2000, Stoyanov and Racheva-Iotova, 2004, and Nolan, 2007).

In this paper, we look at two specific problems associated with the analysis of the distribution of assets financial risk factors: the short series problem and the stagnation problem. To overcome the short series problem, the bootstrap method can be employed (see Hesterberg *et al.*, 2003). Bootstrapping is a method for estimating the distribution of an estimator or test statistic by treating the data as if they were the population of interest. That is, the bootstrap method allows one to “create” from the short series a long enough series such that the series exhibits

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*Key words and phrases.* Passivity, mixed-stable model, forex volatility, power-correlation measures.



multifractality and self-similarity, two characteristics that have been often observed for real-world financial risk factors.

The stagnation problem is due to the fact that a time series for some financial risk factors exhibit extremely strong passivity. That is, for some time periods, the financial risk factor does not change because there are no transactions. Assets such as corporate bonds, real estate, and illiquid stocks exhibit this trait, as well as the market for foreign exchange (forex). The same is valid for some implied volatility time series. The stagnation problem, which can also be referred to as the zero financial return/rate/volatility problems, is potentially more serious than it may seem. In this paper, we provide a model for dealing with this problem. We do so by including the following additional condition into the traditional (continuously distributed random variable modeling) model: The random variable is equal to zero with a certain (rather high) probability, otherwise it is distributed by the continuous (normal, stable etc.) law. More formally, we can say that we are extending a continuous model to the mixed one (probability mixture model).

Our model is applied to various daily FX and implied volatility time series and the distribution used is the stable distribution. After examining more than 20,000 risk factors of a large Hedge Fund (HF) we were offered the selected 14 time series that are difficult to be modeled with other classical methods. Those series are daily implied volatilities and FX-rates. We fit stable distribution using maximal likelihood method (MLE). Goodness of fit is verified by the Anderson-Darling distributional adequacy test. The stability is also tested by the homogeneity test, based on the fundamental property of stable laws. Unfortunately, because of the high probability of zero value (financial risk factor is equal to zero), continuous distribution fitting tests (Anderson-Darling, Kolmogorov-Smirnov, etc) are hardly applicable (they are used only for continuous distributions). Since the mixed-stable is not a continuous model, in this paper the Koutrouvelis goodness-of-fit, test based on the empirical characteristic function and modified  $\chi^2$  (Romanovski) test were used.

Similarly with emerging stock markets Belov *et al.* (2006), the data set we are using exhibits long strings of zeros and heavy tailed distributed values outside the zero-strings.

Section 2.3 deals with the distributional analysis of constancy period lengths of zeros. The empirical study of the 14 time series and modeling experiments have showed that constancy period lengths are distributed by the Hurwitz zeta distribution instead of the commonly used geometric distribution. Considering these results an improved mixed-stable model with dependent states of the log-changes is proposed.

When constructing a portfolio, it is essential to determine relationships between underlying financial risk factors. In classical economics and statistics (i.e., where the data have finite first and second moments), the relationship between random variables (log-volatility changes, log-FX rates, returns etc.) is characterized by covariance or correlation. However under the assumption of stability (non-Gaussian stable models) covariance and correlation (Pearson correlation coefficient) cannot be applied, since the variance (if the index of stability  $\alpha < 2$ ) and the mean (if the index of stability  $\alpha \leq 1$ ) do not exist. In this case, we can apply rank correlation coefficients (ex. Spearman or Kendall [19, 20]) or the contingency coefficient.

Under the assumption of stability, it is reasonable to apply generalized covariance coefficients – codifference [33] or power correlation measures [5].

## 2. METHODOLOGY

Quantitative analysis methods of financial data (returns, rates, log-volatility changes, etc.) usually start with its distributional description and empirical parameter estimation. But in problematic cases, e.g. some analyzed value repeats numerous times, and traditional analysis cannot be accomplished. Theoretically, probability that continuous random variable  $X$  is equal to any value  $a$  supposed to be equal 0, i.e.  $P(X = a) = 0$ , but in practice (in stock and forex markets, computer networks etc.) this rule is broken [2, 3, 4, 5] and [18]. In such cases none of the continuous distributions can be used to describe the given data series. This problem is poorly analyzed in literature, but one approach was proposed by Belovas *et al.* [2]. In this paper we borrow the main ideas from [2] applied to the new data of interest to us.

We analyze the following random variables

$$X_i = \log P_{i+1} - \log P_i$$

where  $P_i$  is either implied volatility value at day  $i$ , or FX-rate at day  $i$ .

We start the analysis of our data set of 14 time series with empirical parameters estimation (mean, standard deviation, skewness, kurtosis etc.) and goodness-of-fit hypothesis testing<sup>1</sup> (Anderson–Darling). Our further analysis includes:

- (1) fit alternative non-Gaussian distributions, e.g. alpha-stable, hyperbolic etc.;
  - in case the data are  $\alpha$ -stable distributed, with alpha less than 2 (second moment of the random variable do not exist), relations between series are discussed;
- (2) analyzing the self-similarity and multifractality of our time-series and calculating Hurst index  $H$ . Recall that for Gaussian processes,  $H = 0.5$  indicates Brownian motion;  $0.5 < H < 1$  indicates long time memory processes and "persistent behavior";  $0 < H < 0.5$  shows "anti-persistent behavior", see for example [33];
- (3) calculating how many times the time series value is equal to zero and fitting the mixed distribution to the data. We are also analyzing the behavior of zeros in the series (how they are distributed and occur in the series, are they random, how they can be simulated, etc.).

**2.1. The stable distributions and an overview of their properties.** Following the well-known definition, see [32, 33], a r.v.  $X$  has stable distribution and denoted

$$X \stackrel{d}{=} S_\alpha(\sigma, \beta, \mu),$$

where  $S_\alpha$  is the probability density function, if  $X$  has a characteristic function of the form:

$$\phi(t) = \begin{cases} \exp \left\{ -\sigma^\alpha \cdot |t|^\alpha \cdot \left( 1 - i\beta \operatorname{sgn}(t) \tan\left(\frac{\pi\alpha}{2}\right) \right) + i\mu t \right\}, & \text{if } \alpha \neq 1 \\ \exp \left\{ -\sigma \cdot |t| \cdot \left( 1 + i\beta \operatorname{sgn}(t) \frac{2}{\pi} \cdot \log |t| \right) + i\mu t \right\}, & \text{if } \alpha = 1 \end{cases}.$$

Each stable distribution is described by 4 parameters: the first one and most important is the stability index  $\alpha \in (0, 2]$ , which is essential when characterizing

<sup>1</sup>Null hypothesis: data are distributed by the normal (Gaussian distribution) with the empirical mean and empirical variance.

financial data. The others, respectively are: skewness  $\beta \in [-1, 1]$ , a position  $\mu \in \mathbf{R}$ , the parameter of scale  $\sigma > 0$ .

The probability density function is

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(t) \cdot \exp(-ixt) dt.$$

In the general case, this function cannot be expressed in closed form. The infinite polynomial expressions of the density function are well known, but it is not very useful for Maximal Likelihood Estimation (MLE) because of the error estimation in the tails, the difficulties with truncating the infinite series, and so on. We use an integral expression of the PDF in standard parameterization

$$p(x, \alpha, \beta, \mu, \sigma) = \frac{1}{\pi\sigma} \int_0^\infty e^{-t^\alpha} \cdot \cos \left( t \cdot \left( \frac{x - \mu}{\sigma} \right) - \beta t^\alpha \tan \left( \frac{\pi\alpha}{2} \right) \right) dt.$$

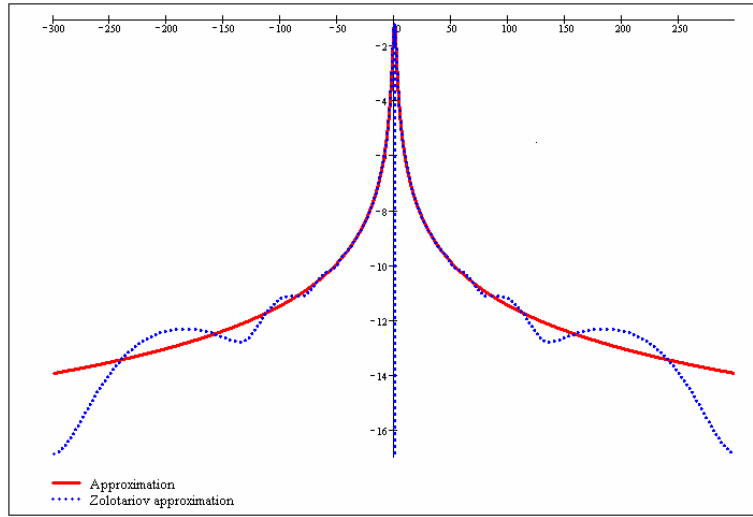


FIGURE 1. Logarithm of the probability density function  $S_{1.5}(1, 0, 0)$

It is important to note that the Fourier integrals are not always convenient to calculate PDF because the integrated function oscillates (Figure 1). That is why a new Zolotarev-type formula is proposed which does not have this problem:

$$p(x, \alpha, \beta, \mu, \sigma) = \begin{cases} \frac{\alpha |\frac{x-\mu}{\sigma}|^{\frac{1}{\alpha-1}}}{2\sigma |\alpha-1|} \int_{-\theta}^1 U_\alpha(\varphi, \theta) \exp \left\{ -|\frac{x-\mu}{\sigma}|^{\frac{\alpha}{\alpha-1}} U_\alpha(\varphi, \theta) \right\} d\varphi, & \text{if } x \neq \mu \\ \frac{1}{\pi\sigma} \cdot \Gamma \left( 1 + \frac{1}{\alpha} \right) \cdot \cos \left( \frac{1}{\alpha} \arctan \left( \beta \cdot \tan \left( \frac{\pi\alpha}{2} \right) \right) \right), & \text{if } x = \mu \end{cases},$$

$$U_\alpha(\varphi, \vartheta) = \left( \frac{\sin \left( \frac{\pi}{2} \alpha (\varphi + \vartheta) \right)}{\cos \left( \frac{\pi\varphi}{2} \right)} \right)^{\frac{\alpha}{1-\alpha}} \cdot \left( \frac{\cos \left( \frac{\pi}{2} ((\alpha-1)\varphi + \alpha\vartheta) \right)}{\cos \left( \frac{\pi\varphi}{2} \right)} \right),$$

where  $\theta = \arctan \left( \beta \tan \frac{\pi\alpha}{2} \right) \frac{2}{\alpha\pi} \cdot \text{sgn}(x - \mu)$  (for properties see [2, 3]).

If  $\mu=0$  and  $\sigma=1$ , then  $p(x, \alpha, \beta) = p(-x, \alpha, -\beta)$ .

A stable r.v. exhibit two important properties (see [18]):

- (1) If  $X_1, X_2, \dots, X_n$  are independent r.v.s. distributed as  $S_\alpha(\sigma, \beta, \mu)$ , then  $\sum_{i=1}^n X_i$  will be distributed as  $S_\alpha(\sigma \cdot n^{1/\alpha}, \beta, \mu \cdot n)$ .
- (2) If  $X_1, X_2, \dots, X_n$  are independent r.v.s. distributed as  $S_\alpha(\sigma, \beta, \mu)$ , then

$$\sum_{i=1}^n X_i \stackrel{d}{=} \begin{cases} n^{1/\alpha} \cdot X_1 + \mu \cdot (n - n^{1/\alpha}), & \text{if } \alpha \neq 1 \\ n \cdot X_1 + \frac{2}{\pi} \cdot \sigma \cdot \beta \cdot n \ln n, & \text{if } \alpha = 1 \end{cases}.$$

Another important property is the following one:

Let  $X_1, X_2, \dots, X_n$  be independent identically distributed random variables and

$$\eta_n = \frac{1}{B_n} \sum_{k=1}^n X_k + A_n,$$

where  $B_n > 0$  and  $A_n$  are constants of scaling and centering. If  $F_n(x)$  is a cumulative distribution function of the r.v.  $\eta_n$ , then the asymptotic distribution of functions  $F_n(x)$ , as  $n \rightarrow \infty$ , is stable. Furthermore, for any stable distribution  $F(x)$ , there exists a series of random variables  $\eta_n$ , such that their distribution functions  $F_n(x)$  converge to  $F(x)$ , as  $n \rightarrow \infty$ .

The  $p$ th moment  $E|X|^p = \int_0^\infty P(|X|^p > y) dy$  of the random variable  $X$  exists and is finite only if  $0 < p < \alpha$ . Otherwise, it does not exist.

**2.1.1. Stable processes.** A stochastic process  $\{X(t), t \in T\}$  is stable if all its finite dimensional distributions are stable [18, 33].

Let  $\{X(t), t \in T\}$  be a stochastic process.  $\{X(t), t \in T\}$  is  $\alpha$ -stable if and only if all linear combinations  $\sum_{k=1}^d b_k X(t_k)$  (here  $d \geq 1$ ,  $t_1, t_2, \dots, t_d \in T$ ,  $b_1, b_2, \dots, b_d$  – real) are  $\alpha$ -stable. A stochastic process  $\{X(t), t \in T\}$  is called the (standard)  $\alpha$ -stable Levy motion if:

- (1)  $X(0)=0$  (almost surely);
- (2)  $\{X(t): t \geq 0\}$  has independent increments;
- (3)  $X(i) - X(s) \sim S_\alpha((t-s)^{1/\alpha}, \beta, 0)$ , for any  $0 \leq s < t < \infty$  and  $0 < \alpha \leq 2$ ,  $-1 \leq \beta \leq 1$ .

Note that the  $\alpha$ -stable Levy motion has stationary increments. As  $\alpha=2$ , we have the Brownian motion.

**2.1.2. Parameter Estimation Methods.** The problem of estimating the parameters of stable distribution is usually severely hampered by the lack of known closed form density functions for almost all stable distributions [3]. Most of the methods in mathematical statistics cannot be used in this case, since these methods depend on an explicit form of the PDF. However, there are numerical methods that have been found useful in practice and are described in [31, 34, 35, 36] and [28].

Given a sample  $x_1, \dots, x_n$  from the stable law, we provide estimates  $\hat{\alpha}$ ,  $\hat{\beta}$ ,  $\hat{\mu}$ , and  $\hat{\sigma}$  of  $\alpha$ ,  $\beta$ ,  $\mu$ , and  $\sigma$  via

- Method of Moments (empirical CF);
- Regression method.
- MLE method

**2.1.3. Comparison of estimation methods.** We simulated a sample of 10 thousand members with the parameters  $\alpha = 1.75$ ,  $\beta = 0.5$ ,  $\mu = 0$  and  $\sigma = 1$  (see [18]). Afterwards we estimated the parameters of a stable random variable with different estimators. All the methods are statistically acceptable, but the maximal likelihood estimator yields the best results. From the practical point-of-view, MLE is the worst method, because it is very time-consuming. For large sets ( $\sim 10,000$  and more) we suggest using the regression (or moments) method to estimate  $\alpha$ ,  $\beta$  and  $\sigma$ , then estimate  $\mu$  by MLM (optimization only by  $\mu$ ). As a starting point you should choose  $\alpha$ ,  $\beta$ ,  $\sigma$  and sample mean, if  $\alpha > 1$  and a median, otherwise, for  $\mu$ . For short time series, use MLE with any starting points (optimization by all 4 parameters).

**2.2. Analysis of process behavior.** Examples of stability analysis can be found in the works of Rachev [6, 17, 37] and Weron [41]. In the latter paper, Weron analyzed the DJIA index (from 1985-01-02 to 1992-11-30. 2000 data points in all). The stability analysis was based on the Anderson–Darling criterion and by the weighted Kolmogorov criterion (D’Agostino), the parameters of stable distribution were estimated by the regression method proposed by Koutrouvelis [21]. The author states that DJIA characteristics perfectly correspond to stable distribution.

We also verify two hypotheses: the first one –  $H_0^1$  is that our sample (with empirical mean  $\hat{\mu}$  and empirical variance  $\hat{\sigma}$ ) follows Gaussian distribution. The second –  $H_0^2$  is that our sample (with parameters  $\alpha$ ,  $\beta$ ,  $\mu$  and  $\sigma$ ) follows the stable (non-Gaussian) distribution. Both hypotheses are examined by two criteria: the Anderson–Darling (A-D) method and Kolmogorov–Smirnov (K-S) method. The first criterion is more sensitive to the difference between empirical and theoretical distribution functions in far quantiles (tails), in contrast to the K-S criterion that is more sensitive to the difference in the central part of distribution.

To prove the stability hypothesis, other researchers [14, 27] applied the method of infinite variance, because non-Gaussian stable r.v.s have infinite variance and thus the set of empirical variances  $S_n^2$  of the random variable  $X$  with infinite variance diverges.

Let  $x_1, \dots, x_n$  be a series of i.i.d.r.v.s  $X$ . Let  $n \leq N < \infty$  and  $\bar{x}_n$  be the mean of the first  $n$  observations,  $S_n^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x}_n)^2$ ,  $1 \leq n \leq N$ . If a distribution has finite variance, then there exists a finite constant  $c < \infty$  such that  $\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x}_n)^2 \rightarrow c$  (almost surely), as  $n \rightarrow \infty$ . Additionally, if the series is simulated by the non-Gaussian stable law, then the series  $S_n^2$  diverges. Fofack [12] has applied this assumption to a series with finite variance (standard normal, Gamma) and with infinite variance (Cauchy and totally skewed stable). In the first case, the series of variances converged quickly and, in the second case, the series of variances oscillated with a high frequency, as  $n \rightarrow \infty$ . Fofack and Nolan [13] applied this method in the analysis of distribution of Kenyan shilling and Morocco dirham exchange rates in the black market. Their results allow us to affirm that the exchange rates of those currencies in the black market change with infinite variance, and even worse – the authors state that distributions of parallel exchange rates of some other countries do not have the mean ( $\alpha < 1$  in the stable case). We present, as an example, a graphical analysis of the variance process of one of our analyzed series (Figure 2).

## STABLE MIXTURE MODEL

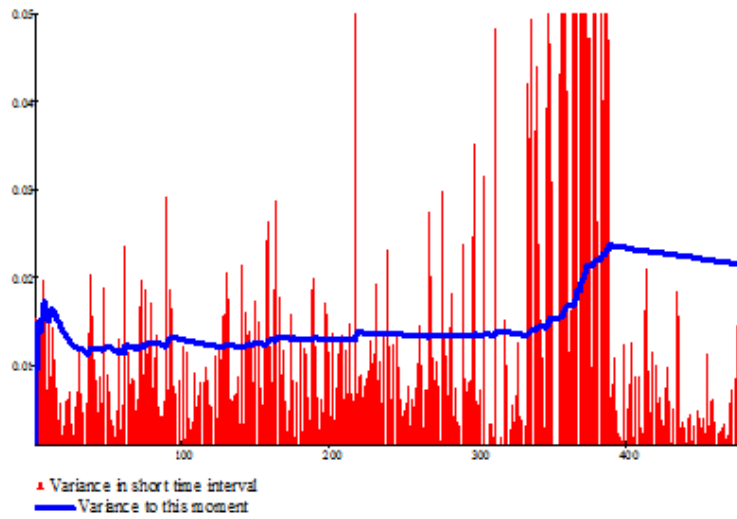


FIGURE 2. Series of empirical variance of the USD.NOK.FXO.0.0.10.25C.CP log-volatility changes

The columns in this graph show the variance at different time intervals (5 days period), and the solid line shows the series of variances  $S_n^2$  (up to time  $n$ ). One can see that, as  $n$  increases, i.e.  $n \rightarrow \infty$ , the series of empirical variance  $S_n^2$  not only diverges, but also oscillates with a high frequency. The same situation is for mostly all the data sets we presented (for more see [18]).

**2.2.1. Stability by homogeneity of the data series and aggregated series.** The third method to verify the stability hypothesis is based on the following property for stable iid (independent identically distributed) rv's. Suppose we have an original financial series (log-volatility changes and log-FX rates)  $X_1, X_2, \dots, X_n$  which we assume are iid and stable distributed. Let us calculate the partial sums  $Y_1, Y_2, \dots, Y_{[n/d]}$ ,

where  $Y_k = \sum_{i=(k-1) \cdot d + 1}^{k \cdot d} X_i$ ,  $k=1 \dots [n/d]$ , and  $d$  is the number of sum components

(freely chosen). The stability implies that original  $X_i$  and derivative series  $Y_i$  must be homogeneous (they must behave similarly and have the same properties). Homogeneity of original and derivative (aggregated) sums was tested by the Smirnov and Anderson criteria ( $\omega^2$ ).

The accuracy of both methods was tested with generated sets, that were distributed by the uniform  $R(-1,1)$ , Gaussian  $N(0,1/\sqrt{3})$ , Cauchy  $C(0,1)$  and stable  $S_{1.75}(1, 0.25, 0)$  distributions. Partial sums were scaled, respectively, by  $\sqrt{d}$ ,  $\sqrt{d}$ ,  $d$ ,  $d^{1/1.75}$ . The test was repeated for a total 100 times. The results of this modeling show that the Anderson criterion (with confidence levels 0.01, 0.05 and 0.1) is more precise than that of Smirnov with the additional confidence level.

It should be noted that these criteria require large samples (of size no less than 200), which is why the original sample must be large enough. The best choice would be if one could satisfy the condition  $n/d > 200$ .

**2.2.2. Self-similarity and multifractality.** Often financial time series exhibit fractionality or self-similarity, see for example [18, 37] and the references therein. The Hurst indicator (or exponent) is used to characterize fractionality.

There are a number of equivalent definitions of self-similarity [39]. The standard one states that a continuous time process  $Y = \{Y(t), t \in T\}$  is self-similar, with the self-similarity parameter  $H$  (Hurst index), if it satisfies the condition:

$$(1) \quad Y(t) \stackrel{d}{=} a^{-H} Y(at), \quad \forall t \in T, \forall a > 0, 0 \leq H < 1,$$

where the equality is in the sense of finite-dimensional distributions. The canonical example of such a process is Fractional Brownian Motion ( $H = 1/2$ ). Since the process  $Y$  satisfying (1) can never be stationary, it is typically assumed to have stationary increments [9].

Figure 3 shows that stable processes are the product of a class of self-similar processes and also that of Levy processes [10]. Suppose a Levy process  $X = \{X(t), t \geq 0\}$ . Then  $X$  is self-similar if and only if each  $X(t)$  is strictly stable. The index  $\alpha$  of stability and the exponent  $H$  of self-similarity satisfy  $\alpha = 1/H$ .

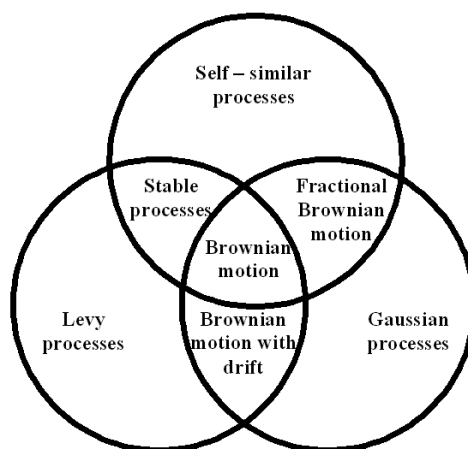


FIGURE 3. Self-similar processes and their relation to Levy and Gaussian processes

Consider the aggregated series  $X^{(m)}$ , obtained by dividing a given series of length  $N$  into blocks of length  $m$  and averaging the series over each block.

$$X^{(m)}(k) = \frac{1}{m} \sum_{i=(k-1)m+1}^{km} X_i, \text{ here } k = 1, 2 \dots [N/m].$$

Self-similarity is often investigated not through the equality of finite-dimensional distributions, but through the behavior of the absolute moments. Thus, consider

$$AM^{(m)}(q) = E \left| \frac{1}{m} \sum_{i=1}^m X(i) \right|^q = \frac{1}{m} \sum_{k=1}^m \left| X^{(m)}(k) - \bar{X} \right|^q$$

## STABLE MIXTURE MODEL

If  $X$  is self-similar, then  $AM^{(m)}(q)$  is proportional to  $m^{\beta(q)}$ , which means that  $\ln AM^{(m)}(q)$  is linear in  $\ln m$  for a fixed  $q$ :

$$(2) \quad \ln AM^{(m)}(q) = \beta(q) \ln m + C(q).$$

In addition, the exponent  $\beta(q)$  is linear with respect to  $q$ . In fact, since  $X^{(m)}(i) \stackrel{d}{=} m^{1-H} X(i)$ , we have

$$(3) \quad \beta(q) = q(H - 1)$$

Thus, the definition of self-similarity is simply that the moments must be proportional as in (2) and that  $\beta(q)$  satisfies (3).

This definition of a self-similar process given above can be generalized to that of multifractal processes. A non-negative process  $X(t)$  is called multifractal if the logarithms of the absolute moments scale linearly with the logarithm of the aggregation level  $m$ . Multifractals are commonly constructed through multiplicative cascades [11]. If a multifractal can take positive and negative values, then it is referred to as a signed multifractal (the term “multiaffine” is sometimes used instead of “signed multifractal”). The key point is that, unlike self-similar processes, the scaling exponent  $\beta(q)$  in (2) is not required to be linear in  $q$ . Thus, signed multifractal processes are a generalization of self-similar processes. To discover whether a process is (signed) multifractal or self-similar, it is not enough to examine the second moment properties. One must analyze higher moments as well.

However this method is only graphical and linearity is only visual.

**2.2.3. Hurst exponent estimation.** There are many methods to evaluate this index, but in literature the following are usually used [39]:

- Time-domain estimators,
- Frequency-domain/wavelet-domain estimators.

The methods: absolute value method (absolute moments), variance method (aggregate variance), R/S method and variance of residuals are known as time domain estimators. Estimators of this type are based on investigating the power law relationship between a specific statistic of the series and the so-called aggregation block of size  $m$ .

The following three methods and their modifications are usually presented as time-domain estimators:

- Periodogram method;
- PWhittle;
- PAbry-Veitch (AV).

The methods of this type are based on the frequency properties of wavelets.

All Hurst exponent estimates were calculated using SELFIS software, which is freeware and can be found on the web page <http://www.cs.ucr.edu/~tkarag>.

**2.3. A mixed stable distribution model.** Let  $Y \sim B(1, p)$  and  $X \sim S_a$  [2]. Let a mixed stable r. v.  $Z$  take the value 0 with probability 1 if  $Y = 0$ , else  $Y = 1$  and  $Z = X$ . Then we can write the distribution function of the mixed stable distribution as

$$(4) \quad P(Z < z) = P(Y = 0) \cdot P(Z < z | Y = 0) + P(Y = 1) \cdot P(Z < z | Y = 1) = p \cdot \varepsilon(z) + (1-p) \cdot S_\alpha(z)$$



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where  $\varepsilon(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$ , is the cumulative distribution function (CDF) of the degenerate distribution. The PDF of the mixed-stable distribution is

$$f(x) = p \cdot \delta(x) + (1 - p) \cdot p_\alpha(x),$$

where  $\delta(x)$  is the Dirac delta function.

**2.3.1. Cumulative density, probability density and characteristic functions of mixed distribution.** For a given set of log-changes  $\{x_1, x_2, \dots, x_n\}$ , let us construct a set of nonzero values  $\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{n-k}\}$ . The implied log-volatility of USD.CAD.FXO.0.0.7.25C.CP is given as an example ( $p = 0.402$ , see Figure 4). Then the likelihood function is given by

$$(5) \quad L(\bar{x}, \theta, p) \sim (1 - p)^k p^{n-k} \prod_{i=1}^{n-k} p_\alpha(\bar{x}_i, \theta)$$

where  $\theta$  is the vector of parameters (in the stable case,  $\theta = (\alpha, \beta, \mu, \sigma)$ ). The function  $(1 - p)^k p^{n-k}$  is easily optimized:  $p_{\max} = \frac{n-k}{n}$ . So we can write the optimal CDF as

$$(6) \quad F(z) = \frac{n-k}{n} S_\alpha(z, \theta_{\max}) + \frac{k}{n} \varepsilon(z),$$

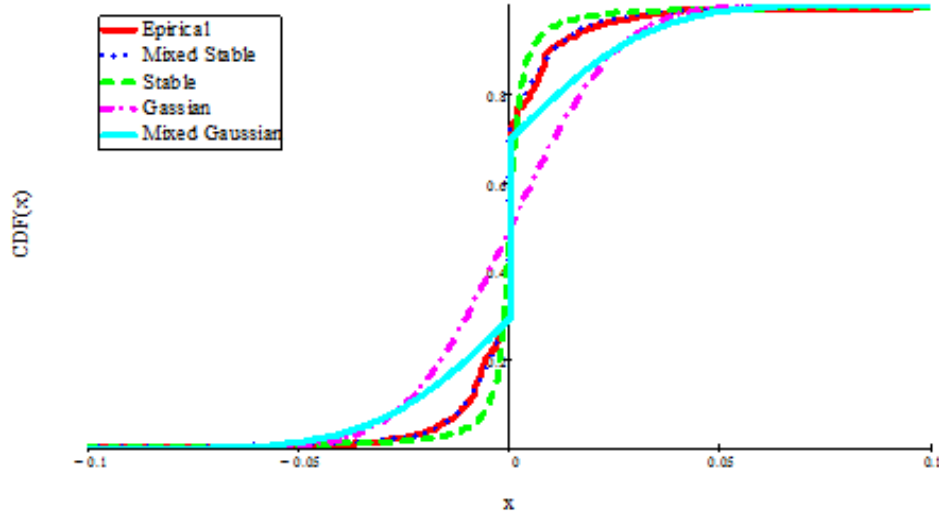


FIGURE 4. CDF of USD.CAD.FXO.0.0.7.25C.CP.

where the vector  $\theta_{\max}$  of parameters is estimated with nonzero data.

The probability density function (see Figure 5)

$$(7) \quad p(z) = \frac{n-k}{n} p_\alpha(z, \theta_{\max}) + \frac{k}{n} \delta(z).$$

## STABLE MIXTURE MODEL

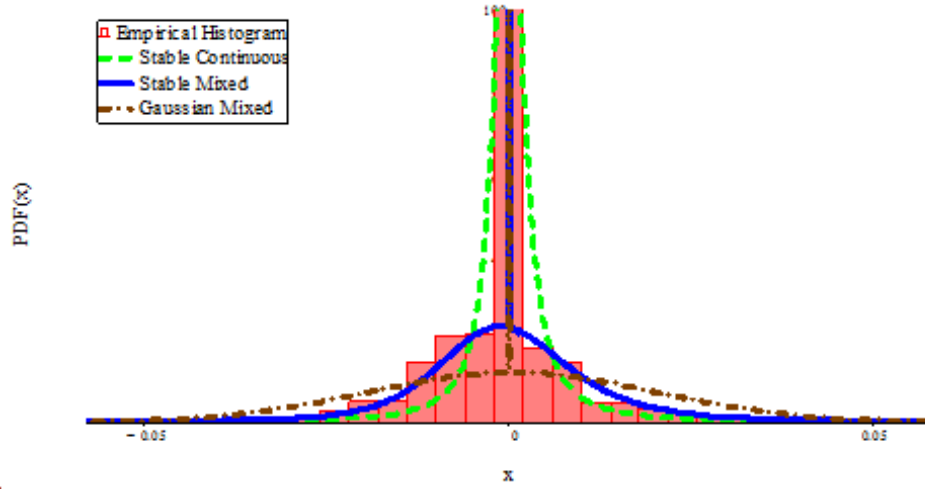


FIGURE 5. PDF and a histogram of USD.CAD.FXO.0.0.7.25C.CP.

Finally we can write down and plot (Figure 6) the characteristic function (CF) of the mixed distribution.

$$\phi_{mix}(t) = \frac{n-k}{n} \cdot \phi(t) + \frac{k}{n}$$

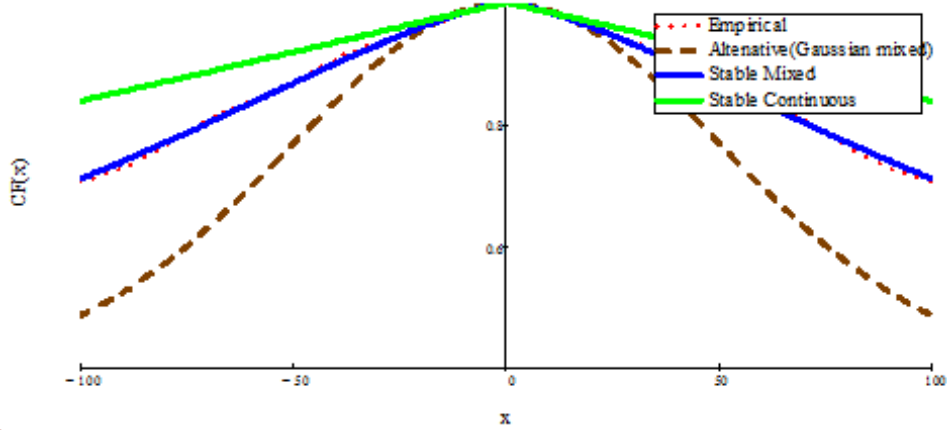


FIGURE 6. Empirical, Gaussian, Stable mixed, and Stable continuous CF of USD.CAD.FXO.0.0.7.25C.CP

The empirical characteristic function is equal to  $\hat{\phi}(t, X) = \frac{1}{n} \sum_{j=1}^n e^{itX_j}$ .

**2.3.2. Mixed model adequacy.** Since we have a discontinuous distribution function, the classic methods (Kolmogorov–Smirnov, Anderson–Darling) do not work, and choose a goodness-of-fit test based on the empirical characteristic function [22, 23], or use a modified  $\chi^2$  (Romanovski) method [20].

The CF-based test of Brown and Saliu [7] is not applicable, since it was developed for symmetric distributions. A new stability test for asymmetric (skewed) alpha-stable distribution functions, based on the characteristic function, should be developed, since the existing tests are not reliable.

A mixed-stable model of risk factor log-changes distribution was proposed in [2]. Since the goodness-of-fit tests for continuous distribution functions cannot be implemented, the tests, based on the empirical characteristic function and a modified  $\chi^2$  test, are used, see [2].

**2.3.3. Modeling of stagnation intervals.** We analyzed (see [18]) the following r.v.s  $X_i = 0$ , if  $P_{i+1} = P_i$  and  $X_i = 1$ , if  $P_{i+1} \neq P_i$ , where  $\{P_i\}$  is a set of exchange rates and  $\{X_i\}$  is a set of discrete states, following our time series (change=1 or not=0).

Empirical study of lengths distribution of zero state runs. Theoretically if states are independent (Bernoulli scheme), then the series of lengths of zero state runs should be distributed by geometrical law. However, the results of empirical tests do not corroborate this assumption. We have fitted the series distribution of lengths of zero state runs by discrete laws (generalized logarithmic, generalized Poisson, Hurwitz zeta, generalized Hurwitz zeta, and discrete stable). The probability mass function of Hurwitz zeta law is

$$P(\xi = k) = \nu_{s,q} (k + q)^{-s},$$

where  $\nu_{s,q} = \left( \sum_{i=0}^{\infty} (i + q)^{-s} \right)^{-1}$ ,  $k \in N$ ,  $q > 0$ ,  $s > 1$ . The parameters of all discrete distributions were estimated by the ML-method.

Transformation and distribution fitting. First of all, we will show how financial data from our data sample are transformed to subsets length of zero state series and then we will fit each of the discrete distributions mentioned in above section. Carvalho, Angeja and Navarro have showed that data in network engineering fit the discrete logarithmic distribution better than the geometrical law. So we intend to test whether such a property is valid for our financial data.

A set of zeros between two units is called a run. The first run is a set of zeros before the first unit and the last one after the last unit. The length of the run is equal to the number of zeros between two units. If there are no zeros between two units, then an empty set has zero length (Figure 7).

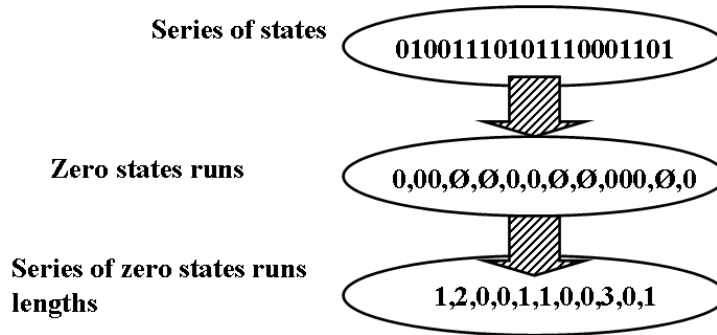


FIGURE 7. Data transformation

To transform our data (from the state series, e.g., 010011101011100110) the two following steps should be taken: (a) extract the zero state runs (e.g., 0.00.0.0.000.0) from the states series; (b) calculate the length of each run (1,2,0.0.1,1,0.0.3,0.1,.). After the transformation, we estimated the parameters of each discrete distribution mentioned above and tested the nonparametric  $\chi^2$  distribution fitting hypothesis. The mixed stable model with dependent states. Since the runs test rejects the randomness hypothesis of the sequence of states, the probability of states (zeros and ones) depends on the position in the sequence. If the lengths of states sequences are distributed by Hurwitz zeta law, then the probabilities of states are

$$P(X_n = 1 | \dots, X_{n-k-1} = 1, \underbrace{X_{n-k} = 0, \dots, X_{n-1} = 0}_k) = \frac{p_k}{1 - \sum_{j=0}^{k-1} p_j}, n \in N, k \in Z_0,$$

where  $p_k$  are probabilities of Hurwitz zeta law;  $P(X_0=1) = p_0$ . It should be noted that  $P(X_n = 0 | \dots) = 1 - P(X_n = 1 | \dots), n, k \in Z_0$ .

With the probabilities of states and distribution of nonzero data, we can generate sequences of log-changes (interchanging in the state sequence units with a stable r.v.) see Figure 8 and [18].

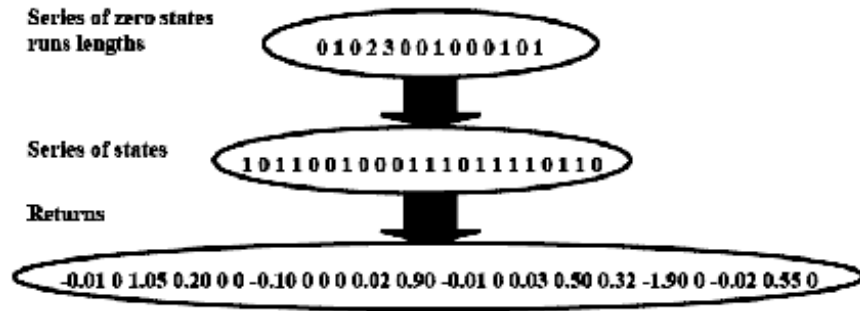


FIGURE 8. Simulation of passive stable series

So, the mixed-stable modeling with dependent states is more advanced than that of independent (Bernoulli) states, and it requires parameter estimation by both the stable  $(\alpha, \beta, \mu, \sigma)$  and Hurwitz zeta  $(q, s)$  law.

**2.4. Relationship measures.** In constructing a financial portfolio, it is essential to determine relationships between different series [18, 30]. However, under the assumption of stability (sets of log-volatility changes and log-FX rates are modeled by stable laws), the classical relationship measures (covariance, correlation) cannot be applied. Therefore the generalized Markowitz problem is solved by generalized relationship measures (covariation, codifference).

In the classical economic statistics (when the distributional law has two first moments, i.e., mean and variance), relations between two random variables are described by covariance or correlation. But if we assume that financial data follow the stable non-Gaussian law (empirical studies corroborate this assumption), covariance and especially correlation (Pearson) cannot be calculated. In a case when

the first ( $\alpha \leq 1$ ) and the second ( $\alpha < 2$ ) moments do not exist, other correlation (rank, e.g., Spearman, Kendall, etc. [19]) and contingency coefficients are proposed. However, in the portfolio selection problem Samorodnitsky and Taqqu [33] suggest better alternatives, even when mean and variance do not exist. They have proposed alternative relation measures: covariation and codifference.

**2.4.1. Codifference.** If  $X_1$  and  $X_2$  are two symmetric i.d. [33] (with  $\alpha_1 = \alpha_2 = \alpha$ ) stable random variables, then the covariation is equal to

$$[X_1, X_2]_\alpha = \int_{S_2} s_1 s_2^{(\alpha-1)} \Gamma(ds),$$

where  $\alpha > 1$ ,  $y^{(\alpha)} = |y|^\alpha \text{sign}(\alpha)$  and  $\Gamma$  is a spectral measure of  $(X_1, X_2)$ .

In such a parameterization, the scale parameter  $\sigma_{X_1}^\alpha$  of symmetric stable r.v. can be calculated from  $[X_1, X_1]_\alpha = \sigma_{X_1}^\alpha$ . If  $\alpha = 2$  (Gaussian distribution), the covariation is equal to half of the covariance  $[X_1, X_2]_2 = \frac{1}{2} \text{Cov}(X_1, X_2)$  and  $[X_1, X_1]_2 = \sigma_{X_1}^2$  becomes equal to the variance of  $X_1$ . However, the covariation norm of  $X \in S_\alpha$  ( $\alpha > 1$ ) can be calculated as  $\|X\| = ([X, X]_\alpha)^{1/\alpha}$ . If  $X \sim S_\alpha(\sigma, 0, 0)$  (SaS case), then the norm is equivalent to the scale parameter of the stable distribution  $\|X\|_\alpha = \sigma$ .

In a general case [29] the codifference is defined through characteristic functions

$$\begin{aligned} \text{cod}_{X,Y} &= \ln(E \exp\{i(X-Y)\}) - \ln(E \exp\{iX\}) - \ln(E \exp\{-iY\}) \\ &= \ln\left(\frac{E \exp\{i(X-Y)\}}{E \exp\{iX\} \cdot E \exp\{-iY\}}\right) = \ln\left(\frac{\phi_{X-Y}}{\phi_X \cdot \phi_{-Y}}\right), \end{aligned}$$

or empirical characteristic functions

$$\text{cod}_{X,Y} = \ln\left(\frac{n \cdot \sum_{j=1}^n e^{i(X_j - Y_j)}}{\sum_{j=1}^n e^{iX_j} \cdot \sum_{j=1}^n e^{-iY_j}}\right)$$

The codifference of two symmetric (SaS) r.v.s  $X$  and  $Y$  ( $0 < \alpha \leq 2$ ) can be expressed through the scale parameters

$$\text{cod}_{X,Y} = \|X\|_\alpha^\alpha + \|Y\|_\alpha^\alpha - \|X - Y\|_\alpha^\alpha$$

If  $\alpha = 2$ , then  $\text{cod}_{X,Y} = \text{Cov}(X, Y)$ .

Samorodnitsky and Taqqu have showed that

$$(1 - 2^{\alpha-1})(\|X\|_\alpha^\alpha + \|Y\|_\alpha^\alpha) \leq \text{cod}_{X,Y} \leq \|X\|_\alpha^\alpha + \|Y\|_\alpha^\alpha,$$

here  $1 \leq \alpha \leq 2$ , and, if we normalize (divide by  $\|X\|_\alpha^\alpha + \|Y\|_\alpha^\alpha$ ), we will get a generalized correlation coefficient.

In the general case [29], the following inequalities

$$\begin{aligned} (1 - 2^{\alpha-1}) \ln\left(\frac{1}{E \exp\{iX\} \cdot E \exp\{-iY\}}\right) &\leq \text{cod}_{X,Y} = \ln\left(\frac{E \exp\{i(X-Y)\}}{E \exp\{iX\} \cdot E \exp\{-iY\}}\right) \\ &\leq \ln\left(\frac{1}{E \exp\{iX\} \cdot E \exp\{-iY\}}\right) \end{aligned}$$

are proper, and if we divide both sides by  $\ln(E \exp\{iX\} \cdot E \exp\{-iY\})$ , we will get the following system of inequalities for the correlation coefficient

$$(1 - 2^{\alpha-1}) \leq \text{corr}_{X,Y} = \frac{\ln\left(\frac{E \exp\{iX\} \cdot E \exp\{-iY\}}{E \exp\{i(X-Y)\}}\right)}{-\ln(E \exp\{iX\} \cdot E \exp\{-iY\})} \leq 1$$

If  $0 < \alpha \leq 1$  this correlation coefficient is only non-negative, and if  $\alpha = 2$ ,  $\beta = 0$ , then  $-1 \leq \text{corr}_{X,Y} = \rho_{X,Y} \leq 1$  is equivalent to the Pearson correlation coefficient. Significance of codifference. The significance of the Pearson correlation coefficient is tested using Fisher statistics and that of the Spearman and Kendall coefficients, respectively, are tested using Student and Gaussian distributions. But it is likely that there are no codifference significance tests created. In such a case, we use the bootstrap method (one of the Monte-Carlo style methods). The following algorithm to test the codifference significance is proposed:

- (1) Estimate stable parameters ( $\alpha$ ,  $\beta$ ,  $\sigma$  and  $\mu$ ) and stagnation probability  $p$  of all equity series;
- (2) Estimate relation matrix of measure  $\rho$  (covariation or codifference) for every pair of equities series;
- (3) Test the significance of each  $\rho_{ij}$  by the bootstrap method:
  - (a) generate a pair of two  $i$ th and  $j$ th mixed-stable (with estimated parameters) series, and proceed to the next step;
  - (b) calculate the  $k$ th relation measure  $\rho_{ij}^k$ , between the  $i$ th and  $j$ th series;
  - (c) repeat (a) and (b) steps for  $k = 1, \dots, N$  (for example, 10000) times;
  - (d) construct ordered series of estimates  $\rho_{ij}^{(k)}$ ;
  - (e) if  $\rho_{ij}^{([N \cdot 0.025])} \leq \rho_{ij} \leq \rho_{ij}^{([N \cdot 0.975])}$ , then the significance of  $\rho_{ij}$  is rejected with the confidence level 0.05, i.e., it is assumed that  $\rho_{ij} = 0$ .
  - (f) repeat 3a–3f steps for each pair of equities  $i$  and  $j$ .

**2.4.2. Generalized power-correlation measures.** In [5] the power correlation measures was introduced, with three standardizations (absolute and median deviation, and universal).

**Definition 1.** *The power-correlation measure is defined as a function of two random variables  $X$  and  $Y$*

$$\rho(X, Y) = 1 - \frac{\sum_{i=1}^N \left| \frac{X_i - \hat{\mu}_X}{\hat{s}_X} - \frac{Y_i - \hat{\mu}_Y}{\hat{s}_Y} \right|^\gamma}{\sum_{i=1}^N \left| \frac{X_i - \hat{\mu}_X}{\hat{s}_X} \right|^\gamma + \left| \frac{Y_i - \hat{\mu}_Y}{\hat{s}_Y} \right|^\gamma},$$

here  $\gamma = \min(\alpha_X, \alpha_Y)$  is an existing moment of r.v.,  $\hat{\mu}_\bullet$  and  $\hat{s}_\bullet$  are standardization constants,  $\alpha_X$  and  $\alpha_Y$  are estimates of stability parameters of random variables  $X$  and  $Y$  respectively.

In the general case,  $\hat{\mu}_\bullet$  is the estimate of the location parameter,  $\hat{s}_\bullet = \hat{\sigma}_\bullet$  is the estimate of the scale parameter. Depending on  $\gamma$  three standardizations was proposed [5]: the universal one (for general case), and two special standardizations: absolute deviation standardization (for  $1 < \gamma < 2$ , when the mean exists) and median standardization (for  $\gamma < 1$ ).

**Definition 2** (Universal standardization). *The centering and normalization constants ( $\hat{\mu}_\bullet$  and  $\hat{s}_\bullet$  respectively) are equal to the estimates of the location and scale parameters respectively.*

This standardization method could be applied with every possible stability index. Note that if  $\gamma=2$ , then  $\hat{\mu}_\bullet$  and  $\hat{s}_\bullet$  can be replaced by the mean and the standard deviation respectively.

**Definition 3** (Absolute deviation standardization). *In the case when  $\gamma \in (1; 2)$ , then the centering and normalization constants ( $\hat{\mu}_\bullet$  and  $\hat{s}_\bullet$  respectively) are equal to the mean  $\bar{\mu}_X$  of underlying series  $X$  and the absolute deviation  $\hat{s}_X = \frac{1}{n} \sum_{i=1}^n |X_i - \bar{\mu}_X|$  respectively.*

**Definition 4** (Median standardization). *In the case when  $\gamma < 1$  (as well as in the general case), then the centering constant  $\hat{\mu}_\bullet$  can be replaced by the median  $m_\bullet$  and the normalization constant  $\hat{s}_\bullet$  by the following normalization constant  $\hat{s}_X = \text{median}|X - m_X|$ .*

The norm of codifference and power correlation measures indicates the strength and direction of a linear relationship between two random variables. Depending on the moment  $\gamma$ , these relation measures are bounded [33]:

$$1 - 2^{\gamma-1} \leq \rho(X, Y) \leq 1.$$

In the general statistical usage, they refer to the departure of two variables from independence. However, if  $\rho(X, Y) = 0$ , we cannot say that random variables  $X$  and  $Y$  are independent.

### 3. DESCRIPTION OF THE DATA

As we already mentioned in the introduction, a large HF will model and forecast around 20,000 risk variables such as equity returns, interest rates, bond yields, FX rates, volatility surfaces, etc. Very often such series exhibit strings of zero values and heavy tailed distributed values otherwise. An example of such daily time series selected from those 20,000 was offered to be analyzed and it is given in Table 1.

TABLE 1. Empirical characteristics of data sets and criterion probabilities of Anderson – Darling

Nr	Data set	Empirical characteristics					Nr. of obs.	Nr. of zero daily data
		$\hat{\mu}$	$\hat{\sigma}$	$\hat{\gamma}_1$	$\hat{\gamma}_2$	Probab. of A-D crit.		
1	CHF.JPY.FXO.0.0.10.25C.CP <sup>a</sup>	0	0.028	1.176	32.34	-	1439	520
2	CHF.JPY.FXO.0.0.7.ATM.CP	-0.0002	0.020	0.642	32.19	-	1439	521
3	EUR.JPY.FXO.0.0.10.25C.CP	0.0001	0.028	2.692	44.12	-	1640	502
4	GBP.NOK.FXO.0.0.3.25C.CP	-0.0001	0.013	1.596	24.46	-	1439	69
5	USD.CAD.FXO.0.0.7.25C.CP	0.0004	0.019	2.040	55.45	-	1443	580
6	USD.GBP.FXO.0.0.10.25C.CP	0.0001	0.012	0.807	19.03	-	1443	564
7	USD.JOD.CCY.CP	0	0.001	-0.101	27.25	-	2488	1773
8	USD.MXN.FXO.0.0.5.25C.CP	-0.0004	0.014	-1.467	32.17	0.999	1640	672
9	USD.MXN.FXO.0.0.5.ATM.CP	-0.0003	0.014	-1.115	31.49	0.999	1640	687
10	USD.NOK.FXO.0.0.10.25C.CP	0.0001	0.021	0.814	16.23	0.999	1443	277
11	USD.NZD.FXO.0.0.3.ATM.CP	0.0004	0.014	6.300	123.1	-	2288	1044
12	USD.NZD.FXO.0.0.7.ATM.CP	0	0.016	-1.45	34.71	0.999	1443	514
13	USD.SEK.FXO.0.0.10.25C.CP	0.0001	0.018	0.695	15.61	0.999	1443	233
14	USD.UAG.CCY.CP.	-0.0004	0.009	-4.350	82.39	-	2488	189

<sup>a</sup> CHF.JPY.FXO.0.0.10.25C.CP means that we analyze JPY/CHF implied volatility of call option, with a delta equal to 0.25, with 10 days to maturity. The probability of A-D crit. shows the  $p$  probability of Anderson-Darling adequacy test with normal distribution. (-) means that A-D goodness-of-fit test is completely rejected for any significance level. Here  $\hat{\mu}$  is an empirical mean,  $\hat{\sigma}$  is a standard deviation,  $\hat{\gamma}_1$  is empirical skewness and  $\hat{\gamma}_2$  is empirical kurtosis.

## STABLE MIXTURE MODEL

In our illustrations, we use a data series for daily foreign exchange rates (series 7 and 14) and option volatilities of foreign exchange rates (other series in Table 1). The first three symbols are ticker of the first currency; the second three symbols are ticker of the second currency; the next three symbols FXO means implied volatility and CCY means the spot exchange rate; the numbers following FXO indicate the time to maturity of the option in terms of years, months, and days (for example, 0.0.10 would mean option with a time to maturity of 0 years, 0 months and 10 days, see note of Table 1). “ATM” means an at-the-money call option and “25C” is a call option with a delta equal to 0.25. We use different pairs of exchange rates with the length of each series being different and ranging from 1,439 observations to 2,488 observations. The average number of observations is 1,694. Most importantly for our study, the number of zero daily data differs, from as few as 4.79% to a high of 71.26% with an average of 33.23%. Information about the data series length is provided in Table 1.

As Table 1 shows, some of the data series are strongly asymmetric ( $\hat{\gamma}_1$ ), and the empirical kurtosis ( $\hat{\gamma}_2$ ) shows that density functions of the series are more peaked than that of the Gaussian distribution. Consequently, consistent with the findings of other studies and Anderson–Darling goodness-of-fit test, we conclude that the Gaussian models are inappropriate. So we proceed to the stability (non-Gaussian) analysis of the series.

**3.1. Application of mixed-stable model.** Since we have found several zeros (repeating value) in our series (see Table 1) we start with the mixed-stable<sup>2</sup> model parameters estimation (see Section 2.3.1). The results of parameter estimation for 14 forex implied volatility and exchange rate series are given in Table 2.

TABLE 2. Stability parameters of 14 series.

Nr.	Series	$\alpha$	$\beta$	$\mu$	$\sigma$	$p$
1.	CHF.JPY.FXO.0.0.10.25C.CP	1.05 <	-0.00387	-4.3E-05	0.001165	0.361
2.	CHF.JPY.FXO.0.0.7.ATM.CP	1.255604	0.021663	-0.00181	0.001915	0.362
3.	EUR.JPY.FXO.0.0.10.25C.CP	1.146631	-0.47731	-0.03653	0.012211	0.306
4.	GBP.NOK.FXO.0.0.3.25C.CP	1.05 <	-0.02908	-0.0016	0.003229	0.048
5.	USD.CAD.FXO.0.0.7.25C.CP	1.066127	0.047113	0.000873	0.001951	0.402
6.	USD.GBP.FXO.0.0.10.25C.CP	1.05 <	0.101149	0.003961	0.002856	0.391
7.	USD.JOD.CCY.CP	1.186602	-0.02678	-6.1E-05	9.78E-05	0.713
8.	USD.MXN.FXO.0.0.5.25C.CP	1.05 <	-0.10462	-0.00289	0.001925	0.410
9.	USD.MXN.FXO.0.0.5.ATM.CP	1.05 <	0.120186	0.002893	0.00174	0.419
10.	USD.NOK.FXO.0.0.10.25C.CP	1.12673	0.131193	0.002027	0.004701	0.192
11.	USD.NZD.FXO.0.0.3.ATM.CP	1.184021	-0.4117	-0.00182	0.001925	0.456
12.	USD.NZD.FXO.0.0.7.ATM.CP	1.186487	0.752788	0.031052	0.008517	0.356
13.	USD.SEK.FXO.0.0.10.25C.CP	1.199201	-0.06614	-0.00175	0.005031	0.161
14.	USD.UAG.CCY.CP	1.475154	0.471805	0.001717	0.001316	0.076

The results of goodness-of-fit test based on the empirical characteristic function and modified  $\chi^2$  (Romanovski) methods are similar (match in 9 and 12 cases) and presented in Table 3. The result shows that in 12 cases out of 14 the goodness-of-fit hypothesis were not rejected for mixed-stable law. More detailed results of stable-mixed model fitting are given in Table 4. One can see that when the number of “zeros” increases, the mixed model fits the empirical data “better”.

<sup>2</sup>Actually the A-D criterion hypotheses of full series stability were rejected in all cases.



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TABLE 3. Results of goodness-of-fit tests (accepted/rejected cases)

Fit Method	Gaussian	Mixed Gaussian	Stable	Mixed-stable
Modified $\chi^2$	0/14	0/14	0/14	9/14
Empirical CF	0/14	0/14	0/14	12/14
Anderson–Darling	0/14	-	0/14	-

TABLE 4. Mixed model fit dependence on the number of zeros in series

Number of “zeros”	Number of such series	Fits mixed model ( $\chi^2$ , %)	Fits mixed model (Empirical CF, %)
< 0.1	2	0	100
0.1-0.2	2	50	50
0.2-0.3	0	-	-
0.3-0.4	5	60	80
0.4-0.5	4	100	100
0.5-0.6	0	-	-
0.6-0.7	0	-	-
0.7-0.8	1	100	100
0.8-0.9	0	-	-

3.1.1. *Analysis of zeros distribution.* As mentioned in Section 2.3.3, theoretically the series of zeros should be distributed by the binomial law and the lengths of these series should be distributed by the geometrical law, however, from Table 5 we can see that other laws fit our data (12 series) much better. It means that zero state series from our data are better described by the Hurwitz zeta distribution.

TABLE 5. Distribution of zero state series

Signif. level	Hurwitz zeta	Generalized Hurwitz zeta	Generalized logarithmic	Discrete stable	Poisson	Generalized Poisson	Geometrical
0.01	92.86%	85.71%	0.00%	21.43%	57.14%	0.00%	0.00%
0.025	92.86%	78.57%	0.00%	7.14%	42.86%	0.00%	0.00%
0.05	78.57%	71.43%	0.00%	7.14%	28.57%	0.00%	0.00%
0.1	71.43%	64.29%	0.00%	7.14%	28.57%	0.00%	0.00%

This result allows us to assume that zero-unit states are not purely independent. The Wald–Wolfowitz runs test [24] corroborates this assumption for almost all series from the given Forex list. The inner series dependence was tested by the Hoel [16] criterion on the order of the Markov chain. It has been concluded that there are no zero order series or Bernoulli scheme series. 99% of given series are higher than 4th-order Markov chains with  $\phi = 0.1\%$  significance level.

3.2. **Analysis of series behavior.** There was performed a test described in Section 2.2.1. The homogeneity of aggregated and full series was tested only by the Anderson criterion. Partial series were calculated by summing  $d=10$  and 15 elements and scaling with  $d^{1/\alpha}$ . Following the fundamental stability statement, one may draw a conclusion that for USD.GBP.FXO.0.0.10.25C.CP and USD.NOK.FXO.0.0.10.25C.CP the hypothesis on stability is acceptable. However, if we remove the zeros from the series we will also see that the hypothesis on stability cannot be rejected for USD.NZD.FXO.0.0.7. ATM.CP, USD.NZD.FXO.0.0.3.ATM.CP, USD.MXN.FXO.0.0.5.ATM.CP, USD.MXN.FXO.0.0.5.25C.CP, USD.JOD.CCY.CP,

USD.GBP.FXO.0.0.10.25C.CP, USD.CAD.FXO.0.0.7.25C.CP and CHF.JPY.FXO.0.0.7.ATM.CP.

Having these results we may then proceed to analysis of multifractality and self-similarity. Result of absolute moments method for full series and series without zeros is given in Table 6.

TABLE 6. Results of multifractality and self-similarity analysis for full series and series without zeros

Nr.	Series	Full series		Without zeros	
		Multifractal	Self-similar	Multifractal	Self-similar
1.	CHF.JPY.FXO.0.0.10.25C.CP	0	0	0	0
2.	CHF.JPY.FXO.0.0.7.ATM.CP	0	0	0	0
3.	EUR.JPY.FXO.0.0.10.25C.CP	0	0	0	0
4.	GBP.NOK.FXO.0.0.3.25C.CP	1	1	1	1
5.	USD.CAD.FXO.0.0.7.25C.CP	1	0	0	0
6.	USD.GBP.FXO.0.0.10.25C.CP	1	0	0	0
7.	USD.JOD.CCY.CP	1	0	0	0
8.	USD.MXN.FXO.0.0.5.25C.CP	1	0	1	0
9.	USD.MXN.FXO.0.0.5.ATM.CP	1	1	0	0
10.	USD.NOK.FXO.0.0.10.25C.CP	1	0	1	0
11.	USD.NZD.FXO.0.0.3.ATM.CP	1	0	0	0
12.	USD.NZD.FXO.0.0.7.ATM.CP	0	0	1	1
13.	USD.SEK.FXO.0.0.10.25C.CP	1	0	1	0
14.	USD.UAG.CCY.CP	1	0	0	0
Total		10	2	5	2

Finally, only 2 indices are self-similar (Table 6): GBP.NOK.FXO.0.0.3.25C.CP and USD.MXN.FXO.0.0.5.ATM.CP. However if we remove the zeros from the series GBP.NOK.FXO.0.0.3.25C.CP. and USD.NZD.FXO.0.0.7.ATM.CP are self-similar.

**3.3. Relation analysis.** In this section tables of following relation measures are given:

- norm of codifference (Table 7);
- Pearson correlation coefficient (Table 8);
- power-correlation coefficient in universal case (Table 9);
- power-correlation coefficient with absolute deviation standardization (Table 10);
- power-correlation coefficient with median standardization (Table 11);

The first one is used only for alpha-stable distributed data. The second one is used in case the second moment of an underlying random variable exists. The three last relation measures may be used for any data, if the highest existing moment of the series is known. The latest measures (as also the first one) are bounded  $1 - 2^{\gamma-1} \leq \rho(X, Y) \leq 1$  (here  $\gamma = \min(\alpha_X, \alpha_Y) > 1$  is an existing moment), and, when  $\gamma \leq 1$ , these measures gives only a non-negative results.

The risk factor sets of CHF.JPY.FXO.0.0.10.25C.CP, CHF.JPY.FXO.0.0.7.ATM.CP, and EUR.JPY.FXO.0.0.10.25C.CP have large correlation, USD.MXN.FXO.0.0.5.25C.CP and USD.MXN.FXO.0.0.5.ATM.CP have large correlation, USD.NOK.FXO.0.0.10.25C.CP and USD.SEK.FXO.0.0.10.25C.CP have large correlation, USD.NZD.FXO.0.0.3.ATM.CP and USD.NZD.FXO.0.0.7.ATM.CP have large correlation. The other pairs have small correlation or are uncorrelated. The factors power-relation measures are given (Tables 9–11).

TABLE 7. Norm of codifference for 14 series

	1 <sup>a</sup>	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1.00	0.66	0.80	0.01	0.00	-0.03	0.00	-0.01	0.00	0.00	0.00	0.05	-0.03	0.00
2	0.66	1.00	0.62	0.05	-0.02	0.05	0.00	0.02	0.04	0.14	0.02	0.05	0.08	0.00
3	0.80	0.62	1.00	0.00	0.01	-0.02	0.00	-0.01	-0.01	0.11	-0.02	0.05	0.10	0.00
4	0.01	0.05	0.00	1.00	-0.01	0.20	0.00	0.10	0.10	0.03	0.11	0.05	0.08	0.00
5	0.00	-0.02	0.01	-0.01	1.00	0.01	0.00	0.03	0.03	0.05	0.01	0.04	0.06	-0.01
6	-0.03	0.05	-0.02	0.20	0.01	1.00	0.00	0.16	0.17	0.10	0.06	0.02	0.07	0.00
7	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
8	-0.01	0.02	-0.01	0.10	0.03	0.16	0.00	1.00	0.98	0.05	0.11	0.01	0.07	0.00
9	0.00	0.04	-0.01	0.10	0.03	0.17	0.00	0.98	1.00	0.05	0.11	0.01	0.07	0.00
10	0.00	0.14	0.11	0.03	0.05	0.10	0.00	0.05	0.05	1.00	-0.02	0.00	0.86	0.00
11	0.00	0.02	-0.02	0.11	0.01	0.06	0.00	0.11	0.11	-0.02	1.00	0.52	0.00	-0.01
12	0.05	0.05	0.05	0.05	0.04	0.02	0.00	0.01	0.01	0.00	0.52	1.00	0.04	-0.01
13	-0.03	0.08	0.10	0.08	0.06	0.07	0.00	0.07	0.07	0.86	0.00	0.04	1.00	0.00
14	0.00	0.00	0.00	0.00	-0.01	0.00	0.00	0.00	0.00	0.00	-0.01	-0.01	0.00	1.00

<sup>a</sup> Note: For numeration see Table 1.

TABLE 8. Pearson correlation coefficient

	1 <sup>a</sup>	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1.00	0.70	0.80	0.02	0.01	-0.05	0.00	-0.02	0.00	-0.01	0.00	0.06	-0.03	-0.02
2	0.70	1.00	0.65	0.06	-0.02	0.06	0.00	0.03	0.05	0.14	0.03	0.05	0.09	-0.02
3	0.80	0.65	1.00	0.00	0.01	-0.03	0.00	-0.02	-0.01	0.12	-0.03	0.06	0.11	-0.02
4	0.02	0.06	0.00	1.00	-0.01	0.20	0.04	0.10	0.10	0.03	0.11	0.05	0.08	-0.01
5	0.01	-0.02	0.01	-0.01	1.00	0.01	0.01	0.03	0.03	0.05	0.01	0.04	0.06	-0.08
6	-0.05	0.06	-0.03	0.20	0.01	1.00	0.01	0.16	0.17	0.13	0.06	0.02	0.09	-0.01
7	0.00	0.00	0.00	0.04	0.01	0.01	1.00	0.01	0.00	0.05	0.02	-0.03	0.05	0.00
8	-0.02	0.03	-0.02	0.10	0.03	0.16	0.01	1.00	0.99	0.06	0.11	0.01	0.08	-0.01
9	0.00	0.05	-0.01	0.10	0.03	0.17	0.00	0.99	1.00	0.06	0.11	0.01	0.08	-0.01
10	-0.01	0.14	0.12	0.03	0.05	0.13	0.05	0.06	0.06	1.00	-0.02	0.00	0.87	-0.01
11	0.00	0.03	-0.03	0.11	0.01	0.06	0.02	0.11	0.11	-0.02	1.00	0.55	0.00	-0.02
12	0.06	0.05	0.06	0.05	0.04	0.02	-0.03	0.01	0.01	0.00	0.55	1.00	0.04	-0.03
13	-0.03	0.09	0.11	0.08	0.06	0.09	0.05	0.08	0.08	0.87	0.00	0.04	1.00	0.01
14	-0.02	-0.02	-0.02	-0.01	-0.08	-0.01	0.00	-0.01	-0.01	-0.01	-0.02	-0.03	0.01	1.00

<sup>a</sup> Note: For numeration see Table 1.

One can easily compare the traditional relation measures (see Tables 7 and 8) with the presented power-correlation measures (see Tables 9–11). The risk factors sets of CHF.JPY.FXO.0.0.10.25C.CP, CHF.JPY.FXO.0.0.7.ATM.CP, and EUR.JPY.FXO.0.0.10.25C.CP have strong relation, USD.MXN.FXO.0.0.5.25C.CP and USD.MXN.FXO.0.0.5.ATM.CP have strong relation, USD.NOK.FXO.0.0.10.25C.CP and USD.SEK.FXO.0.0.10.25C.CP have strong relation, USD.NZD.FXO.0.0.3.ATM.CP and USD.NZD.FXO.0.0.7.ATM.CP have strong relation. The other pairs have weak relation or are unrelated. This means that it is more correct to use power-correlation measures for series that are stable distributed instead of Pearson correlation.

#### 4. CONCLUSIONS

Parameter estimation methods and software have been developed for models with asymmetric stable distributions. The efficiency of estimation methods was tested by simulating the series. Empirical methods are more effective in time, but

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TABLE 9. Power-correlation coefficient in universal case

	1 <sup>a</sup>	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1.00	0.70	0.78	0.26	0.19	0.15	0.10	0.14	0.14	0.22	0.17	0.16	0.21	0.20
2	0.70	1.00	0.61	0.27	0.22	0.21	0.12	0.19	0.19	0.27	0.21	0.18	0.26	0.22
3	0.78	0.61	1.00	0.26	0.19	0.15	0.10	0.14	0.14	0.24	0.17	0.16	0.24	0.20
4	0.26	0.27	0.26	1.00	0.21	0.21	0.12	0.18	0.18	0.22	0.22	0.18	0.25	0.22
5	0.19	0.22	0.19	0.21	1.00	0.16	0.07	0.14	0.14	0.19	0.14	0.14	0.18	0.11
6	0.15	0.21	0.15	0.21	0.16	1.00	0.04	0.22	0.22	0.26	0.15	0.12	0.20	0.07
7	0.10	0.12	0.10	0.12	0.07	0.04	1.00	0.06	0.06	0.08	0.06	0.03	0.07	0.11
8	0.14	0.19	0.14	0.18	0.14	0.22	0.06	1.00	0.99	0.15	0.15	0.11	0.15	0.07
9	0.14	0.19	0.14	0.18	0.14	0.22	0.06	0.99	1.00	0.15	0.14	0.11	0.15	0.07
10	0.22	0.27	0.24	0.22	0.19	0.26	0.08	0.15	0.15	1.00	0.16	0.14	0.83	0.12
11	0.17	0.21	0.17	0.22	0.14	0.15	0.06	0.15	0.14	0.16	1.00	0.67	0.17	0.08
12	0.16	0.18	0.16	0.18	0.14	0.12	0.03	0.11	0.11	0.14	0.67	1.00	0.15	0.07
13	0.21	0.26	0.24	0.25	0.18	0.20	0.07	0.15	0.15	0.83	0.17	0.15	1.00	0.11
14	0.20	0.22	0.20	0.22	0.11	0.07	0.11	0.07	0.07	0.12	0.08	0.07	0.11	1.00

<sup>a</sup> Note: For numeration see Table 1.

TABLE 10. Power-correlation coefficient with absolute deviation standardization

	1 <sup>a</sup>	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1.00	0.69	0.76	0.17	0.13	0.16	0.04	0.12	0.13	0.19	0.15	0.15	0.18	0.10
2	0.69	1.00	0.62	0.19	0.14	0.22	0.03	0.17	0.17	0.23	0.18	0.16	0.21	0.11
3	0.76	0.62	1.00	0.16	0.14	0.17	0.03	0.13	0.13	0.23	0.15	0.16	0.23	0.11
4	0.17	0.19	0.16	1.00	0.18	0.27	0.07	0.22	0.21	0.22	0.25	0.20	0.25	0.17
5	0.13	0.14	0.14	0.18	1.00	0.16	0.03	0.13	0.13	0.18	0.13	0.13	0.17	0.06
6	0.16	0.22	0.17	0.27	0.16	1.00	0.03	0.22	0.22	0.27	0.15	0.12	0.20	0.06
7	0.04	0.03	0.03	0.07	0.03	0.03	1.00	0.04	0.04	0.05	0.04	0.02	0.04	0.06
8	0.12	0.17	0.13	0.22	0.13	0.22	0.04	1.00	0.99	0.15	0.15	0.11	0.15	0.05
9	0.13	0.17	0.13	0.21	0.13	0.22	0.04	0.99	1.00	0.15	0.14	0.11	0.15	0.05
10	0.19	0.23	0.23	0.22	0.18	0.27	0.05	0.15	0.15	1.00	0.17	0.14	0.84	0.10
11	0.15	0.18	0.15	0.25	0.13	0.15	0.04	0.15	0.14	0.17	1.00	0.67	0.17	0.06
12	0.15	0.16	0.16	0.20	0.13	0.12	0.02	0.11	0.11	0.14	0.67	1.00	0.14	0.05
13	0.18	0.21	0.23	0.25	0.17	0.20	0.04	0.15	0.15	0.84	0.17	0.14	1.00	0.08
14	0.10	0.11	0.11	0.17	0.06	0.06	0.06	0.05	0.05	0.10	0.06	0.05	0.08	1.00

<sup>a</sup> Note: For numeration see Table 1.

the maximal likelihood method (MLM) is more effective (for real data) in the sense of accuracy (Anderson-Darling goodness-of-fit test corroborate that). It should be noted that MLM is more sensitive to changes of the parameters  $\alpha$  and  $\sigma$ .

Empirical parameters of 14 risk factors series have been estimated. Most of the series are very asymmetric ( $0.5 < |\gamma_1| < 6$ ), and the empirical kurtosis ( $\gamma_2 \neq 0$ ) suggests that the probability density function of the series is more peaked and exhibits fatter tails than the Gaussian one. The normality hypothesis is rejected by the Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests.

An experimental test of the series homogeneity shows that for the stable series with asymmetry, the Anderson test is more powerful than the Smirnov one. The Anderson test for the 14 series shows that 2 series are homogeneous with their aggregated series and 8 series when the zeros are removed are homogeneous with their aggregate series.

The analysis of self-similarity and multifractality, by the absolute moments method, indicates that 10 series are multifractal and concurrently 2 of them are

TABLE 11. Power-correlation coefficient with median standardization

	1 <sup>a</sup>	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1.00	0.72	0.71	0.12	0.12	0.16	0.01	0.09	0.09	0.13	0.09	0.12	0.11	0.06
2	0.72	1.00	0.62	0.16	0.13	0.22	0.01	0.10	0.10	0.17	0.12	0.14	0.15	0.07
3	0.71	0.62	1.00	0.13	0.12	0.17	0.01	0.07	0.07	0.16	0.11	0.14	0.15	0.07
4	0.12	0.16	0.13	1.00	0.14	0.21	0.01	0.06	0.06	0.22	0.24	0.20	0.25	0.16
5	0.12	0.13	0.12	0.14	1.00	0.18	0.01	0.08	0.08	0.13	0.09	0.12	0.12	0.08
6	0.16	0.22	0.17	0.21	0.18	1.00	0.00	0.11	0.11	0.20	0.09	0.11	0.14	0.09
7	0.01	0.01	0.01	0.01	0.01	0.00	1.00	0.01	0.01	0.00	0.00	0.00	0.00	0.01
8	0.09	0.10	0.07	0.06	0.08	0.11	0.01	1.00	0.99	0.03	0.02	0.03	0.02	0.04
9	0.09	0.10	0.07	0.06	0.08	0.11	0.01	0.99	1.00	0.03	0.02	0.03	0.03	0.04
10	0.13	0.17	0.16	0.22	0.13	0.20	0.00	0.03	0.03	1.00	0.16	0.13	0.82	0.13
11	0.09	0.12	0.11	0.24	0.09	0.09	0.00	0.02	0.02	0.16	1.00	0.52	0.17	0.15
12	0.12	0.14	0.14	0.20	0.12	0.11	0.00	0.03	0.03	0.13	0.52	1.00	0.14	0.11
13	0.11	0.15	0.15	0.25	0.12	0.14	0.00	0.02	0.03	0.82	0.17	0.14	1.00	0.14
14	0.06	0.07	0.07	0.16	0.08	0.09	0.01	0.04	0.04	0.13	0.15	0.11	0.14	1.00

<sup>a</sup> Note: For numeration see Table 1.

self-similar. On the other hand, if we remove the zeros from the series, there remain only 5 multifractal and 2 self-similar series. This is because the series becomes too short for multifractality analysis.

A mixed stable model of log-volatility changes and log-FX rates distribution has been proposed. We introduced the probability density, cumulative density, and the characteristic functions. Empirical results show that this kind of distribution fits the empirical data better than any other.

Since goodness-of-tests tests for continuous distribution functions cannot be implemented, the tests, based on the empirical characteristic function (Koutrouvelis) as well as modified  $\chi^2$ , are used. The experimental tests have shown that, if the stability parameter  $\alpha$  and the number of zeros are increasing, then the validity of the tests is also increasing.

The statistical analysis of the stagnation intervals has been made. Empirical studies showed that the length series of the state runs of our financial data are better described by the Hurwitz zeta distribution, rather than by geometric distribution. Since series of the lengths of each run are not geometrically distributed, the state series must have some internal dependence (Wald-Wolfowitz runs test corroborates this assumption). A new mixed-stable model with dependent states has been proposed and the formulas for probabilities of calculating states (zeros and units) have been obtained. Adequacy tests of this model are hampered by inner series dependence.

The inner series dependence was tested by the Hoel [16] criterion on the order of the Markov chain. It has been concluded that there are no zero order series or Bernoulli scheme series.

When constructing an optimal portfolio, it is essential to determine possible relationships between different data series. However, under the assumption of stability traditional relationship measures (covariance, correlation) which cannot be applied, since  $(1.26 < \alpha < 1.78)$ . In such a case, codifference and power-correlation measures are offered. The significance of these measures can be tested by the bootstrap method.

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## STABLE MIXTURE MODEL

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# Unique Continuation on the Gradient for Second Order Elliptic Equations with Lower Order Terms

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## Abstract

We prove unique continuation results for the gradient of solutions to second order elliptic equations with a strict sign condition on the potential. Some information on the Lebesgue measure of the nodal set of the gradient of solutions is also obtained. Such results are useful in nonlinear partial differential equations. A counterexample of Hartman and Wintner suggests that our results are sharp in some sense. Our method of proof is based in part on  $L^p$ -estimates, a “reverse” Caccioppoli’s type inequality derived herein, a doubling condition established by Garofalo and Lin, and the de Giorgi-Nash-Moser type iteration procedure.

*Keywords:* Elliptic equations; Gradient unique continuation; Sign condition;  $L^p$ -estimates; Doubling condition; Iteration procedure.

## 1 Introduction

Let  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 3$ , be a connected open set. We consider the second order elliptic partial differential equation

$$Hu := -\operatorname{div}(A(x)\nabla u) + b(x) \cdot \nabla u + V(x)u = 0 \quad \text{in } \Omega, \quad (1)$$

where  $|b| \in L^N_{\text{loc}}(\Omega)$ ,  $V \in L^{N/2}_{\text{loc}}(\Omega)$ , and  $A(x) = (a_{ij}(x))$  is a real symmetric matrix with (uniformly) Lipschitz continuous entries such that there exists a constant  $\mu \in (1, \infty)$  with

$$\mu^{-1}|\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \mu|\xi|^2 \quad (2)$$

for all  $x \in \Omega$  and all  $\xi \in \mathbb{R}^N$ .

We are interested in studying different types of unique continuation property on the gradient  $|\nabla u|$  of nontrivial solutions to Eq. (1). We first give some relevant definitions.

By a solution to Eq. (1) we mean a function  $u \in H^1_{\text{loc}}(\Omega)$  such that

$$\int_{\Omega} \langle A(x)\nabla u, \nabla \varphi \rangle + \int_{\Omega} \langle b(x), \nabla u \rangle \varphi + \int_{\Omega} V(x)u\varphi = 0 \quad \text{for all } \varphi \in C^1_0(\Omega). \quad (3)$$

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Note that by the interior regularity properties of (weak) solutions to Eq. (1) (see e.g. [6], Theorem 8.8, or more precisely [12], Chapter 3, Section 10), it follows that  $u \in H_{\text{loc}}^2(\Omega)$ . Actually, it is also known that  $u \in L_{\text{loc}}^\infty(\Omega)$  (see e.g. [12], Chapter 3, Section 14, Theorem 14.1 and Remark 1).

**Definition 1** A function  $v \in L_{\text{loc}}^2(\Omega)$  *vanishes of infinite order at a point*  $x_0 \in \Omega$  if for each  $k \in \mathbb{N}$ ,

$$\int_{|x-x_0|<R} v^2 dx = O(R^k) \quad \text{as } R \rightarrow 0. \quad (4)$$

**Definition 2** The operator  $H$  has the *strong unique continuation property on the gradient* in  $\Omega$  if the only solution to  $Hu = 0$  such that  $|\nabla u|$  vanishes of infinite order at a point  $x_0 \in \Omega$  is  $u \equiv 0$ .

**Definition 3** The operator  $H$  has the *unique continuation property on the gradient* in  $\Omega$  if the only solution to  $Hu = 0$  such that  $|\nabla u|$  vanishes in a subset of  $\Omega$  of positive (Lebesgue) measure is  $u \equiv 0$ .

**Definition 4** The operator  $H$  has the *weak unique continuation property on the gradient* in  $\Omega$  if the only solution to  $Hu = 0$  such that  $|\nabla u|$  vanishes in an open subset of  $\Omega$  is  $u \equiv 0$ .

There is an extensive literature on the unique continuation property of nontrivial solutions,  $u$ , to Eq. (1). Significantly improving upon previous results obtained by Amrein, Berthier and Georgescu [1], Hörmander [9], Protter [16], Schechter and Simon [17] and others, Jerison and Kenig [11] proved sharp unique continuation results in  $L^p$ -spaces ( $V \in L_{\text{loc}}^{N/2}(\Omega)$ ) for nontrivial solutions,  $u$ , to the Schrödinger operator  $-\Delta u + V(x)u$ . (Also see Jerison [10] for a simpler proof.) Their result was extended by Stein [21] to the case of the Lorentz space,  $V \in L_{\text{loc}}^{N/2,\infty}(\Omega)$ , with a “small norm” condition. Their method of proof was linked to Carleman’s type inequalities and the harmonic analysis of the Laplace operator. These results were shown to be sharp in that context by counterexamples constructed by Wolff [22]. The variable coefficients case was taken up in Sogge [20] where it is assumed that  $a_{ij} \in C^\infty(\Omega)$  (also see Hartman and Wintner [8]), and Garofalo and Lin [4, 5] and Hörmander [9] where it is assumed that the entries  $a_{ij}(x)$  are (uniformly) Lipschitz continuous, with additional growth conditions on the potential  $V$  and the drift coefficient  $b$  in Eq. (1). In an other context, the  $(N-1)$ -dimensional Hausdorff measure of the nodal set of solutions was considered by Hardt and Simon in [7]. More references may be found in these papers.

However, to our knowledge, very little has been done concerning unique continuation property on the *gradient*,  $|\nabla u|$ , of nontrivial solutions to Eq. (1). In [4], Garofalo and Lin proved that, if  $b \equiv 0$  and  $V \equiv 0$ , then the gradient,  $|\nabla u|$ , of a solution cannot vanish of infinite order at a point, unless  $u \equiv \text{constant}$ . Since the gradient of the solution was shown in [5] to be an  $A_q$  *weight* of Muckenhoupt, it cannot vanish on a set of positive (Lebesgue) measure either, provided  $u \not\equiv \text{constant}$ .

Their approach cannot be extended to problems with lower order terms, i.e. with  $V \not\equiv 0$  for instance. Actually, they mentioned (without references) in [5], p. 352, that these statements may not be true for  $V \not\equiv 0$ , even when  $V \in C^\infty(\Omega)$  and  $N = 1$ . It appears to us that this

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fact may have been suggested by a 1955 paper of Hartman and Wintner [8] (see e.g. Sections 1 and 2 in [8]).

For problems with lower order terms, an obvious way to prevent constant functions from being (local) eigenfunctions of Eq. (1) is to assume that  $V(x) \neq 0$  a.e. in  $\Omega$ . However, as suggested by a counterexample in [8], p. 452, this condition may not be sufficient for *strong* unique continuation property on the gradient. The counterexample in [8] is of the form  $V(x) = u''(x)/u(x)$ , where  $u(0) = 1$  and  $u(x) = 1 \pm \exp(-x^{-2})$  for  $x \neq 0$ . An analysis of this counterexample shows that the potential  $V$  does not change sign in a neighborhood of the point  $x = 0$ , where the gradient (of the solution) vanishes of infinite order. However,  $V$  does not satisfy a *strict* sign condition in that neighborhood either. Slightly modifying this example by considering  $V(x) = u''(x)/u(x)$ , where now  $u(x) = 1 + x \exp(-x^{-2})$  for  $x \neq 0$ , it is easy to see that  $V$  changes sign in every neighborhood of  $x = 0$ . In either case, the non-constant function  $u$  satisfies the differential equation  $-u'' + V(x)u = 0$  and  $u'(x)$  vanishes of infinite order at  $x = 0$ . Thus, the strong unique continuation property on the gradient does not hold. This discussion clearly shows that the problem of unique continuation of the gradient,  $|\nabla u|$ , has different features than that of unique continuation of the solution,  $u$ , itself.

Therefore, unless otherwise mentioned, we shall assume throughout that the potential  $V$  satisfies the following strict sign condition:

$$V(x) \leq -\varepsilon \quad \text{a.e. in } \Omega, \quad (5)$$

for some real number  $\varepsilon > 0$ . (The case  $V(x) \geq \varepsilon$  a.e. in  $\Omega$  will be treated in a similar way.)

It is the purpose of this paper to show that this (natural) strict sign condition implies the (strong) unique continuation property on the gradient of solutions to Eq. (1) (see Section 2). Furthermore, we obtain some information on the (Lebesgue) measure of the nodal set (the set of zeros) of the gradient of nontrivial solutions. More precisely, we prove that this set has (Lebesgue) measure zero (see Section 3). Such results are also useful in nonlinear boundary value problems (see e.g. Nkashama and Robinson [14, 15] and references therein). The above counterexample of Hartman and Wintner seems to indicate that our results are sharp in the sense that a *strict* sign condition on the potential  $V$  is needed for strong unique continuation property on the gradient.

Let us mention that, in a different setting, namely in scattering theory, a (strict) sign condition (in some sense) plays an important role in proving the absence of positive eigenvalues (that is, eigenvalues embedded in the continuous spectrum) for Schrödinger operators. The eigenfunctions (and their first order derivatives) associated with such eigenvalues are rapidly decreasing at infinity in the  $L^2$ -sense, provided the potential has a certain decaying behavior at infinity. We refer to Hörmander [9], Section 5, and Simon [18, 19].

Our method of proof is based in part on  $L^p$ -estimates, a “reverse” Caccioppoli’s type inequality derived in Section 2, a doubling condition established by Garofalo and Lin [4, 5], and the de Giorgi-Nash-Moser type iteration procedure which we use in Section 3.

Furthermore, unless otherwise indicated, we shall assume throughout that the drift coefficient  $b$  and the potential  $V$  satisfy the following additional growth conditions in the neighborhood of a possible point of singularity (see e.g. Garofalo and Lin [5] and Hörmander [9]).

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For every  $x_0 \in \Omega$ , there exist a constant  $r_0 > 0$ , an increasing function  $h : (0, r_0) \rightarrow \mathbb{R}^+$  with

$$\int_0^{r_0} \frac{h(r)}{r} dr < \infty,$$

and a constant  $C_0 > 0$  such that

$$|b(x)| \leq C_0 \frac{h(|x - x_0|)}{|x - x_0|} \quad \text{and} \quad |V(x)| \leq C_0 \frac{h(|x - x_0|)}{|x - x_0|^2} \quad (6)$$

for all  $x \in B_{r_0}(x_0) \cap \Omega$ , where  $B_{r_0}(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < r_0\}$ . (Note that this condition is in particular satisfied if  $|b| \in L_{\text{loc}}^\infty(\Omega)$  and  $V \in L_{\text{loc}}^\infty(\Omega)$ .)

As mentioned above, it is known that, under the above conditions, with the exception of the sign condition (5), the operator  $H$  enjoys all of the above three types of unique continuation property on the solution  $u$  itself. We refer to de Figueiredo and Gossez [3], Garofalo and Lin [4], [5], Hörmander [9], Jerison [10], Jerison and Kenig [11], Sogge [20] and references therein.

## 2 Strong Unique Continuation

We wish to prove the following main result of this section.

**Theorem 1** *The operator  $H$  has the strong unique continuation property on the gradient in  $\Omega$ ; that is,  $|\nabla u|$  cannot vanish of infinite order at a point, unless  $u \equiv 0$ .*

The following result is an immediate consequence of Theorem 1.

**Corollary 1** *The operator  $H$  has the weak unique continuation property on the gradient in  $\Omega$ .*

A more general result than Corollary 1 actually holds. Indeed, it is obvious that the operator  $H$  has the weak unique continuation property on the gradient in  $\Omega$  if  $V(x) \neq 0$  a.e. in  $\Omega$ .

The following result will be useful in the proof of Theorem 1. (Note that this result does not require the growth conditions (6) on  $b$  and  $V$ .)

**Lemma 1** *Let  $u \in H_{\text{loc}}^1(\Omega)$  be a solution to Eq. (1), and let  $B_{r_0} \subset \Omega$  be an open ball. Then there exist a positive constant  $C_2$ , depending on  $A$  and  $V$ , and a positive constant  $C_3$ , depending on  $u$ ,  $b$ ,  $V$  and  $B_{r_0}$ , such that for every concentric balls  $B_r$  and  $B_{2r}$  contained in  $B_{r_0}$ , we have*

$$\int_{B_r} u^2 \leq C_2 \left[ \frac{1}{r^2} \int_{B_{2r}} |\nabla u|^2 + \int_{B_{2r}} |\nabla u|^2 \right] + C_3 \left( \int_{B_{2r}} |\nabla u|^2 \right)^{1/2}. \quad (7)$$

The constants  $C_2$  and  $C_3$  do not depend on  $r$ .

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*Proof.* Pick  $\varphi \in C_0^\infty(\Omega)$  with  $\varphi \in C_0^\infty(B_{2r})$  such that  $\varphi \equiv 1$  in  $B_r$ , and  $|\nabla\varphi| \leq 2/r$ . Using the function  $u\varphi^2$  as a test function in the definition (3) by density, we get

$$-\int_{\Omega} V(x)u^2\varphi^2 = \int_{\Omega} \langle A(x)\nabla u, \nabla u \rangle \varphi^2 + 2 \int_{\Omega} \langle A(x)\nabla u, \nabla\varphi \rangle u\varphi + \int_{\Omega} \langle b(x), \nabla u \rangle u\varphi^2;$$

that is,

$$-\int_{B_{2r}} V(x)u^2\varphi^2 = \int_{B_{2r}} \langle A(x)\nabla u, \nabla u \rangle \varphi^2 + 2 \int_{B_{2r}} \langle A(x)\nabla u, \nabla\varphi \rangle u\varphi + \int_{B_{2r}} \langle b(x), \nabla u \rangle u\varphi^2.$$

Using Cauchy-Schwarz and Young inequalities, it follows that for every  $\delta > 0$  there is a constant  $C_\delta > 0$  such that

$$\begin{aligned} -\int_{B_{2r}} V(x)u^2\varphi^2 &\leq \int_{B_{2r}} \mu|\nabla u|^2\varphi^2 + 2\delta \int_{B_{2r}} u^2\varphi^2 + C_\delta \int_{B_{2r}} \mu^2|\nabla\varphi|^2|\nabla u|^2 \\ &\quad + |\varphi^2|_{L^\infty(\Omega)} \int_{B_{2r}} (|u||b(x)|)|\nabla u|; \end{aligned}$$

which, by the inequality (5) and Hölder's inequality (used twice), implies that

$$\begin{aligned} \varepsilon \int_{B_{2r}} u^2\varphi^2 &\leq \int_{B_{2r}} (\mu\varphi^2 + \mu^2 C_\delta |\nabla\varphi|^2) |\nabla u|^2 + 2\delta \int_{B_{2r}} u^2\varphi^2 \\ &\quad + |\varphi^2|_{L^\infty(\Omega)} |u|_{L^{2^*}(B_{r_0})} |b|_{L^N(B_{r_0})} \left( \int_{B_{2r}} |\nabla u|^2 \right)^{1/2}, \end{aligned}$$

where  $2^* = 2N/(N-2)$  is the Sobolev critical exponent, and  $\mu > 0$  is the ellipticity constant appearing in the inequality (2). Therefore, choosing  $\delta$  sufficiently small such that  $2\delta < \varepsilon$ , and taking into account the fact that  $\varphi \equiv 1$  in  $B_r$  and  $|\nabla\varphi| \leq 2/r$  in  $B_{2r}$ , we deduce that there exit a constant  $C_2 > 0$ , depending on  $A$  and  $V$ , and a constant  $C_3 > 0$ , depending on  $u$ ,  $b$ ,  $V$  and  $B_{r_0}$ , such that

$$\int_{B_r} u^2 \leq \frac{C_2}{r^2} \int_{B_{2r}} |\nabla u|^2 + C_2 \int_{B_{2r}} |\nabla u|^2 + C_3 \left( \int_{B_{2r}} |\nabla u|^2 \right)^{1/2}.$$

The proof is complete.

**Remark 1** An analysis of the proof of Lemma 1 shows that, if  $|b| \in L_{\text{loc}}^\infty(\Omega)$ , the third term in the right hand side of the inequality (7) may be omitted. Thus, the constant appearing in Lemma 1 would be independent of  $r$  and  $u$  as well. It suffices to use Young's inequality on  $\int_{B_{2r}} (u\varphi)(\varphi|b(x)||\nabla u|)$ . In this case the constant  $C_2 > 0$  also depends on  $|b|_{L^\infty(B_{r_0})}$ .

*Proof. (Theorem 1)* Let  $u \in H_{\text{loc}}^1(\Omega)$  be a solution to Eq. (1). If  $|\nabla u|$  vanishes of infinite order at some point  $x_0$ , it follows from the definition (4) and the inequality (7) that for  $k = 3$ , we have

$$\int_{|x-x_0|<R} u^2 dx = O(R^{3-2}) + O(R^3) + O(R^{3/2}) = O(R) \quad \text{as } R \rightarrow 0,$$

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since  $R \leq 1$ . Moreover, for each  $k \in N$ ,  $k$  even with  $k \geq 4$ , we have

$$\int_{|x-x_0|<R} u^2 dx = O(R^{k-2}) + O(R^k) + O(R^{k/2}) = O(R^{k/2}) \quad \text{as } R \rightarrow 0,$$

since  $R \leq 1$ , and  $k-2 \geq k/2$  for  $k \geq 4$ .

Thus, the solution  $u$  vanishes of infinite order at  $x_0$ . The conclusion of Theorem 1 then follows from the results on strong unique continuation property on the solution  $u$ . We refer to Garofalo and Lin [5] and Hörmander [9] (also see Jerison and Kenig [11], and Sogge [20]). The proof is complete.

Note that if  $b \equiv 0$ , then Theorem 1 holds true when  $a_{ij} \in C^\infty(\Omega)$  and  $V \in L_{\text{loc}}^{N/2}(\Omega)$ , without assuming the growth condition (6) on the potential  $V$ . This follows from Remark 1 and the results of Sogge [20]. (Also see Jerison [10], and Jerison and Kenig [11].)

In order to obtain some information on the Lebesgue measure of the set of zeros of the gradient of nontrivial solutions to Eq. (1), we have, in the spirit of Remark 1, the following result; which may be thought of as a “reverse” Caccioppoli’s type inequality (see Section 3). It is motivated by Poincaré and Wirtinger’s type inequalities.

**Lemma 2** *Let  $u \in H_{\text{loc}}^1(\Omega)$  be a solution to Eq. (1), and let  $B_{r_0} \subset \Omega$  be an open ball, where without loss of generality we assume that  $r_0 \leq 1$ . Then there exists a positive constant  $C_4$ , depending on  $u$ ,  $A$ ,  $b$ ,  $V$  and  $B_{r_0}$ , such that for every concentric balls  $B_r$  and  $B_{2r}$  contained in  $B_{r_0}$ , we have*

$$\int_{B_r} u^2 \leq \frac{C_4}{r^2} \int_{B_{2r}} |\nabla u|^2. \quad (8)$$

The constant  $C_4$  does not depend on  $r$ .

The proof of Lemma 2 will depend in part on the following *doubling condition* which was established in [5].

**Lemma 3** [5] *Let  $u \in H_{\text{loc}}^1(\Omega)$  be a solution to Eq. (1), and let  $B_{r_0} \subset \Omega$  be an open ball, where without loss of generality  $r_0$  is sufficiently small. Then there exists a positive constant  $C_1$ , depending on  $u$ ,  $N$ ,  $A$ ,  $b$ ,  $V$  and  $B_{r_0}$ , such that for every concentric balls  $B_R$  and  $B_{2R}$  contained in  $B_{r_0}$ , we have*

$$\int_{B_{2R}} u^2 \leq C_1 \int_{B_R} u^2. \quad (9)$$

The constant  $C_1$  does not depend on  $R$ .

*Proof.* (Lemma 2) An analysis of the proof of Lemma 1 shows that

$$\begin{aligned} \varepsilon \int_{B_{2r}} u^2 \varphi^2 &\leq \int_{B_{2r}} (\mu \varphi^2 + \mu^2 C_\delta |\nabla \varphi|^2) |\nabla u|^2 + 2\delta \int_{B_{2r}} u^2 \varphi^2 \\ &\quad + |\varphi^2|_{L^\infty(\Omega)} |u|_{L^{2^*}(B_{2r})} |b|_{L^N(B_{r_0})} \left( \int_{B_{2r}} |\nabla u|^2 \right)^{1/2}. \end{aligned}$$

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By the Sobolev Imbedding Theorem, there is a constant  $C' > 0$ , which may be chosen independent of  $r$ , such that

$$\begin{aligned} \varepsilon \int_{B_{2r}} u^2 \varphi^2 &\leq \int_{B_{2r}} (\mu \varphi^2 + \mu^2 C_\delta |\nabla \varphi|^2) |\nabla u|^2 + 2\delta \int_{B_{2r}} u^2 \varphi^2 + C' |\varphi^2|_{L^\infty(\Omega)} \int_{B_{2r}} |\nabla u|^2 \\ &\quad + C' |\varphi^2|_{L^\infty(\Omega)} \left( \frac{1}{r^2} \int_{B_{2r}} u^2 \right)^{1/2} \left( \int_{B_{2r}} |\nabla u|^2 \right)^{1/2}. \end{aligned}$$

Using Young's inequality once more and the doubling condition (9), we get

$$\begin{aligned} \varepsilon \int_{B_{2r}} u^2 \varphi^2 &\leq \int_{B_{2r}} (\mu \varphi^2 + \mu^2 C_\delta |\nabla \varphi|^2) |\nabla u|^2 + 2\delta \int_{B_{2r}} u^2 \varphi^2 + C' |\varphi^2|_{L^\infty(\Omega)} \int_{B_{2r}} |\nabla u|^2 \\ &\quad + C_\gamma C' |\varphi^2|_{L^\infty(\Omega)} \frac{1}{r^2} \int_{B_{2r}} |\nabla u|^2 + \gamma C C' |\varphi^2|_{L^\infty(\Omega)} \int_{B_r} u^2. \end{aligned}$$

Therefore, by choosing  $\delta > 0$  and  $\gamma > 0$  sufficiently small such that

$$2\delta + \gamma C C' |\varphi^2|_{L^\infty(\Omega)} < \varepsilon/2,$$

the inequality (8) follows. The proof is complete.

**Remark 2** It is easy to see that all the above results hold true if the potential  $V \in L_{\text{loc}}^{N/2}(\Omega)$  satisfies the growth condition (6) with  $V$  positive; that is,  $V(x) \geq \varepsilon$  a.e. in  $\Omega$ , for some real number  $\varepsilon > 0$ .  $\square$

### 3 Nodal Set of the Gradient

Next, we will be interested in getting some information on the Lebesgue measure of the nodal set; that is, the zero set of the gradient  $|\nabla u|$  of solutions to Eq. (1). To do so, we assume that  $V \in L_{\text{loc}}^\infty(\Omega)$  and  $|b| \in L_{\text{loc}}^N(\Omega)$ . We have the following result.

**Proposition 1** *Let  $u \in H_{\text{loc}}^1(\Omega)$  be a solution to Eq. (1). If  $|\nabla u| = 0$  on a set  $S \subset \Omega$  of positive (Lebesgue) measure, then both  $|\nabla u|$  and  $|u|$  vanish of infinite order at a.e. point of  $S$ .*

As an immediate consequence of Proposition 1 and Theorem 1 we deduce the following main result of this section.

**Theorem 2** *The operator  $H$  has the unique continuation property on the gradient in  $\Omega$ ; that is,  $|\nabla u|$  cannot vanish on a set of positive (Lebesgue) measure, unless  $u \equiv 0$ .*

To prove Proposition 1, we wish to first derive interior estimates of the  $L^2$ -norm of the second derivatives of an arbitrary function  $u \in H_{\text{loc}}^2(\Omega)$  in terms of the  $L^2$ -norms of the functions  $u$ ,  $|\nabla u|$  and the values of the elliptic operator  $Hu$ . For that purpose, let us write the elliptic operator  $Hu$  in the following equivalent form (for  $u \in H_{\text{loc}}^2(\Omega)$ ):

$$Hu := -a_{ij}(x)D_{ij}u + a_i(x)D_iu + V(x)u,$$

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where  $a_i(x) = -D_j(a_{ij}(x)) + b_i(x)$ ,  $D_i u = \partial u / \partial x_i$ ,  $D_{ij} u = \partial^2 u / \partial x_i \partial x_j$ , etc..., and the summation convention is understood. We have the following interior estimate. (Note that in obtaining this interior estimate we may assume that  $V \in L_{\text{loc}}^{N/2}(\Omega)$ ,  $|b| \in L_{\text{loc}}^N(\Omega)$ , and the growth conditions (6) are not required.)

**Lemma 4** *There exists a constant  $C > 0$  such that for all  $u \in H_{\text{loc}}^2(\Omega)$ ,*

$$\begin{aligned} \int_{\Omega} \sum_{i,j=1}^N |D_{ij} u|^2 \varphi^2 &\leq C \left( \int_{\Omega} (Hu)^2 \varphi^2 + \int_{\Omega} |\nabla \varphi|^2 |\nabla u|^2 + \int_{\Omega} |\nabla u|^2 \varphi^2 \right) \\ &\quad + \int_{\Omega} (V(x)u)^2 \varphi^2, \end{aligned} \quad (10)$$

where  $\varphi \in C_0^\infty(\Omega)$  is any given function. The constant

$$C = C(\mu^{-1}, N, |a_{ij}|_{C^{0,1}(\Omega)}, |b|_{L_{\text{loc}}^N(\Omega)})$$

is independent of  $u$  and the values of  $\varphi$  on its support. (In  $|b|_{L_{\text{loc}}^N(\Omega)}$ , the word “loc” refers to a subset of  $\Omega$  containing the support of  $\varphi$  in  $\Omega$ .)

*Proof.* It suffices to prove (10) for all  $u \in C^\infty(\Omega) \cap H_{\text{loc}}^2(\Omega)$ , since the inequality (10) for  $u \in H_{\text{loc}}^2(\Omega)$  follows from a density argument (see e.g. Theorem 7.9 in [6], p. 154).

Using the above definition of the operator  $Hu$ , we obtain

$$\begin{aligned} \int_{\Omega} (Hu)^2 \varphi^2 &= \int_{\Omega} a_{ij}(D_{ij} u) a_{kl}(D_{kl} u) \varphi^2 + 2 \int_{\Omega} (a_{ij} D_{ij} u)(a_k D_k u + Vu) \varphi^2 \\ &\quad + \int_{\Omega} (a_k D_k u + Vu)^2 \varphi^2. \end{aligned}$$

By twice integrating by parts the first term on the right, we get

$$\begin{aligned} \int_{\Omega} a_{ij}(D_{ij} u) a_{kl}(D_{kl} u) \varphi^2 &= - \int_{\Omega} [a_{ij} a_{kl} D_{kl} u (D_i u) \varphi^2 + D_j (a_{ij} a_{kl} \varphi^2) D_{kl} u D_i u] \\ &= \int_{\Omega} [a_{ij} a_{kl} (D_{ik} u) (D_{jl} u) \varphi^2 - D_j (a_{ij} a_{kl} \varphi^2) D_{kl} u D_i u \\ &\quad + D_k (a_{ij} a_{kl} \varphi^2) D_{lj} u D_i u]. \end{aligned} \quad (11)$$

Using the inequality (7.6) in [12], Chapter 3, Section 7, the first term in the right hand side of (11) may be estimated by

$$\int_{\Omega} a_{ij} a_{kl} D_{ik} u D_{jl} u \varphi^2 \geq \mu^{-2} \int_{\Omega} \sum_{i,j=1}^N |D_{ij} u|^2 \varphi^2.$$

Therefore,

$$\begin{aligned} \frac{1}{\mu^2} \int_{\Omega} \sum_{i,j=1}^N |D_{ij} u|^2 \varphi^2 &\leq \int_{\Omega} (Hu)^2 \varphi^2 \\ &\quad + \int_{\Omega} [D_j (a_{ij} a_{kl} \varphi^2) D_{kl} u D_i u - D_k (a_{ij} a_{kl} \varphi^2) D_{lj} u D_i u] \\ &\quad - 2 \int_{\Omega} (a_{ij} D_{ij} u)(a_k D_k u + Vu) \varphi^2 - \int_{\Omega} (a_k D_k u + Vu)^2 \varphi^2. \end{aligned} \quad (12)$$



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Let us consider the terms containing the derivatives of  $\varphi$ ; that is, part of the second term in the right hand side of the inequality (12). By Young's inequality, we have that, for every  $\delta > 0$ ,

$$\int_{\Omega} a_{ij}a_{kl} (D_{kl}u D_j \varphi - D_{lj}u D_k \varphi) 2\varphi D_i u \leq \delta \int_{\Omega} \sum_{i,j=1}^N |D_{ij}u|^2 \varphi^2 + \delta^{-1} C' \int_{\Omega} |\nabla \varphi|^2 |\nabla u|^2,$$

where the constant  $C' = C'(N, |a_{ij}|_{L^\infty(\Omega)}) > 0$ . Similarly, the remaining terms of the second term in the right hand side of the inequality (12) may be estimated by

$$\int_{\Omega} [D_j(a_{ij}a_{kl})\varphi D_{kl}u - D_k(a_{ij}a_{kl})\varphi D_{lj}u] \varphi D_i u \leq \delta \int_{\Omega} \sum_{i,j=1}^N |D_{ij}u|^2 \varphi^2 + \delta^{-1} C'' \int_{\Omega} |\nabla u|^2 \varphi^2,$$

where the constant  $C'' = C''(N, |a_{ij}|_{C^{0,1}(\Omega)}) > 0$ .

Now, let us consider the term  $\int_{\Omega} a_k^2 |D_k u|^2 \varphi^2$  that appears in the third and last terms in the right hand side of the inequality (12), and let us estimate the term  $\int_{\Omega} |b_k|^2 |D_k u|^2 \varphi^2$  included in it.

Since  $|b| \in L_{\text{loc}}^N(\Omega)$  and  $\varphi$  has compact support in  $\Omega$ , it follows from a result of Brézis and Kato [2], p. 139, Lemma 2.1, that for every  $\delta > 0$ ,

$$\int_{\Omega} |b_k|^2 |\varphi D_k u|^2 \leq \delta \int_{\Omega} |\nabla(\varphi D_k u)|^2 + C_\delta \int_{\Omega} |\varphi D_k u|^2,$$

where the constant  $C_\delta = C_\delta(N, |b_k|_{L_{\text{loc}}^N(\Omega)}) > 0$ . (In  $|b_k|_{L_{\text{loc}}^N(\Omega)}$ , the word “loc” refers to a subset of  $\Omega$  containing the support of  $\varphi$  in  $\Omega$ .) Thus,

$$\int_{\Omega} |b_k|^2 |\varphi D_k u|^2 \leq \delta \int_{\Omega} \sum_{i,j=1}^N |D_{ij}u|^2 \varphi^2 + \delta \int_{\Omega} |\nabla \varphi|^2 |D_k u|^2 + C_\delta \int_{\Omega} |D_k u|^2 \varphi^2.$$

Finally, collecting all the above estimates (and similar ones), where  $\delta > 0$  is chosen sufficiently small, and using the inequality (12), we have that the inequality (10) follows. The proof is complete.

*Proof. (Proposition 1)* For a.e. point  $x_0 \in S$ , let us set

$$\omega_0(R) := \int_{B_R} |\nabla u|^2,$$

where  $B_R = \{x \in \Omega : |x - x_0| < R\}$ . We will show that for every  $k \in \mathbb{N}$ ,

$$\omega_0(R) = O(R^k) \quad \text{as } R \rightarrow 0; \tag{13}$$

which implies that  $|\nabla u|$  vanishes of infinite order at  $x_0$ .

Since  $S$  is a set of positive (Lebesgue) measure, it is known that a.e. point  $x_0 \in S$  is a cluster point in measure (also referred to as a point of density of  $S$ ); that is,

$$\lim_{R \rightarrow 0} \lambda(S \cap B_R) / \lambda(B_R) = 1, \quad \text{and henceforth,} \quad \lim_{R \rightarrow 0} \lambda([\Omega \setminus S] \cap B_R) / \lambda(B_R) = 0,$$

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where as above  $B_R = \{x \in \Omega : |x - x_0| < R\}$ , and  $\lambda(B_R)$  denotes the (Lebesgue) measure of  $B_R$ . Note that the latter is equivalent to saying that for every  $\eta > 0$  there exists  $R_0 = R_0(\eta) > 0$  such that

$$\lambda([\Omega \setminus S] \cap B_R) \leq \eta \lambda(B_R) \quad \text{for all } R \leq R_0. \quad (14)$$

Without loss of generality, we may assume that  $R_0 \leq 1$ . Using Hölder's inequality, the inequality (14), and the Sobolev Imbedding Theorem, we deduce that

$$\begin{aligned} \omega_0(R) &\leq C' (\eta \lambda(B_R \cap [\Omega \setminus S]))^{2/N} |\nabla u|_{L^{2^*}(B_R \cap [\Omega \setminus S])}^2 \\ &\leq C'' \eta^{2/N} R^2 \left( \frac{1}{R^2} |\nabla u|_{L^2(B_R)}^2 + \int_{B_R} \sum_{i,j=1}^N |D_{ij} u|^2 \right) \end{aligned} \quad (15)$$

for all  $R \leq R_0$ , since  $|\nabla u| = 0$  on  $S$ . (The constant  $C'' > 0$  is also independent of  $R$ .)

Moreover, by (local)  $L^2$ -estimates for elliptic operators; i.e. Lemma 4 where  $\varphi$  is picked as a cut-off function; that is  $\varphi \in C_0^\infty(B_{2R})$ ,  $\varphi \equiv 1$  in  $B_R$  and  $|\nabla \varphi| \leq 2/R$ , we have

$$\int_{B_R} \sum_{i,j=1}^N |D_{ij} u|^2 \leq C \left( \frac{1}{R^2} |\nabla u|_{L^2(B_{2R})}^2 + |\nabla u|_{L^2(B_{2R})}^2 + |Vu|_{L^2(B_{2R})}^2 \right), \quad (16)$$

where  $C = C(N, \mu^{-1}, |a_{ij}|_{C^{0,1}(B_{R_0})}, |b|_{L^N(B_{R_0})}) > 0$  is a constant independent of  $R$ , as  $R \rightarrow 0$ .

Using the estimate (16) with  $V \in L_{\text{loc}}^\infty(\Omega)$ , and the fact that, by Lemma 2,

$$\int_{B_{2R}} u^2 \leq \frac{C'}{R^2} \int_{B_{4R}} |\nabla u|^2,$$

for some constant  $C' > 0$  independent of  $R$ , as  $R \rightarrow 0$ , we deduce from the inequality (15) that  $\omega_0(R) \leq K_1 \eta^{2/N} \omega_0(4R)$  for all  $R$  such that  $4R \leq R_0$ ; that is,

$$\omega_0(R/4) \leq K_1 \eta^{2/N} \omega_0(R) \quad \text{for all } R \leq R_0, \quad (17)$$

where  $K_1 > 0$  is a constant independent of  $R$ , as  $R \rightarrow 0$ .

Note that the inequality (17) implies that  $\omega_0$  satisfies the assumptions of Lemma 8.23 in [6], where  $\tau \in (0, 1/4]$  is any number,  $\gamma = K_1 \eta^{2/N}$  and  $\sigma \equiv 0$ . Therefore,  $\omega_0(\tau R) \leq \gamma \omega_0(R)$  for all  $R \leq R_0$ .

Now, we shall use the inequality (17) to prove the assertion (13). To do so, let  $k \in \mathbb{N}$  be given, and let us pick  $\eta > 0$  sufficiently small such that  $(1/4)^k \leq K_1 \eta^{2/N} < 1$ . (Note that, in this way,  $R_0$  depends on  $k$ .) Since  $\tau \leq 1/4$ , it follows that, for any  $\beta \in (0, 1)$ ,

$$\tau^{(1-\beta)^{-1}k} \leq (1/4)^{(1-\beta)^{-1}k} < (1/4)^k \leq K_1 \eta^{2/N} = \gamma,$$

which immediately implies that

$$k \leq (1 - \beta)(\log \gamma / \log \tau).$$

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Therefore, the conclusion of the (short) proof of Lemma 8.23 in [6] (see e.g. [6], p. 202), which uses the de Giorgi-Nash-Moser type iteration procedure, yields

$$\omega_0(R) \leq \frac{1}{\gamma} \omega_0(R_0) \left( \frac{R}{R_0} \right)^{(1-\beta)(\log \gamma / \log \tau)} \leq \frac{1}{\gamma} \omega_0(R_0) \left( \frac{R}{R_0} \right)^k \quad \text{for all } R \leq R_0.$$

Thus, the assertion (13) follows. Finally, the inequality

$$\int_{B_R} u^2 \leq \frac{C'}{R^2} \int_{B_{2R}} |\nabla u|^2,$$

and the assertion (13) imply that for every  $k \in \mathbb{N}$ ,  $k \geq 3$ ,

$$\int_{B_R} u^2 = O(R^{k-2}) \quad \text{as } R \rightarrow 0;$$

which shows that  $u$  also vanishes of infinite order at  $x_0$ . The proof is complete.

**Remark 3** It is easy to see that all the above results hold true if the potential  $V \in L_{\text{loc}}^\infty(\Omega)$  with  $V$  positive; that is,  $V(x) \geq \varepsilon$  a.e. in  $\Omega$ , for some real number  $\varepsilon > 0$ .  $\square$

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# COMPOSITION OPERATORS FROM THE HARDY SPACE TO ZYGMUND-TYPE SPACES ON THE UPPER HALF-PLANE AND THE UNIT DISC

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**ABSTRACT.** The paper characterizes the boundedness of the composition operator  $C_\varphi f(z) = f(\varphi(z))$  from the Hardy space  $H^p(X)$ ,  $p > 0$ , where  $X$  is the upper half-plane  $\Pi_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$  or the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  in the complex plane  $\mathbb{C}$ , to Zygmund-type spaces, where  $\varphi$  is an analytic self-map of  $X$ .

## 1. INTRODUCTION

Let  $\Pi_+$  be the upper half-plane, that is, the set  $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ ,  $\mathbb{D}$  the open unit disc in  $\mathbb{C}$ ,  $dm(z)$  the normalized Lebesgue area measure on  $\mathbb{D}$ , i.e.  $m(\mathbb{D}) = 1$ , and  $H(X)$  the space of all analytic functions on  $X$  which may be  $\Pi_+$  or  $\mathbb{D}$ .

The Hardy space  $H^p(\Pi_+)$ ,  $p > 0$ , consists of all  $f \in H(\Pi_+)$  such that

$$\|f\|_{H^p(\Pi_+)}^p = \sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^p dx < \infty.$$

With this norm  $H^p(\Pi_+)$  is a Banach space when  $p \geq 1$ , while for  $p \in (0, 1)$  it is a Fréchet space with the translation invariant metric  $d(f, g) = \|f - g\|_{H^p(\Pi_+)}^p$ ,  $f, g \in H^p(\Pi_+)$ .

The Hardy space  $H^p(\mathbb{D})$ ,  $p > 0$ , consists of all  $f \in H(\mathbb{D})$  such that

$$\|f\|_{H^p(\mathbb{D})}^p = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

Some information on Hardy's spaces can be found in [4] and [5].

Let  $\mu(z)$  be a positive continuous function on  $X$  (*weight*) and  $n \in \mathbb{N}_0$  be fixed. The  $n$ -th *weighted-type space* on  $X$ , denoted by  $\mathcal{W}_\mu^{(n)}(X)$  consists of all  $f \in H(X)$  such that

$$b_{\mathcal{W}_\mu^{(n)}(X)}(f) := \sup_{z \in X} \mu(z) |f^{(n)}(z)| < \infty.$$

For  $n = 0$  the space is called the weighted-type space and denoted by  $H_\mu^\infty(X)$ , for  $n = 1$  it is called the Bloch-type space and denoted by  $\mathcal{B}_\mu(X)$ , and for  $n = 2$  it is called the Zygmund-type space and denoted by  $\mathcal{Z}_\mu(X)$ .

For the special case when  $X = \Pi_+$ ,  $\mu(z) = \operatorname{Im} z$  and  $n = 0$  is obtained the growth space  $\mathcal{A}_\infty(\Pi_+) = \mathcal{A}_\infty$ , for  $n = 1$  the Bloch space  $\mathcal{B}_\infty(\Pi_+) = \mathcal{B}_\infty$ , and for  $n = 2$  the Zygmund space  $\mathcal{Z}(\Pi_+) = \mathcal{Z}$ , which was introduced in [28]. For Bloch-type spaces on the unit disk, polydisk or the unit ball, see, e.g., [2, 12, 22, 27, 31, 35, 37] and the references therein.

The quantity

$$b_{\mathcal{Z}_\mu(X)} := \sup_{z \in X} \mu(z) |f''(z)|$$

is a seminorm on the Zygmund-type space  $\mathcal{Z}_\mu(X)$  or a norm on  $\mathcal{Z}_\mu(X)/\mathbb{P}_1$ , where  $\mathbb{P}_1$  is the set of all linear polynomials. A natural norm on the Zygmund-type space can be introduced as follows

$$\|f\|_{\mathcal{Z}_\mu(X)} = |f(a)| + |f'(a)| + b_{\mathcal{Z}_\mu(X)}(f),$$

where  $a \in X$ . With this norm the Zygmund-type space becomes a Banach space.

Note that for  $X = \Pi_+$  and  $\mu(z) = y$  is obtained the space  $\mathcal{Z}(\Pi_+)$  on which the following norm can be introduced

$$\|f\|_{\mathcal{Z}(\Pi_+)} := |f(i)| + |f'(i)| + \sup_{z \in \Pi_+} \operatorname{Im} z |f''(z)|.$$

For  $X = \mathbb{D}$  and  $\mu(z) = 1 - |z|^2$  is obtained the classical Zygmund space on the unit disk, on which a norm is introduced as follows

$$\|f\|_{\mathcal{Z}(\mathbb{D})} := |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f''(z)|.$$

For some information on the Zygmund space on the unit disk and some operators on it, see, for example, [1], [7], [10], [11], [14]. For the Zygmund space on the unit ball and some operators on it, see, e.g., [15], [16], [30], [37] and [38].

The following results, regarding the cases  $n = 0, 1, 2$  were proved in [20] and [28].

**Theorem A.** *Assume  $p \geq 1$  and  $\varphi$  is a nonconstant holomorphic self-map of  $\Pi_+$ . Then the following statements true hold.*

(a) *The operator  $C_\varphi : H^p(\Pi_+) \rightarrow \mathcal{A}_\infty(\Pi_+)$  is bounded if and only if*

$$\sup_{z \in \Pi_+} \frac{\operatorname{Im} z}{(\operatorname{Im} \varphi(z))^{\frac{1}{p}}} < \infty.$$

(b) *The operator  $C_\varphi : H^p(\Pi_+) \rightarrow \mathcal{B}_\infty(\Pi_+)$  is bounded if and only if*

$$\sup_{z \in \Pi_+} \operatorname{Im} z \frac{|\varphi'(z)|}{(\operatorname{Im} \varphi(z))^{1+\frac{1}{p}}} < \infty.$$

(c) *The operator  $C_\varphi : H^p(\Pi_+) \rightarrow \mathcal{Z}(\Pi_+)$  is bounded if and only if*

$$\sup_{z \in \Pi_+} \operatorname{Im} z \frac{|\varphi'(z)|^2}{(\operatorname{Im} \varphi(z))^{2+\frac{1}{p}}} < \infty \quad \text{and} \quad \sup_{z \in \Pi_+} \operatorname{Im} z \frac{|\varphi''(z)|}{(\operatorname{Im} \varphi(z))^{1+\frac{1}{p}}} < \infty.$$

A natural problem is to extend Theorem A for the case  $p \in (0, 1)$  and with a general weight  $\mu$ , as well as to obtain the corresponding results in the setting of the unit disc. Here we give some answers to these questions by characterizing the boundedness of the operator  $C_\varphi : H^p(X) \rightarrow \mathcal{Z}(X)$ , where  $p > 0$  and  $X = \Pi_+$  or  $X = \mathbb{D}$ . Let  $X$  and  $Y$  be topological vector spaces whose topologies are given by translation-invariant metrics  $d_X$  and  $d_Y$ , respectively, and  $T : X \rightarrow Y$  be a linear operator. It is said that  $T$  is *metrically bounded* if there is a  $K > 0$  such that  $d_Y(Tf, 0) \leq K d_X(f, 0)$  for all  $f \in X$ . When  $X$  and  $Y$  are Banach spaces, the metrically boundedness coincides with the usual definition of bounded operators between Banach spaces. If we say that an operator is bounded it means that it is metrically bounded.

## COMPOSITION OPERATORS

Somewhat older results on composition and weighted composition operators can be found, e.g., in [3], while some recent results can be found, e.g., in [1, 2, 6, 8, 9, 13, 14, 17, 18, 19, 20, 21, 23, 24, 25, 26, 28, 29, 32, 33, 34, 36, 39] (see also the related references therein).

Throughout this paper, constants are denoted by  $C$ , they are positive and may differ from one occurrence to the other. The notation  $a \preceq b$  means that there is a positive constant  $C$  such that  $a \leq Cb$ . Moreover, if both  $a \preceq b$  and  $b \preceq a$  hold, then one says that  $a \asymp b$ .

## 2. AUXILIARY RESULTS

This section gives several auxiliary results which are used in the proofs of the main results of the paper.

**Lemma 1.** *Assume that  $p > 0$ ,  $n \in \mathbb{N}_0$  and  $w \in \Pi_+$ . Then the function*

$$f_{w,n}(z) = \frac{(\operatorname{Im} w)^{n+\frac{1}{p}}}{(z - \bar{w})^{n+\frac{2}{p}}}, \quad (1)$$

*belongs to  $H^p(\Pi_+)$ . Moreover  $\sup_{w \in \Pi_+} \|f_{w,n}\|_{H^p} \leq \pi^{1/p}$ .*

*Proof.* Let  $z = x + iy$  and  $w = u + iv$ . Then, we have

$$\begin{aligned} \|f_{w,n}\|_{H^p}^p &= \sup_{y>0} \int_{-\infty}^{\infty} |f_{w,n}(x + iy)|^p dx \\ &= (\operatorname{Im} w)^{np+1} \sup_{y>0} \int_{-\infty}^{\infty} \frac{dx}{|z - \bar{w}|^{np} |z - \bar{w}|^2} \\ &\leq v^{np+1} \sup_{y>0} \int_{-\infty}^{\infty} \frac{dx}{((y+v)^2)^{\frac{np}{2}} ((x-u)^2 + (y+v)^2)} \\ &\leq v^{np+1} \sup_{y>0} \frac{1}{(y+v)^{np+1}} \int_{-\infty}^{\infty} \frac{y+v}{(x-u)^2 + (y+v)^2} dx \\ &= \sup_{y>0} \frac{v^{np+1}}{(y+v)^{np+1}} \int_{-\infty}^{\infty} \frac{dt}{t^2 + 1} = \pi, \end{aligned}$$

where we have used the change of variables  $x = u + t(y+v)$ .  $\square$

**Lemma 2.** *Assume that  $p > 0$ ,  $n \in \mathbb{N}_0$  and  $w \in \mathbb{D}$ . Then the function*

$$g_{w,n}(z) = \frac{(1 - |w|^2)^{n+\frac{1}{p}}}{(1 - \bar{w}z)^{n+\frac{2}{p}}}, \quad (2)$$

*belongs to  $H^p(\mathbb{D})$ . Moreover  $\sup_{w \in \mathbb{D}} \|g_{w,n}\|_{H^p} < \infty$ .*

*Proof.* By [4, p. 65], we have

$$\|g_{w,n}\|_{H^p}^p = \sup_{0 < r < 1} \int_0^{2\pi} \frac{(1 - |w|^2)^{np+1}}{|1 - \bar{w}re^{i\theta}|^{np+2}} \frac{d\theta}{2\pi} \leq C \sup_{0 < r < 1} \frac{(1 - |w|^2)^{np+1}}{(1 - |w|r)^{np+1}} = C,$$

as claimed.  $\square$

The following lemma is certainly folklore. We will present a proof here for the completeness and for the benefit of the reader.

**Lemma 3.** Assume that  $p > 0$ ,  $n \in \mathbb{N}_0$  and  $w \in \Pi_+$ . Then there is a positive constant  $C$  independent of  $f$  such that

$$|f^{(n)}(z)| \leq C \frac{\|f\|_{H^p(\Pi_+)}}{y^{n+\frac{1}{p}}}. \quad (3)$$

*Proof.* Let  $z = x + iy \in \mathbb{C}$  be fixed. By the subharmonicity of the function  $|f|^p$ ,  $p > 0$ , we have that

$$\begin{aligned} |f(z)|^p &\leq \frac{4}{\pi y^2} \int_{D(z, y/2)} |f(w)|^p dm(w) \\ &\leq \frac{4}{\pi y^2} \int_{y/2}^{3y/2} \int_{x-y/2}^{x+y/2} |f(\zeta + i\eta)|^p d\zeta d\eta \\ &\leq \frac{4}{\pi y} \|f\|_{H^p(\Pi_+)}^p, \end{aligned} \quad (4)$$

where  $D(z, y/2) = \{z \in \mathbb{C} \mid |z - w| < y/2\}$ .

From (4) it follows that

$$\sup_{|u-z| < y/4} |f(u)| \leq \frac{C}{y^{1/p}} \|f\|_{H^p(\Pi_+)}. \quad (5)$$

On the other hand, by the Cauchy's inequality we have that

$$|f^{(n)}(z)| \leq \frac{C}{y^n} \sup_{|u-z| < y/4} |f(u)|, \quad (6)$$

for some  $C$  independent of  $f$ . From (5) and (6) inequality (3) follows.  $\square$

For the case of the unit disc, we have the following estimate (see, e.g., [5]).

**Lemma 4.** Assume that  $p > 0$ ,  $n \in \mathbb{N}_0$  and  $w \in \mathbb{D}$ . Then there is a positive constant  $C$  independent of  $f$  such that

$$|f^{(n)}(z)| \leq C \frac{\|f\|_{H^p(\mathbb{D})}}{(1 - |z|)^{n+\frac{1}{p}}}. \quad (7)$$

### 3. MAIN RESULTS

In this section we formulate and prove our main results.

**Theorem 1.** Assume  $p > 0$  and  $\varphi$  is a nonconstant holomorphic self-map of  $\mathbb{D}$ . Then  $C_\varphi : H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})$  is bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{\mu(z) |\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2+\frac{1}{p}}} < \infty \quad (8)$$

and

$$\sup_{z \in \mathbb{D}} \frac{\mu(z) |\varphi''(z)|}{(1 - |\varphi(z)|^2)^{1+\frac{1}{p}}} < \infty. \quad (9)$$

Moreover, if the operator  $C_\varphi : H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})/\mathbb{P}_1$  is bounded, then

$$\|C_\varphi\|_{H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})/\mathbb{P}_1} \asymp \sup_{z \in \mathbb{D}} \frac{\mu(z) |\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2+\frac{1}{p}}} + \sup_{z \in \mathbb{D}} \frac{\mu(z) |\varphi''(z)|}{(1 - |\varphi(z)|^2)^{1+\frac{1}{p}}}. \quad (10)$$



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*Proof.* First assume that the operator  $C_\varphi : H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})$  is bounded.

Using the test function  $f(z) \equiv z \in H^p(\mathbb{D})$  (note that  $\|z\|_{H^p(\mathbb{D})} = 1$ ), we obtain

$$\sup_{z \in \mathbb{D}} \mu(z) |\varphi''(z)| = \|C_\varphi(z)\|_{\mathcal{Z}_\mu(\mathbb{D})} \leq \|C_\varphi\|_{H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})} < \infty. \quad (11)$$

By using the test function  $f(z) \equiv z^2 \in H^p(\mathbb{D})$  (note that  $\|z^2\|_{H^p(\mathbb{D})} = 1$ ), we get

$$2 \sup_{z \in \mathbb{D}} \mu(z) |\varphi(z) \varphi''(z) + (\varphi'(z))^2| = \|C_\varphi(z^2)\|_{\mathcal{Z}_\mu(\mathbb{D})} \leq \|C_\varphi\|_{H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})} < \infty. \quad (12)$$

From (11), (12) and the fact that  $\|\varphi\|_\infty \leq 1$ , it follows that

$$\sup_{z \in \mathbb{D}} \mu(z) |\varphi'(z)|^2 \leq \frac{3}{2} \|C_\varphi\|_{H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})} < \infty. \quad (13)$$

For  $w \in \mathbb{D}$ , set

$$f_w(z) = \frac{(1 - |w|^2)^{2+\frac{1}{p}}}{(1 - \bar{w}z)^{2+\frac{2}{p}}}.$$

By Lemma 2 (case  $n = 2$ ) we know that  $f_w \in H^p(\mathbb{D})$  for every  $w \in \mathbb{D}$ . Moreover, we have that

$$\sup_{w \in \mathbb{D}} \|f_w\|_{H^p(\mathbb{D})} \leq C. \quad (14)$$

From (14) and since the operator  $C_\varphi : H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})$  is bounded, for every  $w \in \mathbb{D}$ , we obtain

$$\begin{aligned} \sup_{z \in \mathbb{D}} \mu(z) |f_w''(\varphi(z))(\varphi'(z))^2 + f_w'(\varphi(z))\varphi''(z)| &= \|C_\varphi(f_w)\|_{\mathcal{Z}_\mu(\mathbb{D})} \\ &\leq C \|C_\varphi\|_{H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})}. \end{aligned} \quad (15)$$

We also have that

$$f_w'(z) = C_{p,1} \frac{\bar{w}(1 - |w|^2)^{2+\frac{1}{p}}}{(1 - \bar{w}z)^{3+\frac{2}{p}}} \quad \text{and} \quad f_w''(z) = C_{p,2} \frac{\bar{w}^2(1 - |w|^2)^{2+\frac{1}{p}}}{(1 - \bar{w}z)^{4+\frac{2}{p}}}, \quad (16)$$

for some constants  $C_{p,1}$  and  $C_{p,2}$ .

Replacing (16) in (15) and taking  $w = \varphi(z)$  we obtain

$$\mu(z) \left| C_{p,2} \frac{(\varphi'(z))^2 \overline{(\varphi(z))}^2}{(1 - |\varphi(z)|^2)^{2+\frac{1}{p}}} + C_{p,1} \frac{\varphi''(z) \overline{\varphi(z)}}{(1 - |\varphi(z)|^2)^{1+\frac{1}{p}}} \right| \leq C \|C_\varphi\|_{H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})}$$

and consequently

$$C_{p,1} \frac{\mu(z) |\varphi''(z)| |\varphi(z)|}{(1 - |\varphi(z)|^2)^{1+\frac{1}{p}}} \leq C \|C_\varphi\|_{H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})} + C_{p,2} \frac{\mu(z) |\varphi'(z)|^2 |\varphi(z)|^2}{(1 - |\varphi(z)|^2)^{2+\frac{1}{p}}}. \quad (17)$$

Hence if we show that (8) holds then from the last inequality we get

$$\frac{1}{2} \sup_{|\varphi(z)| > 1/2} \frac{\mu(z) |\varphi''(z)|}{(1 - |\varphi(z)|^2)^{1+\frac{1}{p}}} \leq C \|C_\varphi\|_{H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})} + C_{p,2} \sup_{z \in \mathbb{D}} \frac{\mu(z) |\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2+\frac{1}{p}}}. \quad (18)$$

From (11) we have that

$$\sup_{|\varphi(z)| \leq 1/2} \frac{\mu(z) |\varphi''(z)|}{(1 - |\varphi(z)|^2)^{1+\frac{1}{p}}} \leq \left(\frac{4}{3}\right)^{1+\frac{1}{p}} \sup_{z \in \mathbb{D}} \mu(z) |\varphi''(z)| \leq C \|C_\varphi\|_{H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})}, \quad (19)$$

which along with (18) implies (9).

For  $w \in \mathbb{D}$ , set

$$g_w(z) = \left(3 + \frac{2}{p}\right) \frac{(1 - |w|^2)^{2+\frac{1}{p}}}{(1 - \bar{w}z)^{2+\frac{2}{p}}} - \left(2 + \frac{2}{p}\right) \frac{(1 - |w|^2)^{3+\frac{1}{p}}}{(1 - \bar{w}z)^{3+\frac{2}{p}}}.$$

Then it is easy to see that

$$g'_w(w) = 0 \quad \text{and} \quad g''_w(w) = \frac{C\bar{w}^2}{(1 - |w|^2)^{2+\frac{1}{p}}},$$

and by Lemma 2 (cases  $n = 2$  and  $n = 3$ ) it is easy to see that

$$\sup_{w \in \mathbb{D}} \|g_w\|_{H^p} < \infty.$$

From this, since  $C_\varphi : H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})$  is bounded and by taking  $w = \varphi(z)$ , it follows that

$$C \frac{\mu(z)|\varphi'(z)|^2|\varphi(z)|^2}{(1 - |\varphi(z)|^2)^{2+\frac{1}{p}}} \leq \|C_\varphi(g_w)\|_{\mathcal{Z}_\mu(\mathbb{D})} \leq C\|C_\varphi\|_{H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})}, \quad (20)$$

and consequently

$$\frac{1}{4} \sup_{|\varphi(z)| > 1/2} \frac{\mu(z)|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2+\frac{1}{p}}} \leq C\|C_\varphi\|_{H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})}. \quad (21)$$

From (13) it follows that

$$\sup_{|\varphi(z)| \leq 1/2} \frac{\mu(z)|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2+\frac{1}{p}}} \leq \left(\frac{4}{3}\right)^{2+\frac{1}{p}} \sup_{z \in \mathbb{D}} \mu(z)|\varphi'(z)|^2 \leq C\|C_\varphi\|_{H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})}. \quad (22)$$

Inequalities (21) and (22) imply (8).

Moreover, from (18), (19), (21) and (22) it follows that

$$\sup_{z \in \mathbb{D}} \frac{\mu(z)|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2+\frac{1}{p}}} + \sup_{z \in \mathbb{D}} \frac{\mu(z)|\varphi''(z)|}{(1 - |\varphi(z)|^2)^{1+\frac{1}{p}}} \leq C\|C_\varphi\|_{H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})}. \quad (23)$$

Now assume that conditions (8) and (9) hold. By Lemma 4, we have

$$\begin{aligned} \|C_\varphi f\|_{\mathcal{Z}_\mu(\mathbb{D})} &= |(C_\varphi f)(0)| + |(C_\varphi f)'(0)| + \sup_{z \in \mathbb{D}} \mu(z)|(C_\varphi f)''(z)| \\ &= |f(\varphi(0))| + |f'(\varphi(0))||\varphi'(0)| + \sup_{z \in \mathbb{D}} \mu(z)|f''(\varphi(z))(\varphi'(z))^2 + f'(\varphi(z))\varphi''(z)| \\ &\leq C\|f\|_{H^p(\mathbb{D})} + C\|f\|_{H^p(\mathbb{D})} \left( \sup_{z \in \mathbb{D}} \frac{\mu(z)|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2+\frac{1}{p}}} + \sup_{z \in \mathbb{D}} \frac{\mu(z)|\varphi''(z)|}{(1 - |\varphi(z)|^2)^{1+\frac{1}{p}}} \right). \end{aligned}$$

From this and by conditions (8) and (9), it follows that the operator  $C_\varphi : H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})$  is bounded. Moreover, if we consider the space  $\mathcal{Z}_\mu(\mathbb{D})/\mathbb{P}_1$ , we have that

$$\|C_\varphi\|_{H^p(\mathbb{D}) \rightarrow \mathcal{Z}_\mu(\mathbb{D})/\mathbb{P}_1} \leq C \left( \sup_{z \in \mathbb{D}} \frac{\mu(z)|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2+\frac{1}{p}}} + \sup_{z \in \mathbb{D}} \frac{\mu(z)|\varphi''(z)|}{(1 - |\varphi(z)|^2)^{1+\frac{1}{p}}} \right). \quad (24)$$

From (23) and (24), we obtain asymptotic relation (10).  $\square$

The following theorem can be proved similar to Theorem 1, and its proof is somewhat simpler, hence we omit it.

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**Theorem 2.** Assume  $p > 0$  and  $\varphi$  is a nonconstant holomorphic self-map of  $\mathbb{D}$ . Then the following statements true hold.

(a) The operator  $C_\varphi : H^p(\mathbb{D}) \rightarrow H_\mu^\infty(\mathbb{D})$  is bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{\mu(z)}{(1 - |\varphi(z)|^2)^{\frac{1}{p}}} < \infty.$$

(b) The operator  $C_\varphi : H^p(\mathbb{D}) \rightarrow \mathcal{B}_\mu(\mathbb{D})$  is bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{\mu(z)|\varphi'(z)|}{(1 - |\varphi(z)|^2)^{1+\frac{1}{p}}} < \infty.$$

The next result is an extension of Theorem 1 in [28]. It is proved similarly, and by using the test functions in Lemma 1. Hence, we also omit its proof.

**Theorem 3.** Assume  $p > 0$  and  $\varphi$  is a nonconstant holomorphic self-map of  $\Pi_+$ . Then the following statements true hold.

(a) The operator  $C_\varphi : H^p(\Pi_+) \rightarrow H_\mu^\infty(\Pi_+)$  is bounded if and only if

$$\sup_{z \in \Pi_+} \frac{\mu(z)}{(\operatorname{Im} \varphi(z))^{\frac{1}{p}}} < \infty.$$

(b) The operator  $C_\varphi : H^p(\Pi_+) \rightarrow \mathcal{B}_\mu(\Pi_+)$  is bounded if and only if

$$\sup_{z \in \Pi_+} \frac{\mu(z)|\varphi'(z)|}{(\operatorname{Im} \varphi(z))^{1+\frac{1}{p}}} < \infty.$$

(c) The operator  $C_\varphi : H^p(\Pi_+) \rightarrow \mathcal{Z}_\mu(\Pi_+)$  is bounded if and only if

$$\sup_{z \in \Pi_+} \frac{\mu(z)|\varphi'(z)|^2}{(\operatorname{Im} \varphi(z))^{2+\frac{1}{p}}} < \infty \quad \text{and} \quad \sup_{z \in \Pi_+} \frac{\mu(z)|\varphi''(z)|}{(\operatorname{Im} \varphi(z))^{1+\frac{1}{p}}} < \infty.$$

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# ON CARLESON MEASURES AND $F(p, q, s)$ SPACE ON THE UNIT BALL

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ABSTRACT. We characterize Carleson-type measures on the unit ball of  $\mathbb{C}^n$  in terms of  $\alpha$ -Bloch and  $F(p, q, s)$  functions.

## 1. INTRODUCTION

Let  $z = (z_1, \dots, z_n)$  and  $w = (w_1, \dots, w_n)$  be points in the complex vector space  $\mathbb{C}^n$ . By  $\langle z, w \rangle = \sum_{k=1}^n z_k \bar{w}_k$  we denote the complex inner product of  $z$  and  $w$ , and  $|z| = \sqrt{\langle z, z \rangle}$ . Let  $\mathbb{D}$  denote the open unit disc in the complex plane  $\mathbb{C}$ ,  $\mathbb{B}$  the open unit ball in  $\mathbb{C}^n$ ,  $dV$  the normalized Lebesgue measure on  $\mathbb{B}$  (i.e.  $V(\mathbb{B}) = 1$ ),  $d\sigma$  the normalized surface measure on the boundary  $\partial\mathbb{B} = S$  of  $\mathbb{B}$ , and  $H(\mathbb{B})$  the class of all functions analytic on  $\mathbb{B}$ . By  $\delta_{i,j}$ ,  $i, j \in \mathbb{N}_0$  we denote the function equal to one when  $i = j$  and zero if  $i \neq j$ . For  $a \in \mathbb{B}$ , let  $\varphi_a(z)$  be the biholomorphic involutive automorphisms of  $\mathbb{B}$  taking 0 to  $a$  (see, e.g., [14]).

For an  $f \in H(\mathbb{B})$ , the radial derivative  $\Re f$  of  $f$  is defined by

$$\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z).$$

If  $f(z) = \sum_{k=0}^{\infty} P_k(z)$  is the homogeneous polynomial expansion of function  $f$  then it is easy to see that  $\Re f(z) = \sum_{k=0}^{\infty} k P_k(z)$  ([14]).

Let  $\alpha > 0$ . The  $\alpha$ -Bloch space  $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbb{B})$  is the space of all  $f \in H(\mathbb{B})$  such that

$$b_\alpha(f) = \sup_{z \in \mathbb{B}} (1 - |z|^2)^\alpha |\Re f(z)| < \infty.$$

It is known that  $\mathcal{B}^\alpha$  is a normed space under the norm  $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + b_\alpha(f)$ .

Let  $\mathcal{B}_0^\alpha = \mathcal{B}_0^\alpha(\mathbb{B})$  denote the subspace of  $\mathcal{B}^\alpha$  consisting of those  $f \in \mathcal{B}^\alpha$  for which

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha |\Re f(z)| = 0.$$

This space is called the little  $\alpha$ -Bloch space. For  $\alpha = 1$  the  $\alpha$ -Bloch space and the little  $\alpha$ -Bloch space become the Bloch space  $\mathcal{B}$  and the little Bloch space  $\mathcal{B}_0$ . Some information on Bloch-type spaces on the unit ball can be found, for example, in [4, 7, 8, 15, 21].

For  $p \in (0, \infty)$ ,  $-n-1 < q < \infty$ ,  $s \geq 0$ , and  $\mu$  a positive measure on  $\mathbb{B}$  the space  $F(p, q, s, \mu) = F(p, q, s, \mu)(\mathbb{B})$  is defined as follows

$$\|f\|_{F(p,q,s,\mu)}^p = |f(0)|^p + \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\Re f(z)|^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s d\mu(z) < \infty.$$

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For  $d\mu(z) = dV(z)$  we obtain the space  $F(p, q, s)$  (see e.g. [9, 22]).

For any  $\zeta \in S$  and  $r > 0$  let

$$Q_r(\zeta) = \{z \in \mathbb{B} : |1 - \langle z, \zeta \rangle| < r\},$$

be the  $n$ -dimensional analogues of Carleson squares in  $\mathbb{D}$ .

In [20], among others, was proved the following result.

**Theorem A.** *Suppose  $n + \alpha + 1 > 0$  and  $\mu$  is a positive Borel measure on  $\mathbb{B}$ . Then the following conditions are equivalent.*

(a) *There exists a constant  $C > 0$  such that*

$$\mu(Q_r(\zeta)) \leq Cr^{n+1+\alpha}$$

*for all  $\zeta \in S$  and all  $r > 0$ .*

(b) *For some  $s > 0$  there exists a constant  $C > 0$  such that*

$$\int_{\mathbb{B}} \frac{(1 - |z|^2)^s}{|1 - \langle z, w \rangle|^{n+1+\alpha+s}} d\mu(w) \leq C$$

*for all  $z \in \mathbb{B}$ .*

Based on Theorem A we introduce the following definition.

**Definition 1.** Assume that  $\mu$  is a positive measure on  $\mathbb{B}$ , and  $a \in (0, \infty)$ . We say that  $\mu$  is a *bounded  $a$ -Carleson measure* if there exists a constant  $C > 0$  such that

$$\mu(Q_r(\zeta)) \leq Cr^a$$

for all  $\zeta \in S$  and all  $r > 0$ .

Here we characterize  $a$ -Carleson measures on the unit ball  $\mathbb{B}$  in terms of  $\alpha$ -Bloch and  $F(p, q, s)$  functions, extending some the one-dimensional results in [13].

In this paper, constants are denoted by  $C$ , they are positive and may differ from one occurrence to the other. The notation  $a \preceq b$  means that there is a positive constant  $C$  such that  $a \leq Cb$ . If both  $a \preceq b$  and  $b \preceq a$  hold, then one says that  $a \asymp b$ .

## 2. AUXILIARY RESULTS

In order to prove the main results of this paper, we need some auxiliary results which are incorporated in the lemmas which follow. First lemma is folklore.

**Lemma 1.** *Assume  $f \in H(\mathbb{B})$ ,  $\alpha \in (0, \infty)$  and  $m \in \mathbb{N}$ , or  $\alpha \in (1, \infty)$  and  $m = 0$ . Then  $f \in \mathcal{B}^\alpha$  if and only if*

$$\sup_{z \in \mathbb{B}} |\Re^m f(z)| (1 - |z|^2)^{\alpha+m-1} < \infty. \quad (1)$$

Moreover,

$$\sup_{z \in \mathbb{B}} |\Re f(z)| (1 - |z|^2)^\alpha \asymp |f(0)| \delta_{m,0} + \sup_{z \in \mathbb{B}} |\Re^m f(z)| (1 - |z|^2)^{\alpha+m-1}. \quad (2)$$

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**Lemma 2.** ([23]) *Let  $p \in (0, \infty)$ . If  $(n_k)_{k \in \mathbb{N}}$  is an increasing sequence of positive integers satisfying  $n_{k+1}/n_k \geq \lambda > 1$  for all  $k \in \mathbb{N}$ , then there is a positive constant  $A$  depending only on  $p$  and  $\lambda$  such that*

$$\frac{1}{A} \left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} \leq \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^{\infty} a_k e^{in_k \theta} \right|^p d\theta \right)^{1/p} \leq A \left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2}$$

for any sequence of numbers  $(a_k)_{k \in \mathbb{N}}$ .

**Lemma 3.** ([13]) *Assume that  $\beta \in (0, \infty)$ ,  $\lambda \in (1, \infty)$  and  $e^{-\beta/\lambda} \leq r_0 < 1$ . Then there is a positive constant  $C$ , depending only on  $\lambda$ ,  $\beta$  and  $r_0$ , such that*

$$\sum_{k=1}^{\infty} \lambda^{k\lambda\beta} r^{\lambda^{k+1}} \geq \frac{C}{(1-r^2)^{\lambda\beta}},$$

for all  $r_0 \leq r < 1$ .

An analytic function on  $\mathbb{B}$  with the homogeneous expansion  $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$  is said to have Hadamard gaps if  $n_{k+1}/n_k \geq \lambda > 1$  for all  $k \in \mathbb{N}$ . The next result is a natural generalization of a one-dimensional result by Yamashita from [19] (see, e.g. [17]). For related results see also [2, 3, 10, 12, 16, 18, 19].

**Theorem B.** *Assume that  $\alpha > 0$  and  $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$  is an analytic function on  $\mathbb{B}$  with Hadamard gaps. Then the following statements hold true*

- (a)  $f \in \mathcal{B}^\alpha$  if and only if  $\limsup_{k \rightarrow \infty} \|P_{n_k}\|_\infty n_k^{1-\alpha} < \infty$ .
- (b)  $f \in \mathcal{B}_0^\alpha$  if and only if  $\lim_{k \rightarrow \infty} \|P_{n_k}\|_\infty n_k^{1-\alpha} = 0$ .

The following Ryll-Wojtaszczyk-type lemma was proved in [1, Theorem 4].

**Lemma 4.** *There are  $\tau \in (0, 1)$  and  $k_0 \in \mathbb{N}$ , depending only on  $n$ , and homogeneous polynomials  $(P_{2^k}^{(l)}(z))_{k \in \mathbb{N}}$ ,  $l \in \{1, \dots, k_0\}$ , such that*

$$\max_{1 \leq l \leq k_0} \max_{\zeta \in S} |P_{2^k}^{(l)}(\zeta)| \leq 1 \quad \text{and} \quad \max_{1 \leq l \leq k_0} \min_{\zeta \in S} |P_{2^k}^{(l)}(\zeta)| \geq \tau. \quad (3)$$

Assume that  $f \in H(\mathbb{B})$ . Let  $f_\zeta(w) = f(\zeta w)$ ,  $\zeta \in S$ , where  $\zeta$  is fixed and  $w \in \mathbb{D}$ , be a slice function. By some calculation we see that

$$f'_\zeta(w) = \zeta_1 \frac{\partial f}{\partial z_1}(w\zeta) + \dots + \zeta_n \frac{\partial f}{\partial z_n}(w\zeta) = \frac{1}{w} \Re f(w\zeta). \quad (4)$$

Hence  $\Re^m f(w\zeta) = w(w(\dots(w(f'_\zeta(w))' \dots))' )'$ .

On the other hand, if  $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$ , we also have that

$$f'_\zeta(w) = \sum_{k=1}^{\infty} n_k P_{n_k}(\zeta) w^{n_k-1},$$

so that

$$w f'_\zeta(w) = \sum_{k=1}^{\infty} n_k P_{n_k}(\zeta) w^{n_k} = \sum_{k=1}^{\infty} n_k P_{n_k}(w\zeta),$$

and consequently

$$\Re^m f(w\zeta) = w(w(\dots(w(f'_\zeta(w))' \dots))' )' = \sum_{k=1}^{\infty} n_k^m P_{n_k}(\zeta) w^{n_k}. \quad (5)$$

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## 3. MAIN RESULTS

In this section we prove the main results of this paper.

**Theorem 1.** *Assume that  $\mu$  is a positive measure on  $\mathbb{B}$  and  $p \in (0, \infty)$ . If  $\alpha \in (0, \infty)$  and  $m \in \mathbb{N}$  or  $\alpha \in (1, \infty)$  and  $m = 0$ , then*

$$\int_{\mathbb{B}} \frac{d\mu(z)}{(1 - |z|^2)^{\alpha p}} < \infty \quad (6)$$

if and only if

$$\int_{\mathbb{B}} |\Re^m f(z)|^p (1 - |z|^2)^{p(m-1)} d\mu(z) \leq C \|f\|_{\mathcal{B}^\alpha}^p, \quad (7)$$

for every  $f \in \mathcal{B}^\alpha$  (or every  $f \in \mathcal{B}_0^\alpha$ ), where  $C$  is a positive constant independent of function  $f$ .

*Proof.* Assume that (6) holds. Let

$$I_m = \int_{\mathbb{B}} |\Re^m f(z)|^p (1 - |z|^2)^{p(m-1)} d\mu(z).$$

From (2), we have that

$$I_m \leq C \|f\|_{\mathcal{B}^\alpha}^p \int_{\mathbb{B}} (1 - |z|^2)^{p(m-1) - p(\alpha+m-1)} d\mu(z) = C \|f\|_{\mathcal{B}^\alpha}^p \int_{\mathbb{B}} \frac{d\mu(z)}{(1 - |z|^2)^{\alpha p}},$$

that is, (7) holds.

Now assume that (7) holds. Let  $z = r\zeta = |z|\zeta$ ,  $\zeta \in S$ , and

$$g_m^{(l)}(z) = \sum_{k=1}^{\infty} 2^{k(\alpha-1)} \rho_m^{2^k} P_{2^k}^{(l)}(z) = \sum_{k=1}^{\infty} 2^{k(\alpha-1)} \rho_m^{2^k} r^{2^k} P_{2^k}^{(l)}(\zeta), \quad (8)$$

for  $l = 1, \dots, k_0$ , and  $m \in \mathbb{N}$ , where  $(\rho_m)_{m \in \mathbb{N}}$  is a positive sequence increasingly converging to one and  $(P_{2^k}^{(l)}(z))_{k \in \mathbb{N}}$ ,  $l \in \{1, \dots, k_0\}$ , are from Lemma 4. Then by Theorem B (b) and (3), we have  $g_m^{(l)} \in \mathcal{B}_0^\alpha$  for each  $m \in \mathbb{N}$  and  $l \in \{1, \dots, k_0\}$ .

On the other hand, by (5), we have that for each  $m \in \mathbb{N}$  and  $l \in \{1, \dots, k_0\}$

$$\Re^m g_m^{(l)}(r\zeta) = \sum_{k=1}^{\infty} 2^{k(\alpha-1+m)} \rho_m^{2^k} r^{2^k} P_{2^k}^{(l)}(\zeta). \quad (9)$$

Replacing the function  $f$  in (7) by the functions

$$g_m^{l,\theta}(z) = g_m^{(l)}(e^{i\theta}z), \quad l = 1, \dots, k_0, \quad m \in \mathbb{N}, \quad \theta \in [0, 2\pi], \quad (10)$$

summing such obtained inequalities in  $l$  from 1 to  $k_0$ , then integrating it in  $\theta$  from 0 to  $2\pi$ , noticing that  $M_{l,1} := \sup_{\theta \in [0, 2\pi], m \in \mathbb{N}} \|g_m^{l,\theta}\|_{\mathcal{B}^\alpha} < \infty$ , for each  $l \in \{1, \dots, k_0\}$ , applying Fubini's theorem, Lemma 2 and Lemma 4, and an elementary inequality, for  $d\mu_p(z) = (1 - |z|^2)^{p(m-1)} d\mu(z)$ , we obtain



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$$\begin{aligned}
 C \sum_{l=1}^{k_0} M_{l,1}^p &\geq \sum_{l=1}^{k_0} \int_{\mathbb{B}} \left( \int_0^{2\pi} |\Re^m g_m^{l,\theta}(|z|\zeta e^{i\theta})|^p d\theta \right) d\mu_p(z) \\
 &\geq \sum_{l=1}^{k_0} \int_{\mathbb{B}} \left( \int_0^{2\pi} \left| \sum_{k=1}^{\infty} 2^{k(\alpha-1+m)} \rho_m^{2^k} (|z|e^{i\theta})^{2^k} P_{2^k}^{(l)}(\zeta) \right|^p d\theta \right) d\mu_p(z) \\
 &\geq C \sum_{l=1}^{k_0} \int_{\mathbb{B}} \left( \sum_{k=1}^{\infty} 2^{2k(\alpha-1+m)} \rho_m^{2^{k+1}} |z|^{2^{k+1}} |P_{2^k}^{(l)}(\zeta)|^2 \right)^{p/2} d\mu_p(z) \\
 &\geq C \int_{\mathbb{B}} \left( \sum_{k=1}^{\infty} \sum_{l=1}^{k_0} 2^{2k(\alpha-1+m)} \rho_m^{2^{k+1}} |z|^{2^{k+1}} |P_{2^k}^{(l)}(\zeta)|^2 \right)^{p/2} d\mu_p(z) \\
 &\geq C \int_{\mathbb{B}} \left( \sum_{k=1}^{\infty} 2^{2k(\alpha-1+m)} \rho_m^{2^{k+1}} |z|^{2^{k+1}} \max_{1 \leq l \leq k_0} \min_{\zeta \in S} |P_{2^k}^{(l)}(\zeta)|^2 \right)^{p/2} d\mu_p(z) \\
 &\geq C \tau^p \int_{\mathbb{B}} \left( \sum_{k=1}^{\infty} 2^{2k(\alpha-1+m)} \rho_m^{2^{k+1}} |z|^{2^{k+1}} \right)^{p/2} d\mu_p(z). \tag{11}
 \end{aligned}$$

By Lemma 3, we have that for  $|z| \in [r_0, 1)$  and for some  $r_0 \in (0, 1)$

$$\sum_{k=1}^{\infty} 2^{2k(\alpha-1+m)} \rho_m^{2^{k+1}} |z|^{2^{k+1}} \geq \frac{C}{(1 - \rho_m^2 |z|^2)^{2(\alpha-1+m)}}. \tag{12}$$

From (11) and (12), we have that

$$\infty > \sum_{l=1}^{k_0} M_{l,1}^p \geq C \int_{\mathbb{B} \setminus B(0, r_0)} \frac{d\mu(z)}{(1 - \rho_m^2 |z|^2)^{\alpha p}},$$

from which along with Fatou's lemma condition (6) easily follows.  $\square$

Now we give a necessary condition for a Carleson measure on the Bloch space.

**Theorem 2.** *Assume that*

$$\sqrt[p]{\int_{\mathbb{B}} |f(z)|^p d\mu(z)} \leq C_1 \|f\|_{\mathcal{B}} \tag{13}$$

for every  $f \in \mathcal{B}^1$ . Then

$$\int_{\mathbb{B}} \left( \ln \frac{1}{1 - |z|} \right)^{p/2} d\mu(z) < \infty. \tag{14}$$

*Proof.* Let  $\tau$ ,  $k_0$  and  $(P_{2^k}^{(l)}(z))_{k \in \mathbb{N}}$ ,  $l \in \{1, \dots, k_0\}$ , be as in Lemma 4 and

$$h_l(z) = \sum_{n=0}^{\infty} P_{2^n}^{(l)}(z), \quad l = 1, \dots, k_0.$$

Replacing the functions  $h_{l,\theta}(z) = h_l(e^{i\theta}z)$ ,  $l = 1, \dots, k_0$ , into (13), summing such obtained inequalities in  $l$  from 1 to  $k_0$ , then integrating it in  $\theta$  from 0 to  $2\pi$ , noticing

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that  $M_{l,2} := \sup_{\theta \in [0,2\pi]} \|h_{l,\theta}\|_{\mathcal{B}} < \infty$ , using Fubini's theorem Lemma 2 and Lemma 4, as in the proof of Theorem 1 we get

$$\begin{aligned} C \sum_{l=1}^{k_0} M_{l,2}^p &\geq \sum_{l=1}^{k_0} \int_{\mathbb{B}} \int_0^{2\pi} \left| \sum_{k=0}^{\infty} (|z|e^{i\theta})^{2^n} P_{2^k}^{(l)}(\zeta) \right|^p d\theta d\mu(z) \\ &\geq C\tau^p \int_{\mathbb{B}} \left( \sum_{n=0}^{\infty} |z|^{2^{n+1}} \right)^{p/2} d\mu(z) \\ &\geq C\tau^p \int_{\mathbb{B} \setminus B(0,r_0)} \left( \ln \frac{1}{1-|z|} \right)^{p/2} d\mu(z), \end{aligned} \quad (15)$$

since we have that

$$\begin{aligned} \ln \frac{1}{1-|z|} &= \sum_{n=1}^{\infty} \frac{|z|^n}{n} = \sum_{k=1}^{\infty} \sum_{l=2^{k-1}}^{2^k-1} \frac{|z|^l}{l} \leq \sum_{k=1}^{\infty} \left( \sum_{l=2^{k-1}}^{2^k-1} \frac{1}{l} \right) |z|^{2^{k-1}} \\ &\leq \sum_{k=1}^{\infty} |z|^{2^{k-1}} = |z| + \sum_{k=0}^{\infty} |z|^{2^{k+1}} \leq C \sum_{k=0}^{\infty} |z|^{2^{k+1}}, \end{aligned}$$

for  $|z| > 1/2$ . From (15), inequality (14) easily follows.  $\square$

**Remark 1.** One-dimensional case of Theorem 2 was probably proved for the first time by Limperis in his thesis [11, Theorem 3.1 i)].

**Remark 2.** Regarding an inverse of Theorem 2, in [6] was proved that if  $p > 0$  and  $f \in \mathcal{B}(\mathbb{D})$  then there is a constant depending only of  $p$  such that

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq C \|f\|_{\mathcal{B}}^p \left( \ln \frac{e}{1-r} \right)^{p/2}. \quad (16)$$

By using the functions  $f_{\zeta}(w) = f(w\zeta)$ ,  $w \in \mathbb{D}$ ,  $\zeta \in S$ , and Proposition 1.4.7 in [14], from (16) we get that there is a constant depending only of  $p$  such that

$$\int_S |f(r\zeta)|^p d\sigma(\zeta) \leq C \|f\|_{\mathcal{B}}^p \left( \ln \frac{e}{1-r} \right)^{p/2}, \quad (17)$$

for every  $f \in \mathcal{B}(\mathbb{B})$ . Inequality (17) seems folklore. It appears as Problem 3.19 in [21], but nor K. Zhu or the present author have the exact reference for inequality (17). From Theorem 2, inequality (17) and by using polar coordinates we immediately obtain the following corollary, similar to the corresponding one-dimensional result (see [11, p. 18]). Hence we omit the details.

**Corollary 1.** Assume  $p > 0$  and  $\mu$  is a radially symmetric measure on  $\mathbb{B}$ . Then condition (13) holds for every  $f \in \mathcal{B}(\mathbb{B})$ , if and only if inequality (14) holds.

**Theorem 3.** Assume that  $\mu$  is a positive measure on  $\mathbb{B}$  and  $p, \alpha \in (0, \infty)$  and  $m \in \mathbb{N}$ . Then the following conditions are equivalent

- (a)  $\mu$  is a bounded  $\alpha p$ -Carleson measure.
- (b)  $\Re^{(m-1)} : \mathcal{B}^{\alpha} \rightarrow F(p, p(m-1), \alpha p, \mu)$  is bounded.
- (c)  $\Re^{(m-1)} : \mathcal{B}_0^{\alpha} \rightarrow F(p, p(m-1), \alpha p, \mu)$  is bounded.

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*Proof.* (a)  $\Rightarrow$  (b) By applying Lemma 1, we have that

$$\begin{aligned} \|f\|_{F(p,p(m-1),\alpha p,\mu)}^p &\leq C \|f\|_{\mathcal{B}^\alpha}^p \int_{\mathbb{B}} \frac{(1-|z|^2)^{p(m-1)}}{(1-|z|^2)^{p(\alpha+m-1)}} (1-|\varphi_a(z)|^2)^{\alpha p} d\mu(z) \\ &= C \|f\|_{\mathcal{B}^\alpha}^p \int_{\mathbb{B}} \frac{(1-|a|^2)^{\alpha p}}{|1-\langle z, a \rangle|^{2\alpha p}} d\mu(z). \end{aligned} \quad (18)$$

From (18) and Theorem A, the boundedness of the operator  $\mathfrak{R}^{(m-1)} : \mathcal{B}^\alpha \rightarrow F(p, p(m-1), \alpha p, \mu)$  follows.

(b)  $\Rightarrow$  (c) This implication is obvious.

(c)  $\Rightarrow$  (a) By using the functions  $g_m^{l,\theta}$ , defined in (10) which belong to  $\mathcal{B}_0^\alpha$ , similar to Theorem 1 is obtained

$$\int_{\mathbb{B}} \frac{(1-|\varphi_a(z)|^2)^{\alpha p}}{(1-\rho_m^2|z|^2)^{\alpha p}} d\mu(z) \leq C \sum_{l=1}^{k_0} M_{l,1}^p, \quad m \in \mathbb{N}. \quad (19)$$

From (19), by the monotone convergence theorem and Theorem A, the implication follows, finishing the proof of the theorem.  $\square$

**Added in proofs.** After this paper was accepted for publication E. Doubtsov informed me that some related results can be found in his preprint [5]. Among others he also obtained Theorem 2 and Corollary 1, but his motivation stemmed from a paper from 2008 which had rediscovered Limperis' results. I would also like to express my thanks to him for clarifying me a detail in Lemma 4.

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# Solution to a Singular Integro-Differential Equation

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## Abstract

In this paper we find a method of solving a singular integro-differential equation of the type

$$\frac{dy}{dt} + a(t)Hy = f(t) \quad (1)$$

in the space of semi-almost periodic distributions where  $a(t)$  and  $f(t)$  are almost periodic functions on the real line composed of trigonometric polynomials with finitely many terms and the operator  $H$  the operator of Hilbert transform. Solution to (1) when  $a(t)$ ,  $f(t)$  are any a.p. function remains an open problem.

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## Definitions and Preliminaries

**The space of almost periodic functions and distributions.** Let  $L$  be the space of functions consisting of trigonometric polynomials of the type  $\sum_k A_k e^{i\lambda_k t}$  consisting of finitely many terms,  $\lambda_k$ 's are all real numbers. Let  $M$  be the set of all continuous functions defined on  $\mathbb{R}$  which are the uniform limit of a sequence of trigonometric polynomials then the space of functions  $L \cup M$  is the space of almost periodic functions where an almost periodic function on the real line is defined as follows: We say that a continuous function  $f$  defined on  $\mathbb{R}$  is an almost periodic function if, for an  $\varepsilon > 0$ , there exists a positive real number  $\ell = \ell(\varepsilon)$  such that in every interval of length  $\ell$  there exists at least one  $\tau > 0$  for which

$$|f(t + \tau) - f(t)| < \varepsilon \quad \forall \quad t \in (-\infty, \infty).$$

It was proved by H. Bohr that the space  $L \cup M$ , which is also denoted by  $B$ , is the space of almost periodic functions and any almost periodic function defined on  $\mathbb{R}$  is an element of  $L \cup M$ . So an almost periodic function defined on  $\mathbb{R}$  is either a trigonometric polynomial with finitely many terms in it or a uniformly continuous function on the whole real line as a uniform limit of trigonometric polynomials on the whole real line. Further details about a.p. functions can be looked into [1].

The linear space (system)  $L$  is metrized as follows: If:

$$f(t) = \sum_{r=1}^m A_r e^{i\lambda_r t}, \quad g(t) = \sum_{s=1}^h B_s e^{i\mu_s t}$$

we define the scalar product of trigonometric polynomials  $f$  and  $g$  by

$$\begin{aligned}\langle f, g \rangle &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t) \overline{g(t)} dt \\ &= \lim_{T \rightarrow \infty} \sum_{r=1}^m \sum_{s=1}^n A_r \overline{B_s} \frac{1}{2T} \int_{-T}^T e^{it(\lambda_r - \mu_s)} dt \\ &= \sum_{r=1}^m \sum_{s=1}^n \delta(\lambda_r - \mu_s) A_r \overline{B_s}\end{aligned}\quad (2)$$

$$\|f\|^2 = \sum_{r=1}^m |A_r|^2 \quad (3)$$

The norm defined by (3) is actually a norm as can be easily shown. Let  $B^2$  be a space obtained by completing the space  $L$  by the norm defined by (3). Then the space  $B^2$  is not a separable space and the space  $B^2$  contains the space  $B$ . Let  $f(t)$  and  $g(t)$  be two a.p. functions belonging to the span of the orthogonal set  $\{e^{\lambda_m t i}\}_{m=1}^{\infty}$ . It can now be easily shown that if a.p.

$$f(t) = \sum_{m=1}^{\infty} a_m e^{\lambda_m t i}, \quad g(t) = \sum_{m=1}^{\infty} b_m e^{\lambda_m t i},$$

then  $f(t) = g(t)$  if and only if  $a_m = b_m \forall m = 1, 2, 3, \dots$

It is this property that we will make use of in solving the system (1). It is a very difficult problem to solve the system

$$\frac{dy}{dt} + a(t)Hy = f(t)$$

in its most general form stated when  $a(t)$  and  $f(t)$  are any almost periodic functions. The case when  $a(t)$  and  $f(t)$  are constant functions has been discussed in [4]. We will therefore consider the cases when  $a(t)$  and  $f(t)$  are non-constant a.p. functions. We will take some special cases of  $a(t)$  and  $f(t)$  and generate the solutions which may give some clue to solving the general case.

We define, as in [4], the space  $B_{ap}^{ac}$  as the space of infinitely differentiable almost periodic functions defined on the real line in the sense of H. Bohr. An element  $\phi(t)$  of  $B_{ap}^{ac}$  has the form

$$\phi(t) = \sum_{n=-\infty}^{\infty} a_n e^{i\lambda_n t}$$

such that  $\sum_{n=-\infty}^{\infty} a_n \lambda_n^k$  is absolutely convergent for each  $k = 0, 1, 2, \dots$ . Therefore,  $\phi^{[k]}(t) = \sum_{n=-\infty}^{\infty} a_n (i\lambda_n)^k e^{i\lambda_n t}$  and the series representing  $\phi^{[k]}(t)$  is not only an a.p. function but is also absolutely and uniformly convergent on  $\mathbb{R}$ . It is for this reason that the space consisting of functions like  $\phi^{(k)}(t)$  is denoted by  $B_{ap}^{ac}$ , the lower suffix  $ap$  represents almost periodic and the upper suffix  $ac$  stands for absolutely convergent.

Let us now denote the space  $B_{ap}^{ac}$  by  $\Phi$  and let  $\Phi_k$  stand for the space of almost periodic functions obtained by differentiating  $k$  times each element of  $\Phi$ . Thus  $\Phi_0$  will stand for the space  $\Phi$  or  $B_{ap}^{ac}$ . Now one can observe that  $\Phi = \Phi_0 \supset \Phi_1 \supset \Phi_2 \supset \dots$

The space  $\Phi$  is equipped with the topology generated by the sequence of seminorms  $\{\gamma_k\}_{k=0}^{\infty}$ , where

$$\gamma_k(\phi) = \sqrt{\lim_{N \rightarrow \infty} \frac{1}{2N} (P) \int_{-N}^N |\phi^{[k]}(t)|^2 dt}. \quad (\text{see [7], p. 8})$$

A continuous linear functional over  $\Phi$  is called a semi-almost periodic distribution. A simple function defined on the real line is said to belong to the space  $\Phi'$  if it can be identified as a continuous linear functional over the space  $\Phi$ . It has been proved in [4] that the space of almost periodic distribution as defined by Laurent Schwartz in [6] is contained in the space  $\Phi'$  of semi-almost periodic distributions. Each of the spaces  $\Phi_i$  is an incomplete inner product space with inner product defined by

$$(\theta, \phi) = \lim_{N \rightarrow \infty} \frac{1}{2N} \int_{-N}^N \theta(t) \overline{\phi(t)} dt \quad \forall \theta, \phi \in \Phi_i.$$

Let us denote the completion of the space  $\Phi_i$  with respect to the norm  $\|\theta\|$  by  $\tilde{\Phi}_i$  where

$$\|\theta\| = \sqrt{\lim_{N \rightarrow \infty} \frac{1}{2n} \int_{-N}^N |\theta(t)|^2 dt} \quad \forall \theta \in \Phi_i.$$

It was shown in [4] that a continuous linear functional over the space  $\Phi$  is a finite linear combination of finite order generalised derivative (distributional derivatives) of elements of  $\tilde{\Phi}_0, \tilde{\Phi}_1, \tilde{\Phi}_2, \dots$ . This means that if  $f \in \Phi'$  then

$$f = \sum_{i=0}^m (-1)^i g_i^{(i)}, \quad g_i \in \tilde{\Phi}_i.$$

i.e.

$$\langle f, \phi \rangle = \left\langle \sum_{i=0}^m (-1)^i g_i^{(i)}, \phi \right\rangle \quad \forall \phi \in \Phi.$$

The Hilbert transform of a function  $f$  defined on  $\mathbb{R}$  is defined by

$$(Hf)(x) = \frac{1}{\pi} (P) \int_{-\infty}^{\infty} \frac{f(t)}{x-t} dt$$

provided the integral exists. It is a well-known fact that if  $f \in L^p(-\infty, \infty)$   $p > 1$ , then  $(Hf)(x)$  exists a.e. and  $\in L^p(-\infty, \infty)$  and that  $\|Hf\|_p \leq c_p \|f\|_p$ ,  $c_p$  is a constant  $> 0$ , depending upon  $p$  and not upon  $f$ . It is proved in [7] that

$$H^2 f = -f \quad \text{a.e.}$$

The Hilbert transform  $Hf$  of a generalized function  $f \in (D_{L^p})'$ ,  $p > 1$  is defined by

$$\langle Hf, \phi \rangle = \langle f, -H\phi \rangle \quad \forall \phi \in (D_{L^p}).$$

Here  $Hf$  is an element of  $(D_{L^p})'$ . From this we get the inversion formula

$$H^2 f = -f \quad \text{if } f \in (\mathcal{D}_{L^b})'$$

and

$$H^2 f = -f + c \quad \text{if } f \in \Phi'.$$

One can see that the Hilbert transform of a constant a.p. distribution is zero.

Motivation for the Hilbert transform of a periodic function and distributions comes from a result in [9] and [4]. The differentiation of  $f \in \Phi'$  is defined in the usual way, i.e.  $\langle Df, \phi \rangle = \langle f, -D\phi \rangle \quad \forall \phi \in \Phi$ . Another important result that we have is

$$\frac{d}{dt}(Hf) = Hf' \quad \forall f \in \Phi'.$$

We now proceed to solve the singular integro-differential equation

$$\frac{dy}{dt} + a(t)Hy = f(t) \quad (4)$$

in the space of semi-almost periodic distributions when  $a(t)$  and  $f(t)$  are a.p. functions. So far we have not been able to find the solution to the general case of (4) for any a.p. functions  $a(t)$  and  $f(t)$  and therefore it remains an open problem all by itself. It is also an open problem to describe a physical situation wherein this differential equation appears. When  $a(t)$  is a constant we solved such problems in [2] and [4] by differentiating both sides of the above equation with respect to  $t$  and then eliminating  $H$  from the new resulting singular integro-differential equation and the above singular integro-differential equation. But in the situation when  $a(t)$  is not a constant by eliminating  $H$  we come across an identity such as  $0 = 0$ . So the method of eliminating  $H$  does not work in this case. We therefore take some special cases and solve them, thereby we generate a technique of solving the general case.

We now present the solutions to some special cases of the integro-differential equation given by (4). We are taking a case when  $a(t)$  is a constant but  $f(t)$  is not. In this example we will recall some techniques used in [4].

**Example 1:** Solve:

$$\frac{dy}{dt} + kHy = \sin(\lambda t) \quad (5)$$

where  $k$  is a constant; in the space of a.p. distributions. We first solve the associated homogeneous differential equation

$$\frac{dy}{dt} + kHy = 0. \quad (6)$$

Operating both sides of (5) by  $H$  we get

$$\begin{aligned} H \frac{dy}{dt} + kH^2y &= 0 \\ \frac{d}{dt}(Hy) + k(-y + c) &= 0, \quad c \text{ is an arbitrary constant} \\ \frac{d}{dt} \left( -\frac{1}{k} \left( \frac{dy}{dt} \right) \right) - ky + kc &= 0 \\ \frac{d^2y}{dt^2} + k^2y &= k^2c. \end{aligned}$$

Therefore, using standard techniques of solving linear d.e. we get

$$y = A \cos kt + B \sin kt + c \quad (7)$$

It is easily verified that (7) satisfies (6).

It appears that we have three arbitrary constants; but we can reduce the solution to two arbitrary constants as the system (6) is a homogeneous one. If (7) is the solution to (6) then

$$y = \frac{1}{C} [A \cos kt + B \sin kt] + 1$$

is also a solution. Thus the general solution to the associated homogeneous system (6) is

$$y = a \cos kt + b \sin kt + 1 \quad (8)$$



$$a = \frac{A}{c}, \quad b = \frac{B}{c}.$$

We now proceed to find a particular solution to (5). The system (5) can be written in the form

$$\frac{dy}{dt} + kHy = \frac{e^{ti\lambda} - e^{-ti\lambda}}{2}. \quad (9)$$

Therefore, we can guess a form of the p.s. as follows

$$y = a_0 + \sum_{n=1}^{\infty} (a_n e^{i\lambda n t} + b_n e^{-i\lambda n t}). \quad (10)$$

Eliminating  $y$  from (9) and (10) we get,

$$\sum_{n=1}^{\infty} a_n i n \lambda e^{i\lambda n t} + \sum_{n=1}^{\infty} b_n i n \lambda e^{-i\lambda n t} + k \left[ \sum_{n=1}^{\infty} a_n (-i) e^{i\lambda n t} + \sum_{n=1}^{\infty} b_n (-i) (-1) e^{-i\lambda n t} \right] = \frac{e^{t\lambda i} - e^{-t\lambda i}}{2i}.$$

Equating the coefficients of the like powers of  $e^{\lambda t i}$  and  $e^{-\lambda t i}$  we get

$$\begin{aligned} a_1 i \cdot 1\lambda - k a_1 i &= \frac{1}{2i} \\ k b_1 i - b_1 1 \cdot i \lambda &= -\frac{1}{2i} \\ a_1 &= \frac{1}{2(\lambda - k)i^2} = -\frac{1}{2(\lambda - k)} \\ b_1 &= -\frac{1}{2i^2(k - \lambda)} = -\frac{1}{2(\lambda - k)}. \end{aligned}$$

It is assumed that  $\lambda \neq k$ . For  $n > 1$  we have

$$\begin{aligned} a_n i n \lambda - k i a_n &= 0 \\ a_n (n\lambda - k) &= 0, \quad n = 2, 3, 4, \dots \\ b_n k &= b_n n \lambda \\ b_n (n\lambda - k) &= 0 \quad n = 2, 3, 4, \dots \end{aligned}$$

Assuming that for no integer  $n \geq 2$  we have  $n\lambda = k$  then

$$a_n = 0, \quad b_n = 0 \quad \forall \quad n > 1.$$

Therefore, a particular solution of the system (9) is

$$y = -\frac{1}{2(\lambda - k)} e^{i\lambda t} - \frac{1}{2(\lambda - k)} e^{-i\lambda t} + a_0 = a_0 - \frac{1}{\lambda - k} \cdot \cos \lambda t.$$

Therefore, the general solution to (9) is

$$y = a \cos kt + b \sin kt - \frac{\cos \lambda t}{\lambda - k} + a_0 + 1$$

or

$$y = a \cos kt + b \sin kt - \frac{\sin \lambda t}{\lambda - k} + c, \quad c = a_0 + 1$$

$a, b, c$  are all arbitrary constants and  $\lambda \neq k$ .

If for some  $n = n_0$ ,  $n\lambda - k = 0$  i.e.  $n_0\lambda - k = 0$  then  $a_{n_0}$  and  $b_{n_0}$  are arbitrary constants and  $a_n, b_n$  are all zero for  $n = 2, 3, 4, \dots, n_0 - 1, n_0 + 1, \dots$  then a particular solution is

$$y = \frac{e^{i\lambda n_0 t}}{2(\lambda - k)i} - \frac{1}{2i(\lambda - k)}e^{-i\lambda n_0 t} + a_0 - \frac{\cos \lambda t}{\lambda - k} = \frac{\sin(\lambda n_0 t)}{\lambda - k} + a_0 - \frac{\cos \lambda t}{\lambda - k}.$$

so the general solution to (9) is

$$y = \frac{\sin(\lambda n_0 t)}{\lambda - k} - \frac{\cos(\lambda t)}{\lambda - k} + a \cos kt + b \sin kt + c$$

where  $a, b, c$  are arbitrary constants.

Lastly we consider the case  $\lambda = k$ . In this case the differential equation (10) takes the form

$$\frac{dy}{dt} + \lambda Hy = \sin \lambda t. \quad (11)$$

The general solution to the associated homogeneous differential equation (11) is

$$y = A \cos(\lambda t) + B \sin \lambda t + C$$

where  $A, B, C$  are arbitrary constants.

We now proceed to find a particular solution to the system (11).

Let us take a particular solution to be of the form

$$y = a_0 + \sum_{n=1}^{\infty} a_n e^{i\lambda n t} + \sum_{n=1}^{\infty} b_n e^{-i\lambda n t} \quad (12)$$

Eliminating  $y$  from (12) and (11) we get

$$\sum_{n=1}^{\infty} a_n i \lambda n e^{i\lambda n t} - \sum_{n=1}^{\infty} b_n i \lambda n e^{-i\lambda n t} + \lambda \left[ \sum_{n=1}^{\infty} a_n (-i) e^{i\lambda n t} + \sum_{n=1}^{\infty} b_n (i) e^{-i\lambda n t} \right] = \frac{e^{\lambda t i} - e^{-\lambda t i}}{2i}.$$

We arrive at a contradiction  $0 = \frac{1}{2}$  by equating the coefficients of  $e^{\lambda t i}$  and  $e^{-\lambda t i}$ .

Therefore there does not exist a solution to (11) in the span of  $1, e^{\pm \lambda t i}, e^{\pm 3\lambda t i}, \dots$ . Assume that  $\sin \lambda t$  belongs to the span of  $1, e^{\pm \mu_1 t i}, e^{\pm \mu_2 t i}, \dots$  and none of  $\mu_i = \lambda$  then a solution to (11) exists in the span of  $1, e^{\pm \mu_1 t i}, e^{\pm \mu_2 t i}, e^{\pm \mu_3 t i}, \dots$  and this solution is given by

$$y = A \cos \lambda t + B \sin \lambda t + C + \sum_{n=-\infty}^{\infty} \frac{-a_n i}{\mu_n - \lambda \operatorname{sgn}(\mu_n)} e^{\mu_n t i} \quad [4, p. 207].$$

It is assumed that

$$\sin \lambda t = \sum_{n=-\infty}^{\infty} a_n e^{\mu_n t i}$$

and none of  $\mu_n = \lambda$ . We have to choose a basis so that none of  $\mu_n = \lambda$ .

Solutions to such S.I.D.E. generally have two arbitrary constants in the space of a.p. functions. But in the space of semi almost periodic distributions we have three arbitrary constants. This is because our inversion formula for Hilbert transform in this case is

$$H^2 f = -f + c.$$

So far our method has been very traditional and simple but the method of eliminating  $H$  does not work when  $a(t)$  in (9) is not a constant function. So we will make a slight deviation in our technique to solve (9) in this case. Our technique will be demonstrated by the following example.

**Example 2.** Solve the following singular integro-differential equation in the space of semi-almost periodic distributions

$$\frac{dy}{dt} + e^{i\lambda t}Hy = e^{i\lambda t}, \quad \lambda > 0. \quad (13)$$

We first solve the associated homogeneous integro-differential equation

$$\frac{dy}{dt} + e^{i\lambda t}Hy = 0. \quad (14)$$

We propose the solution to this singular integro-differential equation to be of the form

$$y = \sum_{n=1}^{\infty} a_n e^{i\lambda n t} + \sum_{n=1}^{\infty} b_n e^{-i\lambda n t} + a_0 \quad (15)$$

$a_0$ ,  $a_n$  and  $b_n$  are all constants to be determined.

Eliminating  $y$  from (14) and (15) we get

$$\sum_{n=1}^{\infty} [a_n \lambda n e^{i\lambda n t} - a_n e^{i\lambda(n+1)t}] + \sum_{n=1}^{\infty} [b_n e^{-i\lambda(n-1)t} - b_n \lambda n e^{-i\lambda n t}] = 0.$$

Now equating the coefficients of the like powers of  $e^{\lambda t i}$  we get

$$\begin{aligned} a_1 &= 0, \quad b_1 = 0 \\ a_{n+1} \lambda(n+1) - a_n &= 0, \quad n = 1, 2, 3, \dots \\ b_{n+1} - b_n \lambda n &= 0, \quad n = 1, 2, 3, \dots \end{aligned}$$

Combining these results we get

$$\begin{aligned} a_n &= 0, \quad b_n = 0 \quad \forall n = 1, 2, 3, \dots \\ a_0 &\text{ is an arbitrary constant.} \end{aligned}$$

So the solution to (13) in the space of semi-almost periodic distribution is  $y = a_0$ .

We now find a particular solution to (12). Let us take such a solution to be of the form

$$y = a_0 + \sum_{n=1}^{\infty} a_n e^{i\lambda n t} + \sum_{n=1}^{\infty} b_n e^{-i\lambda n t} \quad (16)$$

$a_0$ ,  $a_i$  and  $b_i$  are constants to be determined. Eliminating  $y$  from (12) and (14) we get

$$\sum_{n=1}^{\infty} [a_n \lambda n e^{i\lambda n t} - a_n e^{i\lambda(n+1)t}] + \sum_{n=1}^{\infty} [b_n e^{-i\lambda(n-1)t} - b_n \lambda n e^{-i\lambda n t}] = -ie^{i\lambda t}.$$

Coefficient of  $e^{-i\lambda 0 t}$  :  $b_1 = 0$

Coefficient of  $e^{i\lambda 1 t}$  :  $a_1 \lambda 1 = -i$  so  $a_1 = \frac{-i}{\lambda}$

Coefficient of  $e^{i\lambda(n+1)t}$  :  $n = 1, 2, 3, \dots$   $a_{n+1}\lambda(n+1) - a_n = 0$ ,  $n = 1, 2, 3, \dots$

Coefficient of  $e^{-i\lambda(n)t}$  :  $n = 1, 2, 3, \dots$   $b_{n+1} - b_n\lambda n = 0$ ,  $n = 1, 2, 3, \dots$

Using these recurrence relations we get

$$\begin{aligned} b_1, b_3, b_n, \dots & \text{ zero } \quad \text{i.e. } b_1 = 0 \quad \forall i = 1, 2, 3, \dots \\ b_2 = b_1\lambda 1 = 0, \quad b_3 = b_2\lambda 2 = b_1\lambda^2 2! = 0 \\ b_4 = b_3\lambda 3 = b_1\lambda^3 3! = 0 \quad & \text{and so on} \end{aligned}$$

so

$$b_n = \lambda^{n-1}(n-1)!b_1 = 0, \quad n = 2, 3, 4, \dots$$

This relation is also true for  $n = 1$ .

$$\begin{aligned} a_{n+1} &= \frac{a_n}{\lambda(n+1)} \\ a_2 &= \frac{a_1}{\lambda 2}, \quad a_3 = \frac{a_2}{\lambda 3} = \frac{a_1}{\lambda^2 3!} = \frac{-i}{\lambda^3 3!} \\ \text{so} \quad a_n &= \frac{-i}{\lambda^n n!}, \quad n = 1, 2, 3, \dots \end{aligned}$$

Therefore the solution to the system is

$$y = a_0 - \sum_{n=1}^{\infty} \frac{i}{\lambda^n n!} e^{i\lambda n t}, \quad a_0 \text{ is an arbitrary constant.}$$

**Example 3.** Solve

$$\frac{dy}{dt} + \sin \lambda t H y = \cos \lambda t \quad (17)$$

in the space of semi-almost periodic distributions. We first solve the associated homogeneous differential equation

$$\frac{dy}{dt} + \sin \lambda t H y = 0.$$

We can write this S.A.P.D.E. in the form

$$\frac{dy}{dt} + \frac{e^{\lambda t i} - e^{-\lambda t i}}{2i} H y = 0. \quad (18)$$

Take

$$y = a_0 + \sum_{n=1}^{\infty} a_n e^{i\lambda n t} + \sum_{n=1}^{\infty} b_n e^{-i\lambda n t}. \quad (19)$$

Eliminating  $y$  from (18) and (19) we get

$$\sum_{n=1}^{\infty} a_n i\lambda n e^{i\lambda n t} + \sum_{n=1}^{\infty} b_n (-i\lambda n) e^{-i\lambda n t} + \frac{e^{\lambda t i} - e^{-\lambda t i}}{2i} \left[ \sum_{n=1}^{\infty} a_n (-i) e^{i\lambda n t} + \sum_{n=1}^{\infty} b_n i e^{-i\lambda n t} \right] = 0$$

$$\begin{aligned} & \sum_{n=1}^{\infty} a_n i \lambda n e^{i \lambda n t} - \sum_{n=1}^{\infty} \frac{a_n}{2} e^{i \lambda (n+1) t} + \sum_{n=1}^{\infty} \frac{a_n}{2} e^{i \lambda (n-1) t} \\ & + \sum_{n=1}^{\infty} b_n (-i \lambda n) e^{-i \lambda n t} + \sum_{n=1}^{\infty} \frac{b_n}{2} e^{-i \lambda (n-1) t} - \sum_{n=1}^{\infty} \frac{b_n}{2} e^{-i \lambda (n+1) t} = 0. \end{aligned}$$

Equating the coefficients of the like powers of  $e^{\lambda t i}$  we get

$$\text{Coefficient of } e^{\lambda t i 0}: \quad \frac{a_1}{2} + \frac{b_1}{2} = 0 \quad \text{so } b_1 = -a_1.$$

$$\text{Coefficient of } e^{\lambda t i}: \quad a_1 i \lambda + \frac{a_2}{2} = 0 \quad a_2 = -2a_1 \lambda i.$$

$$\text{Coefficient of } e^{-\lambda t i}: \quad \frac{b_2}{2} - i b_1 \lambda = 0 \quad b_2 = 2i b_1 \lambda = -2i a_1 \lambda.$$

$$\text{Coefficient of } e^{\lambda n t i}: \quad a_n i \lambda n + \frac{a_{n+1}}{2} - \frac{a_{n-1}}{2} = 0 \quad a_{n+1} = a_{n-1} - 2i \lambda n a_n.$$

$$\text{Coefficient of } e^{-\lambda n t i}: \quad -b_n i \lambda n + \frac{b_{n+1}}{2} - \frac{b_{n-1}}{2} = 0 \quad b_{n+1} = b_{n-1} + 2b_n i n \lambda.$$

$$\begin{aligned} a_3 &= a_1 - 2i \lambda 2 a_2 = a_1 - 2i \lambda 2 (-2a_1 \lambda i) = a_1 + [8 \lambda^2 i^2] a_1 = a_1 (1 - 8 \lambda^2) = a_1 (1 - 2^2 \lambda^2 2!) \\ b_3 &= b_1 + 2b_2 i 2 \lambda = b_1 - 2(2i a_1 \lambda) i 2 \lambda = b_1 + 8 \lambda^2 = a_1 [-1 + 2^2 2! \lambda^2] \\ b_4 &= b_2 + 2b_3 i 3 \lambda = 3i 2 \lambda [-1 + 2^2 2! \lambda^2] a_1 - 2a_1 \lambda i = i \lambda a_1 [(-3! + \lambda^2 2^3 3!)] - 2a_1 \lambda \\ a_4 &= a_2 - 2i \lambda 3 a_1 [-1 + 2^2 \lambda^2 2!] = -2a_1 \lambda i - 2i \lambda 3 a_1 [1 + 2^2 \lambda^2 2!], \end{aligned}$$

and so on. Therefore, the solution to the associated homogeneous S.I.D.E. (18) is

$$\begin{aligned} y &= a_0 + a_1 e^{\lambda t i} - 2a_1 \lambda i e^{2 \lambda t i} + a_1 (1 - 2^2 \lambda^2 2!) e^{3 \lambda t i} + a_1 [-2 \lambda - 2.3 i \lambda - 2^3 \lambda^3 3! i] e^{4 \lambda t i} + \dots \\ &+ (-a_1) e^{-\lambda t i} - 2i a_1 \lambda e^{-2 \lambda t i} + a_1 (1 - 2^2 2! \lambda^2) e^{-3 \lambda t i} + a_1 [-2 \lambda - 2.3 \lambda i - 2^3 \lambda^3 3! i] e^{-4 \lambda t i} + \dots \\ &= f(t) \quad (\text{say}). \end{aligned}$$

We now proceed to find a particular solution to (17) of the form

$$y = \sum_{n=1}^{\infty} a_n e^{i \lambda n t} + \sum_{n=1}^{\infty} b_n e^{-i \lambda n t}. \quad (20)$$

Eliminating  $y$  from (20) and (17) we get

$$\begin{aligned} & \sum_{n=1}^{\infty} [a_n i \lambda n e^{i \lambda n t} - \frac{a_n}{2} e^{i \lambda (n+1) t} + \frac{a_n}{2} e^{i \lambda (n-1) t}] \\ & + \sum_{n=1}^{\infty} \left[ \frac{b_n}{2} e^{-i \lambda (n-1) t} - b_n (-i \lambda n) e^{-i \lambda n t} - \sum_{n=1}^{\infty} \frac{b_n}{2} e^{-i \lambda (n+1) t} \right] = \frac{e^{\lambda t i} + e^{-\lambda t i}}{2}. \end{aligned}$$

So equating the coefficients of like powers of  $e^{\lambda t i}$  we get

$$\text{Coefficient of } e^{\lambda t i 0}: \quad \frac{a_1}{2} + \frac{b_1}{2} = 0$$

$$\text{Coefficient of } e^{\lambda t i}: \quad a_1 i \lambda + \frac{a_2}{2} = \frac{1}{2} \quad a_2 = 1 - 2a_1 i \lambda$$

$$\text{Coefficient of } e^{-\lambda ti}: \quad \frac{b_2}{2} - ib_1\lambda = \frac{1}{2}, \quad b_2 = 1 + 2b_1\lambda i = 1 - 2a_1\lambda i.$$

$$\text{Coefficient of } e^{\lambda nti}: \quad n \geq 2 \quad a_n i \lambda n - \frac{a_{n-1}}{2} + \frac{a_{n+1}}{2} = 0, \quad a_{n+1} = a_{n-1} - 2a_n i \lambda n.$$

$$\text{Coefficient of } e^{-n\lambda ti}: \quad -ib_n n \lambda - \frac{b_{n-1}}{2} + \frac{b_{n+1}}{2} = 0, \quad b_{n+1} = b_{n-1} + 2ib_n n \lambda$$

$$\begin{aligned} n = 2, \quad a_3 &= a_1 - 2a_2 i \lambda 2 = a_1 - 4a_2 i \lambda = a_1 - 4i\lambda[1 - 2a_1 i \lambda] \\ b_3 &= b_1 + 2i2b_2 \lambda = b_1 + 4ib_2 \lambda = b_1 + 4i\lambda(1 - 2a_1 i \lambda) \quad \text{and so on.} \end{aligned}$$

Therefore a particular solution to (17) is

$$y = a_0 + a_1 e^{\lambda ti} + (1 - 2a_1 i \lambda) e^{2\lambda ti} + \cdots + a_1 e^{-\lambda ti} + (1 - 2a_1 i \lambda) e^{-2\lambda ti} + \cdots = g(t)$$

Here  $a_0$  and  $a_1$  are arbitrary constants.

Therefore, the general solution to the system (18) is

$$y = f(t) + g(t).$$

We can see that the solution to the system involves three arbitrary constants.

**Example 4:** Solve the integro-differential equation

$$\frac{dy}{dt} + (e^{2ti} + e^{3ti})Hy = e^{ti} + 2e^{2ti} \quad (21)$$

in the space of semi-almost periodic distributions.

We have discussed such problems in (5) when the coefficient of  $Hy$  in (21) has only one exponential term. We are discussing this problem with two exponential terms in the coefficient of  $Hy$ . A more general case of  $a(t)$  can be discussed similarly. Our space in this case is closed with respect to multiplication by  $e^{2ti}$  and  $e^{3ti}$ . The space has to be closed with respect to the operator  $H$  and the operator  $\frac{d}{dt}$  of differentiation. The space spanned by  $1, e^{ti}, e^{2ti}e^{3ti}, \dots$  will satisfy all our conditions. There are other spaces such as the space spanned by  $1, e^{ti}, e^{2ti}e^{3ti}, \dots, e^{-ti}, e^{-2ti}e^{-3ti}, \dots$  which also satisfy these conditions. We will find out first solutions to (21) in the span of  $1, e^{ti}, e^{2ti}, \dots$ . In our definition of span we are also taking infinite linear combinations, i.e. our elements of the span are trigonometric polynomials with infinitely many terms which represent either a.b. functions, a.b. distributions or semi-almost periodic distributions. We first find solutions of the associated homogeneous integro-differential equation

$$\frac{dy}{dt} + (e^{2ti} + e^{3ti})Hy = 0 \quad (22)$$

in the span of  $1, e^{ti}, e^{2ti}e^{3ti}, \dots$ . In the basis of the span, terms such as  $e^{-ti}, e^{-2ti}e^{-3ti}, \dots$  are not included so we take a solution of (21) of the form

$$y = a_0 + \sum_{n=1}^{\infty} a_n e^{int}. \quad (23)$$

So from (23) and (22) we get

$$\begin{aligned}
& \sum_{i=1}^{\infty} a_n i n e^{int} + (e^{2ti} + e^{3ti}) \sum_{n=1}^{\infty} a_n (-i) e^{int} = 0 \\
& \sum_{n=1}^{\infty} a_n n e^{int} - \sum_{n=1}^{\infty} a_n e^{i(n+2)ti} - \sum_{n=1}^{\infty} a_n e^{i(n+3)t} = 0 \\
& \sum_{n=1}^{\infty} n a_n e^{int} - \sum_{n=3}^{\infty} a_{n-2} e^{int} - \sum_{n=4}^{\infty} a_{n-3} e^{int} = 0 \\
& 1a, e^{it} + 2a_2 e^{i2t} + 3a_3 e^{i3t} - a_1 e^{i3t} - \left[ \sum_{n=4}^{\infty} (a_{n-3} + a_{n-2} - n a_n) \right] e^{int} = 0.
\end{aligned}$$

Equating coefficients of the like powers of  $e^{ti}$  we get

$$\begin{aligned}
& \text{Coefficient of } e^{ti}: 1a_1 = 0 \quad \text{so } a_1 = 0 \\
& \text{Coefficient of } e^{2ti}: 2a_2 = 0 \quad \text{so } a_2 = 0 \\
& \text{Coefficient of } e^{3ti}: 3a_3 - a_1 = 0 \quad \text{so } a_3 = 0 \\
& \text{Coefficient of } e^{int}: n \geq 4 \quad a_{n-3} + a_{n-2} - n a_n = 0 \\
& \quad \quad \quad a_n = \frac{a_{n-2} + a_{n-3}}{n} \\
& \quad \quad \quad a_4 = \frac{a_2 + a_1}{4} = 0.
\end{aligned}$$

Thus  $a_n = 0 \forall n = 1, 2, 3, 4, \dots$ . Therefore the only solution to (22) is  $y = a_0$ ,  $a_0$  is an arbitrary constant. Now we proceed to find a solution to (22).

Taking the solution of the form

$$y = a_0 + \sum_{n=1}^{\infty} a_n t^{nti}$$

and substituting it in (22) we get

$$1a, e^{it} + 2a_2 e^{i2t} + (3a_3 - a_1) e^{i3t} + \sum_{n=4}^{\infty} (a_{n-2} + a_{n-3} - n a_n) e^{int} = e^{ti} + 2e^{2ti}.$$

So equating coefficients of the like powers of  $e^{ti}$  we get

$$\begin{aligned}
a_1 &= 1, \quad a_2 = 1, \quad a_3 = \frac{1}{3} \quad \text{and} \\
a_n &= \frac{a_{n-2} + a_{n-3}}{n}, \quad n \geq 4 \\
a_4 &= \frac{a_2 + a_1}{4} = \frac{1}{2} \\
a_5 &= \frac{a_3 + a_2}{5} = \frac{1 + \frac{1}{3}}{5} = \frac{4}{3 \cdot 5} = \frac{4}{15} \\
a_6 &= \frac{a_4 + a_3}{6} = \frac{(\frac{1}{2} + \frac{1}{3})}{6} = \frac{5}{36} \\
a_7 &= \frac{a_4 + a_5}{7} = \frac{\frac{1}{2} + \frac{4}{15}}{7} = \frac{23}{210}.
\end{aligned}$$

and so on. Thus our solution is

$$y = a_0 + e^{ti} + e^{2ti} + \frac{1}{3}e^{3ti} + \frac{1}{2}e^{4ti} + \dots$$

Here  $a_0$  is an arbitrary constant.

The series solution we obtained is absolutely convergent. We have only one arbitrary constant. Normally in such problems we have two arbitrary constants. If we select the larger space of almost periodic functions or distributions we will get another arbitrary constant in our solutions. To do so we may find solution in the span of

$$1, e^{\pm ti}, e^{\pm 2ti}, e^{\pm 3ti}, e^{\pm 4ti}, \dots$$

and we will have to get our solution of the form

$$y = a_0 + \sum_{n=1}^{\infty} a_n e^{nti} + \sum_{n=1}^{\infty} b_n e^{-nti}.$$

However calculation in this case will be very complex.

Now substituting for  $y$  in (22) we get

$$\begin{aligned} \sum_{n=1}^{\infty} n a_n e^{nti} - \sum_{n=1}^{\infty} n b_n e^{-nti} + (e^{2ti} + e^{3ti}) \left[ \sum_{n=1}^{\infty} -a_n e^{nti} + \sum_{n=1}^{\infty} b_n e^{-nti} \right] = 0 \\ b_1 e^{ti} + b_2 + b_1 e^{2ti} + b_2 e^{ti} + \sum_{n=1}^{\infty} [n a_n e^{nti} - a_n e^{(n+2)ti} - a_n e^{(n+3)ti}] \\ b_3 + b_3 e^{-ti} + \left[ \sum_{n=4}^{\infty} b_n e^{-(n-2)ti} + \sum_{n=4}^{\infty} b_n e^{-(n-3)ti} - \sum_{n=1}^{\infty} n b_n e^{-nti} \right] = 0 \end{aligned}$$

Equating coefficients of like powers of  $e^{ti}$  we get

$$\begin{aligned} b_{n+2} + b_{n+3} - n b_n &= 0, & n \geq 1 \\ n a_n - a_{n-2} - a_{n-3} &= 0, & n \geq 4 \\ b_{n+3} &= (n b_n - b_{n+2}), & n \geq 1 \\ a_1 + b_1 + b_2 &= 0 \\ b_2 + b_3 &= 0 \\ 2a_2 + b_1 &= 0 \end{aligned}$$

$$\begin{aligned} b_2 &= -(a_1 + b_1) \\ b_3 &= (a_1 + b_1) \\ b_4 &= 1b_1 - b_3 = -a_1 \\ b_5 &= 2b_2 - b_4 = -2(a_1 + b_1) + a_1 = -2b_1 - a_1 = -(a_1 + 2b_1) \\ b_6 &= 3b_3 - b_5 = a_1 + 2b_1 + 3(a_1 + b_1) = 4a_1 + 5b_1 \\ b_7 &= 4b_4 - b_6 = -4a_1 - 4a_1 - 5b_1 = -8a_1 - 5b_1 \end{aligned}$$

and so on.

$$a_n = \frac{a_{n-2} + a_{n-3}}{n}, \quad n \geq 4.$$



Also,

$$\begin{aligned} a_2 &= -\frac{b_1}{2}, \quad 3a_3 - a_1 = 0, \quad a_3 = \frac{a_1}{3} \\ a_4 &= \frac{a_2 + a_1}{4} = \frac{-\frac{b_1}{2} + a_1}{4} = \frac{2a_1 - b_1}{8} \\ a_5 &= \left( \frac{2a_1 - 3b_1}{30} \right) \quad \text{and so on.} \end{aligned}$$

We now get three arbitrary constants  $a_1$ ,  $b_1$  and  $a_0$ , in solution to the homogeneous part of S.I.D.E. equation, i.e. of (22).

$$\begin{aligned} y &= a_0 + a_1 e^{ti} + a_2 e^{2ti} + a_3 e^{3ti} + \dots + b_1 e^{-ti} + b_2 e^{-2ti} + b_3 e^{-3ti} \\ y &= a_0 + a_1 e^{ti} - \frac{b_1}{2} e^{2ti} + \frac{a_1}{3} e^{3ti} + \dots + b_1 e^{-ti} - (a_1 + b_1) e^{-2ti} + (a_1 + b_1) e^{-3ti} + \dots \end{aligned}$$

Coefficients  $a_0$ ,  $a_1$  and  $b_1$  give three linearly independent solutions. Sometimes it happens that only one of the solutions obtained is convergent and the other is not. In this case the divergent series is either an asymptotic expansion of a solution or it is an almost periodic distribution or a semi-almost periodic distribution.

Solve:

$$\frac{dy}{dt} + (e^{ti} + e^{\frac{t}{2}i} + e^{\frac{t}{3}i})Hy = 0 \quad (24)$$

This S.I.D.E. can be written in the form

$$\frac{dy}{dt} + (e^{\frac{6t}{6}i} + e^{\frac{3t}{6}i} + e^{\frac{2t}{6}i})Hy = 0. \quad (25)$$

So the space in which we are looking for a solution has to be closed with respect to the following operations

- (i) The operator  $H$
- (ii) The addition operation
- (iii) The operator  $\frac{d}{dt}$  of differentiation
- (iv) Multiplication by  $e^{\frac{1}{6}ti}$ .

So we have a solution of the form

$$y = a_0 + \sum_{n=1}^{\infty} a_n e^{\frac{nti}{6}}. \quad (26)$$

Using (25) and (26) we get

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a_n n i}{6} e^{\frac{nti}{6}} - i \left( e^{\frac{6ti}{6}} + e^{\frac{3ti}{6}} + e^{\frac{2ti}{6}} \right) \sum_{n=1}^{\infty} a_n e^{\frac{nti}{6}} &= 0 \\ \sum_{n=1}^{\infty} \frac{a_n n}{6} e^{\frac{nti}{6}} - \left( e^{\frac{6ti}{6}} + e^{\frac{3ti}{6}} + e^{\frac{2ti}{6}} \right) \sum_{n=1}^{\infty} a_n e^{\frac{nti}{6}} &= 0 \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{na_n}{6} e^{\frac{n+i}{6}} - \sum_{n=1}^{\infty} a_n \left[ e^{\frac{(n+6)ti}{6}} + e^{\frac{(n+3)ti}{6}} + e^{\frac{(n+2)ti}{6}} \right] = 0$$

$$\sum_{n=1}^{\infty} \frac{na_n}{6} e^{\frac{n+i}{6}} - \sum_{n=7}^{\infty} (a_{n-6} + a_{n-3} + a_{n-2}) e^{\frac{n+i}{6}}$$

$$- a_1 e^{\frac{3ti}{6}} - a_2 e^{\frac{4ti}{6}} - a_3 e^{\frac{5ti}{6}} - a_4 e^{\frac{6ti}{6}} - a_1 e^{\frac{4ti}{6}} - a_2 e^{\frac{5ti}{6}} - a_3 e^{\frac{6ti}{6}} = 0.$$

Equating the coefficients of the like powers of  $e^{\frac{ti}{6}}$  we get

$$\begin{aligned} \frac{a_1 1}{6} &= 0, & \frac{a_2 2}{6} &= 0, & \frac{a_3 3}{6} &= a_1 & \text{so } a_1 = a_2 = a_3 = 0 \\ \frac{a_4 4}{6} &= a_2 + a_1 = 0 & \text{so } a_4 &= 0 \\ \frac{a_5 5}{6} &= a_2 + a_3 = 0 & \text{so } a_5 &= 0 \\ \frac{a_6 6}{6} &= a_3 + a_4 = 0 & \text{so } a_6 &= 0 \\ a_{n-6} + a_{n-3} + a_{n-2} &= \frac{a_n n}{6} & n &= 7, 8, 9, 10, \dots \\ a_n &= \frac{6}{n} (a_{n-2} + a_{n-3} + a_{n-6}) \\ a_7 &= \frac{6}{7} (a_5 + a_4 + a_1) = 0 & \text{and so on.} \\ a_n &= 0 \quad \forall n = 1, 2, 3, \dots \end{aligned}$$

Thus we have  $y = a_0$ , where  $a_0$  is an arbitrary constant. In fact this is the only solution in the space chosen.

Solve:

$$\frac{dy}{dt} + \left( e^{ti} + e^{\frac{ti}{2}} + e^{\frac{ti}{3}} \right) Hy = -e^{\frac{3ti}{6}} - 2e^{\frac{4ti}{6}} + \frac{1}{6} e^{\frac{ti}{6}}.$$

Take

$$y = a_0 + \sum_{n=1}^{\infty} a_n e^{\frac{n+i}{6}}.$$

as in the previous example.

$$\sum_{n=1}^{\infty} \frac{na_n}{6} e^{\frac{n+i}{6}} - \sum_{n=7}^{\infty} (a_{n-6} + a_{n-3} + a_{n-2}) e^{\frac{n+i}{6}}$$

$$- a_1 e^{\frac{3ti}{6}} - a_2 e^{\frac{4ti}{6}} - a_3 e^{\frac{5ti}{6}} - a_4 e^{\frac{6ti}{6}} - a_1 e^{\frac{4ti}{6}} - a_2 e^{\frac{5ti}{6}} - a_3 e^{\frac{6ti}{6}} = -e^{\frac{3ti}{6}} - 2e^{\frac{4ti}{6}} + \frac{1}{6} e^{\frac{ti}{6}}$$

$$\frac{1a_1}{6} e^{\frac{1ti}{6}} + \frac{2a_2}{6} e^{\frac{2ti}{6}} + \left( \frac{3a_3}{6} - 1 \right) e^{\frac{3ti}{6}} + \left( \frac{4a_4}{6} - a_1 - a_2 \right) e^{\frac{4ti}{6}} + \left( \frac{5a_5}{6} - a_2 - a_3 \right) e^{\frac{5ti}{6}}$$

$$+ \left( \frac{6a_6}{6} - a_4 - a_3 \right) e^{\frac{6ti}{6}} - \sum_{n=7}^{\infty} (a_{n-6} + a_{n-3} + a_{n-2}) e^{\frac{n+i}{6}} = -e^{\frac{3ti}{6}} - 2e^{\frac{4ti}{6}} + \frac{1}{6} e^{\frac{ti}{6}}.$$

$$\begin{aligned}
a_1 &= 1, & \frac{2a_2}{6} &= 0 & \text{so } a_2 &= 0 \\
\frac{3a_3}{6} - a_1 &= -1, & \frac{3a_3}{6} &= 0 & \text{so } a_3 &= 0 \\
\frac{4a_4}{6} - a_2 - a_1 &= -2, & \frac{4a_4}{6} &= -1 & \text{so } a_4 &= -\frac{6}{4} \\
\frac{5a_5}{6} - a_2 - a_3 &= 0, & & & \text{so } a_5 &= 0 \\
\frac{6a_6}{6} - a_4 - a_3 &= 0, & & & \text{so } a_6 &= a_4 = -\frac{6}{4} \\
a_n &= \frac{6}{n}(a_{n-2} + a_{n-3} + a_{n-6}) \\
a_7 &= \frac{6}{7}(a_5 + a_4 + a_1) = -\frac{3}{7} \\
a_8 &= \frac{6}{8}(a_6 + a_5 + a_2) = \frac{6}{8}\left(-\frac{6}{4} + 0 + 0\right) = -\frac{9}{8}
\end{aligned}$$

and so on...

Therefore,

$$y = a_0 + 1e^{\frac{ti}{6}} - \frac{6}{4}e^{\frac{4ti}{6}} - \frac{6}{4}e^{\frac{6ti}{6}} - \frac{3}{7}e^{\frac{7ti}{6}} - \frac{9}{8}e^{\frac{8ti}{6}} + \dots \quad (27)$$

Solution to the singular integro differential equation

$$\frac{dy}{dt} + (ae^{\lambda ti} + be^{\mu ti})Hy = 0 \quad (28)$$

will be of the form

$$y = \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} a_{pq} e^{(m\lambda + n\mu)ti}. \quad (29)$$

Eliminating  $y$  from (28) and (29) and equating the coefficients of the like powers of the exponential terms the values of the coefficients  $a_{pq}$  can be calculated. This solution can be used to solve

$$\frac{dy}{dt} + (ae^{\lambda ti} + be^{\mu ti})Hy = f(t) \quad (30)$$

where  $f(t)$  is an a.p. function (with two exponential terms) for simplicity.

For the sake of definiteness, let us take  $f(t) = e^{cti} + e^{diti}$  where  $c, d$  are real numbers. We can find a particular solution to the system (30) by eliminating  $y$  from (29) and (30) and then equating the coefficients of the like powers of the exponential terms.

Let this solution be

$$y = g(t)$$

and assume that the solution to the system (28) is

$$y = h(t)$$

then the general solution to the system (30) is

$$y = f(t) + h(t).$$

This solution is in the space of semi-almost periodic space of distributions. The details of calculation is being left to the readers.

### Discussions

Solutions that we obtain to the system (1) in the form of a trigonometric series do not always converge. In the case when it does not converge it may either be an almost periodic distribution or a semi-almost periodic distribution. We have found solutions to the system (1) only in very special cases of coefficient  $a(t)$  and  $b(t)$ . The general case needs special consideration and we will look into that eventually. Laurent Schwartz has proved in [6] that a distribution is an almost periodic distribution if and only if it can be expressed as the finite linear combination of the finite order distributional derivative of almost periodic functions. According to this theorem the series  $\sum_{n=1}^{\infty} n^2 e^{nti}$  though a divergent series is an a.p. distribution as it can be expressed in the form

$$\frac{d^4}{dt^4} \left[ \sum_{n=1}^{\infty} \frac{-1}{n^2} e^{nti} \right].$$

The series under the differentiation sign is an almost periodic function. The differentiation is being done in distributional sense.

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# A New Hybrid Model for Intraday Spot Foreign Exchange Trading Accounting for Heavy Tails and Volatility Clustering

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## Abstract

Intraday Spot Foreign Exchange market is extremely volatile and cannot be explained by macro fundamentals. Models of market microstructure have a better forecasting quality, while still cannot fully explain the exchange rates fluctuations, especially over short term and on high frequency data. In this paper, we construct a new model for explaining forex market movements on minute data. This model involves price data on two different time frames, one macro fundamental variable and accounts for volatility clustering through a GARCH approach. Alpha-stable distributions appropriately describe the behavior of residuals. The model is constructed in two variants - for market makers observing the orders flow and for traders who only have the information about the price. In both cases, the new model outperforms other previously studied models.

**Keywords:** Foreign Exchange Market, Intraday Trading, Heavy Tails, Volatility Clustering, Market Microstructure

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## Introduction

Foreign exchange market is different from other financial markets in many respects and specific models are developed to describes its behavior. By definition, there are no insiders or particularly informed traders on forex. The exchange rate movements are theoretically defined by a relative state of two countries' economies, but macro based model still largely fail to explain short term price movements on forex. Model of market microstructure generally do slightly better, while still have low forecasting quality on high frequency data.

The present research develops a new hybrid model based on market microstructure, but also involving a macro fundamental as an explanatory variable. The model uses price data on two different time frames to account for both recent changes and longer term trends. The model uses GARCH approach to mind volatility clustering, and alpha-stable distributions explain the heavy-tailed behavior of residuals. The model is constructed in two versions, one for market makers having information about the order flow, another is for traders only observing price movements. Both models outperform previously studied macro, micro and hybrid models.

The remainder of the paper is organized as follows. The section 1 gives an overview of models for market microstructure. Section 2 describes previous attempts to construct a hybrid model. Section 3 presents and tests the new proposed model. Section 4 concludes.

## 1 Models of the Spot Foreign Exchange Market Microstructure

Micro-based models attempt to describe the behavior of the market, discover laws of its functioning, short-term reactions. As this analysis is based on early signs and leading indications, it is closer to technical analysis on one side and is more used for short term trades. It is for instance not uncommon to have a long term bullish trend on the market, confirmed by fundamental analysis of macroeconomic variables, while many investors and traders take advantage of short term downturns on the market, forecasted based on technical analysis and short term market movements. As a consequence, this type of analysis is the most used for intraday trading.

### 1.1 Orders Flow Model

Evans and Lyons [6] concentrate on the orders flow. They assume that it contains information on relevant fundamentals for two reasons:

- Traders who aim making profit on the foreign exchange market initiate trades when they believe they have information they can take advantage of, and

- Market participants who cover their other activity through forex market operations, all together, represent the current state and direction of economy.

Note that it is crucial to take into account the *signed* orders flow, rather than unsigned. Imagine a trader approaching a market maker to sell 10 lots of EUR vs. USD. Another trader comes in the same time with a request to buy 20 lots of EUR vs. USD. The unsigned order flow merely indicates that the volume of transactions reached 30 lots. The signed order flow however, will be  $-10 + 20$ , which gives  $+10$  as an outcome, and provides the valuable information to the market maker that the market expects a rise of the price.

The relevant expectation (that of the market maker):

$$s_t = (1 - b) \sum_{i=0}^{\infty} b^i E_t^m f_{t+i}, \quad (1)$$

where  $E_t^m$  is the expectation conditioned on market makers' information at the start of period  $t$  - this difference is crucial, as micro-based models attempt to explain the process of incorporation of available information into prices. In practice, the one estimates the model consisting in two independently estimated equations:

$$\Delta s_{t+1} = b + ax_t^{AGG} + e_{t+1} \quad (2)$$

and

$$\Delta s_{t+1} = b + \sum_{j=1}^6 a_j x_{j,t}^{DIS} + e_{t+1}, \quad (3)$$

where

- $x_t^{AGG}$  is aggregated order flow from six last periods of time,
- $x_{j,t}^{DIS}$  is the order flow from segment  $j$ .

The first equation estimates the forecasting power of the aggregated flow, then the second detects the input of each of the disaggregated flow individually.

Evans and Lyons [6] compare the true, ex-ante forecasting performance of a micro-based model against both a standard macro model and a random walk. The forecasting is examines for a short term period, one day to one month. Over 3 years of testing, it is shown that the micro-based model consistently outperform both random walk and the macromodel. This does not imply that past macro analysis has overlooked key fundamentals: finding consistent with exchange rate being driven by standard fundamentals.

An important reserve to be put on this models is whether the actual demand of the market is adequately reflected by the order flow. It is not uncommon to have a situation when there is an actual demand, but there is no transaction

generated due to eventual limits on credit lines, trading hours or technical limitations or issues.

One should also analyze if any type of transactions are equally informative and important for the price formation. The aggregated order flow may come from many small traders each of which has its own source of information and beliefs about the market. This same net position can be generated by one transaction of an institutional client or a small bank - his source of information may be more reliable on one side, but if it is erroneous, it will generate a more important distortion on price than that serving retail clients.

Also, an important question to ask, whenever market makers know the identity of their clients, is whether orders flow generated by corporations, financial institutional and banks are more susceptible to influence the market makers perception of the market and thus the price formation.

The last remark is related to the type of orders received by the market makers. There are two types of orders: market order with immediate execution and pending orders - revocable or irrevocable, with or without expiration date, with a prespecified non market price at which the order is placed if the market hits the target price before the order expired. While it is that the first type of orders should be taken into account, the answer is less obvious for the pending orders. Should they be taken into account at all? On one side they do reflect the traders' expectation regarding future prices and even provide more details in the form of a target price. On the other hand, especially if pending orders can be revoked without financial penalty, these beliefs of traders are not backed by real money and may simply reflect an attempt of an arbitrage strategy trying to catch the price "just in case". If the answer on the first question is still positive and pending orders should be taken into account, an important timing question arises - at which moment these positions have an actual influence on the market? Does it happen in the moment when the position is placed and the information about it is already available to the market maker? Or this order should rather be incorporated in the order flow in the moment of its activation when a trader commits real money into the operation? The answer on this second question has more influence if pending orders are allowed to stay active for longer time periods, eventually several days or even longer.

## 1.2 Evidence of Orders Flow Model on the Modern Foreign Exchange Market

To test the model described in the previous section, we take data from the foreign exchange market for the period from 1st March 2009 till 31st May 2009. The data reflects the volume of trades going through the dealing desk of a market maker. For each day of trades, the following data is analyzed:

1. Total number of lots of each currency pair bought by traders through the company,
2. Total number of lots of each currency pair sold by traders through the company,



3. Average price (quote) of the currency pair for each trading day, calculated as  $\frac{Open+High+Low+Close}{4}$ .

The analysis was done for seventy available currency pairs. The trades are taken from a sample with trading hours weekly from Sunday 23.00 till Friday 23.00. For the daily data, Sunday evening data was merged with Monday data, this providing 25 hours of trading information for Mondays and 23 hours of trading for Fridays.

Figure 1 shows the plots of daily exchange rate variations vs. scaled order flow of a market maker.

Several currency pairs were omitted in the experiment due to lack of regular data. We assume that the current order flow  $t$  is already known to the market maker. We also suspect the volatility clustering, i.e. current change in price depends on previous changes in price. We thus estimate the improved model 4 for the remaining currency pairs.

$$\Delta s_{t+1} = b + \sum_{j=0}^6 a_j x_{j,t}^{DIS} + \sum_{j=0}^6 c_j \Delta s_{t-j} + e_{t+1}, \quad (4)$$

The estimation process was iterative, whenever a coefficient was not significant, the corresponding variable was eliminated. The  $R^2$  for the estimated models are given in the table 1 for each currency pair.

As can be observed, the orders flow model have decent explanatory power of price changes of actively traded currency pairs. It will be thus admitted as basic model for further exploration. Its step-by-step analysis is presented below.

**Comparison with random walk.** We want to know if the suggested model is any better than a simple random walk. We estimate the equation 5.

$$\Delta s_t = a_0 + e_t, \quad (5)$$

where  $e_t \sim N(\mu, \sigma)$ . The results of this calculation are have shown  $R^2$  at zero for all the currency pairs without exception. In other words, the suggested orders flow model outperforms the random walk approximation, in average by 15.9% taking into account the currency pairs for which an appropriate orders flow model was found.

**In-sample and out-of-sample data.** Orders flow model estimated for the period 1st March 2009 till 31st May 2009 is now applied to out of sample daily data from 1st June 2009 till 15th June 2009.  $R^2$  for in-sample and out-of-sample data are compared in the table 2.

As can be observed, for those cases where the model could explain the out-of-sample variations, the  $R^2$  was well comparable for the in-sample data.

**High-frequency data.** As our primary purpose is to build a model for intraday trading, higher frequency data is to be analyzed. The same procedure as described above, is now applied to hourly and minute-by-minute data over the same three month period of time. For the hourly data, last 6 periods are taken into account in the model. The number of past periods is increased to 30 for the minute-by-minute data. For each model, variables with non significant

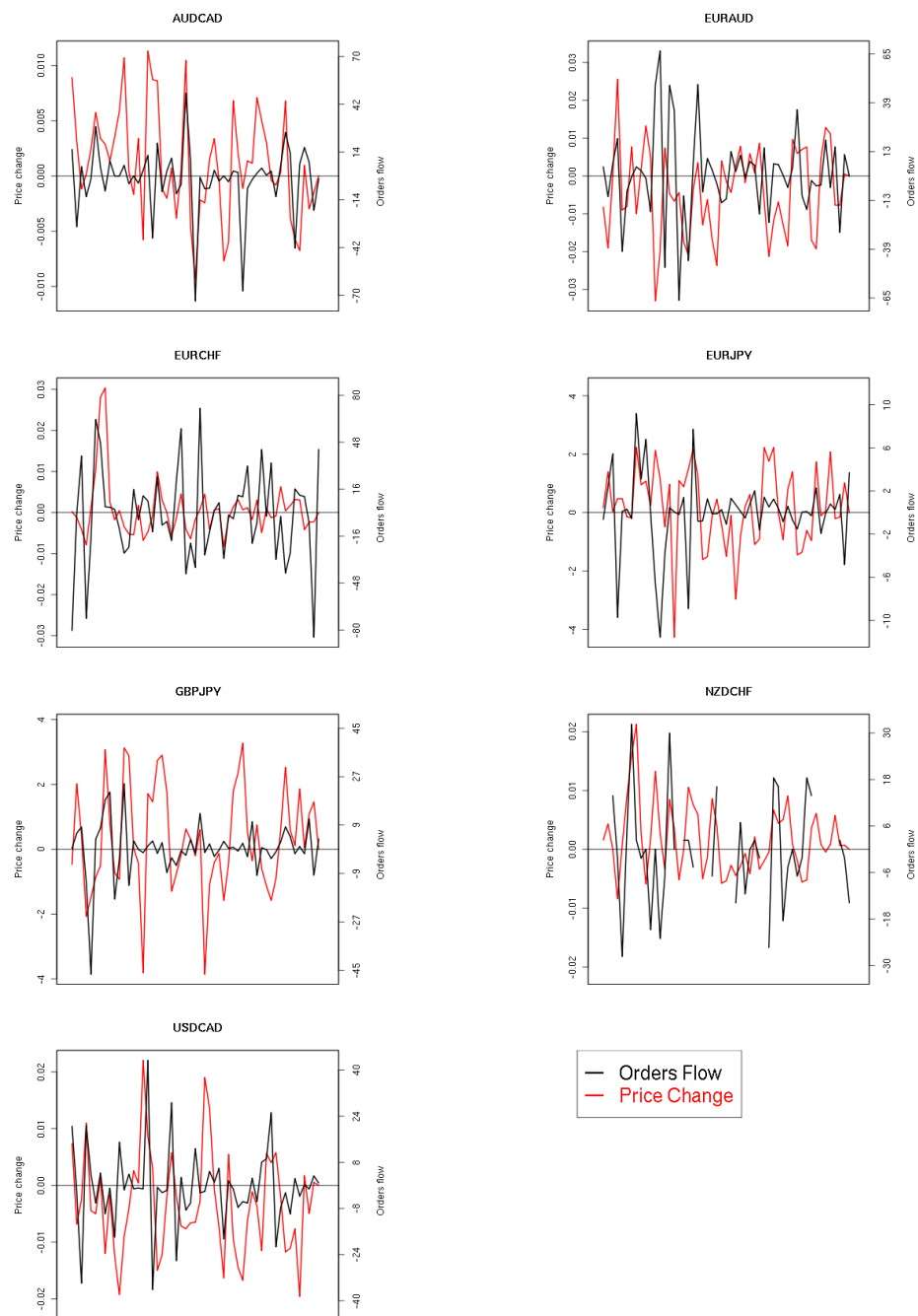


Figure 1: Daily exchange rate variations vs. orders flow.

Table 1:  $R^2$  of the Orders Flow model estimated on daily data.

Currency pair	$R^2$ , %	Currency pair	$R^2$ , %
AUDCAD	20.24	EURZAR	33.87
AUDCHF	45.55	GBPAUD	35.84
AUDJPY	25.69	GBPCAD	25.89
AUDNZD	35.84	GBPCHF	43.89
AUDUSD	24.69	GBPJPY	38.24
CADCHF	29.24	GBPNZD	36.63
CADJPY	34.64	GBPUSD	45.56
CHFJPY	25.15	NOKJPY	46.96
CHFSGD	43.62	NZDCAD	29.79
EURAUD	15.95	NZDCHF	37.22
EURCAD	19.66	NZDJPY	23.39
EURCHF	45.21	NZDUSD	28.63
EURGBP	35.31	USDCAD	31.23
EURJPY	29.22	USDCHF	32.95
EURNZD	25.44	USDJPY	29.48
EURSGD	36.88	USDTRY	40.17
EURTRY	27.48	USDZAR	64.26
EURUSD	43.53		

Table 2:  $R^2$  of the Orders Flow model for in-sample and out-of-sample data.

Currency pair	In-sample	Out-of-sample	Currency pair	In-sample	Out-of-sample
AUDCAD	20.24	32.59	EURZAR	33.87	12.13
AUDCHF	45.55	0	GBPAUD	35.84	0
AUDJPY	25.69	14.60	GBPCAD	25.89	0
AUDNZD	35.84	0	GBPCHF	43.89	34.64
AUDUSD	24.69	4.66	GBPJPY	38.24	41.32
CADCHF	29.24	0	GBPNZD	36.63	0
CADJPY	34.64	0	GBPUSD	45.56	41.93
CHFJPY	25.15	0	NOKJPY	46.96	0
CHFSGD	43.62	0	NZDCAD	29.79	3.21
EURAUD	15.95	9.18	NZDCHF	37.22	11.81
EURCAD	19.66	0	NZDJPY	23.39	0
EURCHF	45.21	0	NZDUSD	28.63	0
EURGBP	35.31	44.60	USDCAD	31.23	0
EURJPY	29.22	16.90	USDCHF	32.95	39.28
EURNZD	25.44	16.61	USDJPY	29.48	0
EURSGD	36.88	0	USDTRY	40.17	2.40
EURTRY	27.48	0	USDZAR	64.26	0
EURUSD	43.53	1.34			

Table 3:  $R^2$  of the Orders Flow model for daily, hourly and minute-by-minute data.

Currency pair	Daily data	Hourly data	Minute data	Currency pair	Daily data	Hourly data	Minute data
AUDCAD	20.24	18.49	10.41	EURZAR	33.87	4.09	-
AUDCHF	45.55	21.91	11.56	GBPAUD	35.84	19.18	15.37
AUDJPY	25.69	20.79	19.06	GBPCAD	25.89	18.49	14.60
AUDNZD	35.84	18.27	7.21	GBPCHF	43.89	27.63	16.84
AUDUSD	24.69	20.83	15.18	GBPJPY	38.24	20.90	22.35
CADCHF	29.24	21.82	9.82	GBPNZD	36.63	22.33	-
CADJPY	34.64	22.34	16.9	GBPUSD	45.56	21.36	19.31
CHFJPY	25.15	22.71	15.88	NOKJPY	46.96	9.60	-
CHFSGD	43.62	19.89	9.96	NZDCAD	29.79	19.75	6.16
EURAUD	15.95	21.09	15.07	NZDCHF	37.22	22.08	7.57
EURCAD	19.66	19.94	12.10	NZDJPY	23.39	22.24	14.57
EURCHF	45.21	26.80	8.71	NZDUSD	28.63	22.46	9.64
EURGBP	35.31	24.80	12.93	USDCAD	31.23	22.77	11.26
EURJPY	29.22	22.49	21.38	USDCHF	32.95	43.59	10.47
EURNZD	25.44	22.54	11.18	USDJPY	29.48	22.44	11.42
EURSGD	36.88	4.98	14.29	USDTRY	40.17	19.76	-
EURTRY	27.48	24.27	-	USDZAR	64.26	4.22	1.79
EURUSD	43.53	15.24	14.40				

coefficients are iteratively removed. The calculated  $R^2$  are shown in the table 3.

The results obtained at this step are indeed interesting. As can be observed, the coefficient of determination steadily declines as the frequency of the data increases.

**Analyzing the residuals.** For each equation, we now calculate the series of residuals and approximate the distribution of these series

- first by normal distribution,
- then by alpha-stable distribution.

Several residual patterns and their estimated distributions are presented on the figures 2.

To evaluate quantitatively, which distribution is more appropriate to describe the data, we will use the Integral of difference [12].

$$I = \frac{1}{2} \int_{-\infty}^{\infty} |f_{X,e}(x) - f_{X,th}(x)| dx, \quad (6)$$

where

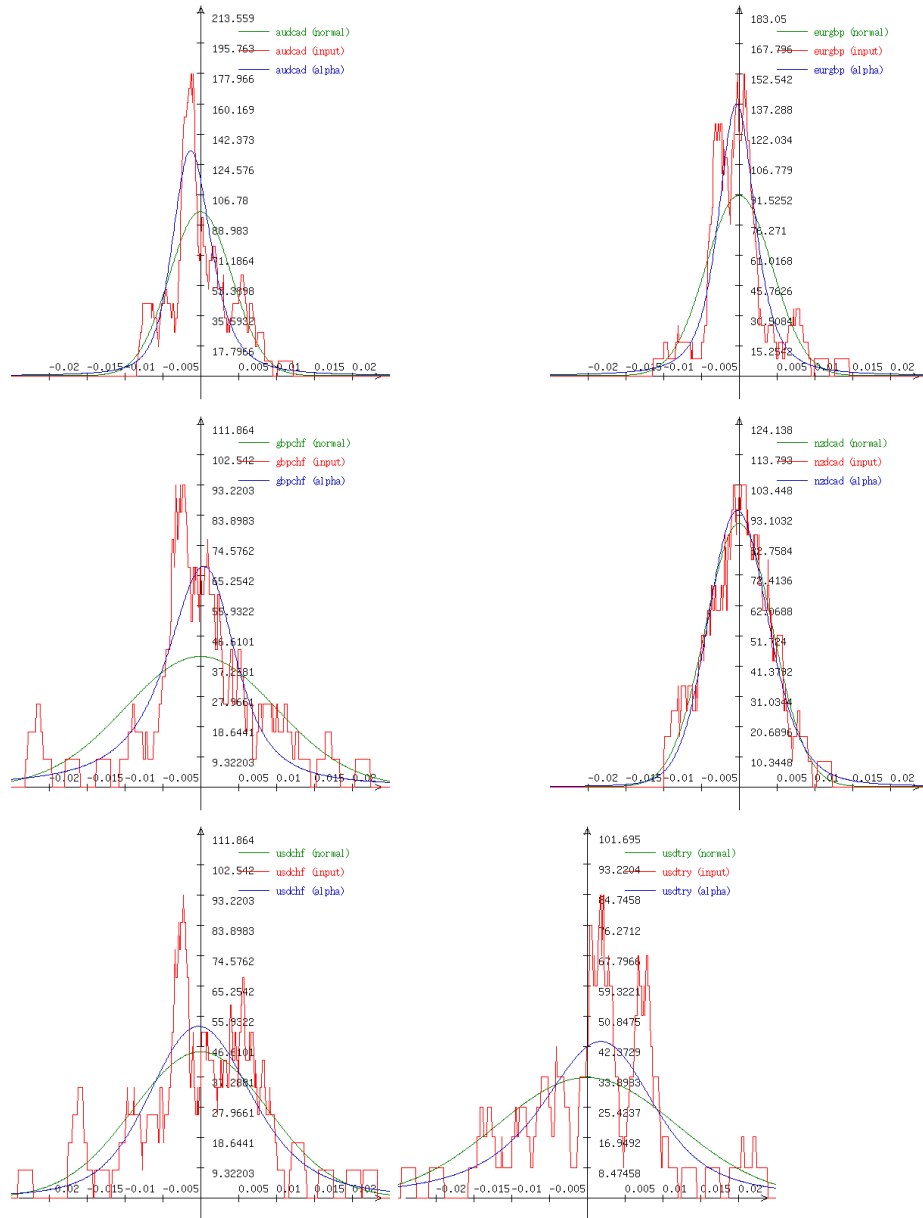


Figure 2: Empirical distribution functions (input) and their approximations with normal and alpha-stable distributions.

- $f_{X,e}$  is the empirical frequency,
- $f_{X,th}$  is the estimated, theoretical, probability distribution function.

This integral is always  $0 \leq I \leq 1$  and can be interpreted as a part of the residuals' behavior, unexplained by the suggested distribution. In our particular case, we calculate

$$I_n = \frac{1}{2} \int_{-\infty}^{\infty} |f_{X,e}(x) - f_{X,normal}(x)| dx \quad (7)$$

$$I_\alpha = \frac{1}{2} \int_{-\infty}^{\infty} |f_{X,e}(x) - f_{X,\alpha-stable}(x)| dx \quad (8)$$

The table 4 shows the values the integrals 7 and 8, for each currency pair, as well as the residuals distribution selected on the basis of this calculation.

As can be observed, for all series with no exceptions, the  $\alpha$ -stable distribution described the behavior of residuals more appropriately than the respective normal distribution.

**Institutional vs. retail traders.** To test whether institutional investors are better informed than individual traders, or, better to say, if their expectations are taken into account more seriously by market makers, we re-estimate the model 2 separately for the following groups of traders:

- Big traders: balance on the trading account exceeding 1 000 000 USD or equivalent in another currency,
- Medium traders: trading balance between 200 000 and 999 999 USD or equivalent,
- Retail traders: trading balance below 199 999 USD.

The  $R^2$  for each group of clients calculated on the minute data is presented in the table 5.

Observing the results, we conclude that the predicting power of the order flow does not change depending on the financial size of traders generating the this order flow.

**Instant execution and pending orders.** Finally, in all the previous analysis pending orders were not taken into account at all until they become an active order, e.g. until the trader commits real money into a position, following his estimations of the market evolution. However, pending orders themselves give additional information to the market makers about the price changes the trades expect.

We re-estimate the model on the minute data and two variant of order flow calculation:

- Initially used order flow of instant execution transactions,
- Alternative order flow of both instant execution and pending orders transactions.

Table 4: Comparison of normal and  $\alpha$ -stable residuals distribution estimation using the Integral of difference, for minute data.

Currency pair	Integral of difference		Better res. distr.
	Normal	$\alpha$ -stable	
AUDCAD	0.0814	0.0448	$\alpha$ -stable
AUDCHF	0.0952	0.0555	$\alpha$ -stable
AUDJPY	0.1108	0.0776	$\alpha$ -stable
AUDNZD	0.0874	0.0473	$\alpha$ -stable
AUDUSD	0.1063	0.0545	$\alpha$ -stable
CADCHF	0.1193	0.0519	$\alpha$ -stable
CADJPY	0.1222	0.0783	$\alpha$ -stable
CHFJPY	0.1169	0.0756	$\alpha$ -stable
CHFSGD	0.1206	0.0484	$\alpha$ -stable
EURAUD	0.1033	0.0413	$\alpha$ -stable
EURCAD	0.1096	0.0462	$\alpha$ -stable
EURCHF	0.1947	0.0468	$\alpha$ -stable
EURGBP	0.1399	0.0426	$\alpha$ -stable
EURJPY	0.1304	0.0855	$\alpha$ -stable
EURNZD	0.1286	0.0504	$\alpha$ -stable
EURSGD	0.1075	0.0472	$\alpha$ -stable
EURTRY	-	-	-
EURUSD	0.1481	0.0540	$\alpha$ -stable
EURZAR	-	-	-
GBPAUD	0.0881	0.0564	$\alpha$ -stable
GBPCAD	0.0986	0.0508	$\alpha$ -stable
GBPCHF	0.1342	0.0426	$\alpha$ -stable
GBPJPY	0.1367	0.0937	$\alpha$ -stable
GBPNZD	-	-	-
GBPUSD	0.1483	0.0626	$\alpha$ -stable
NOKJPY	-	-	-
NZDCAD	0.1888	0.0554	$\alpha$ -stable
NZDCHF	0.1338	0.0488	$\alpha$ -stable
NZDJPY	0.1252	0.0803	$\alpha$ -stable
NZDUSD	0.1476	0.0768	$\alpha$ -stable
USDCAD	0.1904	0.0857	$\alpha$ -stable
USDCHF	0.1716	0.0603	$\alpha$ -stable
USDJPY	0.1665	0.0740	$\alpha$ -stable
USDTRY	-	-	-
USDZAR	0.4074	0.2974	$\alpha$ -stable



Table 5:  $R^2$  (%) estimated for big, medium and retail clients, on the minute-by-minute data. "-" means transactions data is not available.

Currency pair	Big	Medium	Retail	Currency pair	Big	Medium	Retail
AUDCAD	10.35	10.36	10.42	EURZAR	-	-	-
AUDCHF	11.55	11.55	11.56	GBPAUD	15.39	15.40	15.38
AUDJPY	19.05	19.03	19.06	GBPCAD	14.53	14.57	14.60
AUDNZD	7.24	7.19	7.21	GBPCHF	16.79	16.86	16.83
AUDUSD	15.20	15.20	15.19	GBPJPY	22.35	22.34	22.36
CADCHF	9.83	9.81	9.82	GBPNZD	-	-	-
CADJPY	16.90	16.90	16.90	GBPUSD	19.27	19.29	19.31
CHFJPY	15.89	15.89	16.89	NOKJPY	-	-	-
CHFSGD	-	9.93	9.95	NZDCAD	6.10	6.14	6.16
EURAUD	-	15.05	15.09	NZDCHF	7.51	-	7.57
EURCAD	12.10	12.07	12.10	NZDJPY	14.56	14.58	14.57
EURCHF	8.72	8.71	8.69	NZDUSD	9.63	9.71	9.62
EURGBP	12.92	12.90	12.93	USDCAD	11.26	11.29	11.26
EURJPY	21.34	21.34	12.38	USDCHF	10.56	10.45	10.46
EURNZD	11.18	11.17	11.18	USDJPY	11.48	11.43	11.42
EURSGD	-	-	14.29	USDTRY	-	-	-
EURTRY	-	-	-	USDZAR	-	1.81	1.79
EURUSD	14.17	15.09	14.40				

Using the alternative order flow, we assume that the market makers base their price estimation taking into account the pending orders. Thus the pending orders are considered when they are placed and become known to the market maker, instead of the moment real money are committed into the trading operation.

The results of the estimation using the alternative order flow gave exactly the same coefficients and coefficients of determination as the initial model using the order flow for instant execution positions only.

### 1.3 Other Models for Forex Market and Special Cases

After having discussed both macro and micro approach to modeling the exchange rates dynamics, it is important to note that these two model types are not incompatible. In is typically assumed for macro approach that all the relevant information is publicly known and is reflected in current market prices. If any of these assumptions is relaxed, the order flow does explain a part of the exchange rates variations.

On the other hand, the micro approach does not claim the macro fundamentals do not define exchange rates. It rather says the order flow is more dynamic and forecasts those fluctuations better. The flow of orders merely reflects the belief of market participants materialized in form of their real money put into play.

A core distinction between the two approaches is the role of trades in price determination.

Fundamentals have little to no importance for intraday trading, and that the exchange rates are too much more volatile than any fundamental. As a remedy to that situation, a hybrid model, taking into account both short term and long term variations, was proposed by [8]. The model has the following form:

$$\delta P_t = f(i, m, z) + g(X, I, Z) + \varepsilon_t, \quad (9)$$

where

- $f(i, m, z)$  is the macro component of the model,
- $g(X, I, Z)$  is the micro component of the model,
- $i$  - nominal interest rates,
- $m$  - money supply,
- $z$  - other macro determinants,
- $X$  - order flow,
- $I$  - dealer's net positions,
- $Z$  - other micro determinants.

## 1.4 Portfolio Shifts Model

Macro models are typically estimated on a monthly frequency data and have the form

$$\Delta p_t = f(\Delta i, \Delta m, \dots) + \varepsilon_t, \quad (10)$$

where

- $\Delta p_t$  is the change in the log nominal exchange rate of the month,
- $\Delta i$  is the change in domestic and foreign interest rates  $i$ ,
- $\Delta m$  is the change of money supply over the month,
- $\varepsilon_t$  is the residual.

In this model, there is no place for the order flow in determining the price, any of its effects would be absorbed by the residual  $\varepsilon_t$ .

Micro approach generally leads to the following form of the model

$$\Delta p_t = g(\Delta x, \Delta I, \dots) + \nu_t, \quad (11)$$

where

- $\Delta p_t$  is the rate change over two transactions,
- $\Delta x$  is the change in order flow,
- $\Delta I$  change in the net dealer position,
- $\nu_t$  is the residual.

Lyons and Evans [5] propose a new model which combines both macro and micro approach:

$$\Delta p_t = f(\Delta i, \dots) + g(\Delta x, \dots) + \eta_t. \quad (12)$$

The main difficulty in using this model is that the macro part of it is usually estimated based on monthly data, while the micro-part is often determined on high-frequent values - daily, hourly or even tick-by-tick. A fair and meaningful trade-off can be using daily data for both macro and micro variables, getting more frequent data for the first and aggregating the latter.

The two processes assumed in the portfolio shifts models are the following:

- As a portfolio shift occurs, it is not publicly known. It is manifested in orders on the forex market, the initial volume of which goes through market makers and then are completed by inter-dealer operations. The market learns about the shift by observing these operations.
- The shift is important enough to move the market price.

If the demand is perfectly elastic, then currencies are perfect substitutes and the Portfolio Shifts model approaches the Portfolio Balance model. But in the opposite case the portfolio shifts model is different. It has a constant asset supply and defines demand components - driven by public and non-public information. The later is reflected by portfolio shifts.

Lyons and Evans [5] estimate the Portfolio Shifts model in the following form:

$$\Delta P_t = r_t + \lambda \Delta x_t, \quad (13)$$

or

$$\Delta P_t = \beta_1 \Delta(i_t - i_t^*) + \beta_2 \Delta x_t + \eta_t \quad (14)$$

where

- $\Delta P_t$  is the change of the price between periods  $t - 1$  and  $t$ ,
- $r_t$  is the public information increment,
- $\lambda$  is a positive constant,
- $\Delta x_t$  is the order flow,
- $i_t$  is nominal dollar interest rate,
- $i_t^*$  is nominal non dollar interest rate,
- $\beta_1, \beta_2$  are parameters,
- $\eta_t$  is the residual.

Tests show that this model produces better than random-walk results for both in-sample and out-of-sample data, the forecasting being more precise over shorter period of time (39 days) rather than for a longer period of 89 days.

### 1.5 Evidence of the Portfolio Shifts Model

To test if the Portfolio Shifts Models is applicable to the Foreign Exchange, we estimate the following equation

$$\Delta s_{t+1} = b + \sum_{j=0}^6 a_j \sum_{i=0}^6 x_{j+i,t}^{DIS} + \sum_{j=0}^6 c_j \Delta s_{t-j} + dr_t + e_{t+1}, \quad (15)$$

Overnight LIBOR rates announced daily, taken from [www.dowjonesclose.com/liborrates.html](http://www.dowjonesclose.com/liborrates.html), state of 29th June 2009, are taken for the values of the public information  $r_t$ . The iterative estimation was done in the same way as in the previous chapter. The table 6 allows for comparison of the  $R^2$  between Orders Flow model and Portfolio Shifts model.

It can be observed that the addition of the macro economic variable does consistently improve the quality of modeling.

Table 6: Orders Flow model and Portfolio Shifts model estimated on daily data

Currency pair	$R^2$ Port- folio Shifts, %	$R^2$ Or- ders Flow, %	Currency pair	$R^2$ Port- folio Shifts, %	$R^2$ Or- ders Flow, %
AUDCAD	22.23	20.24	EURZAR	34.00	33.87
AUDCHF	46.17	45.55	GBPAUD	35.84	35.84
AUDJPY	26.31	25.69	GBPCAD	27.13	25.89
AUDNZD	42.67	35.84	GBPCHF	43.89	43.89
AUDUSD	25.04	24.69	GBPJPY	39.37	38.24
CADCHF	29.25	29.24	GBPNZD	39.51	36.63
CADJPY	35.05	34.64	GBPUSD	46.38	45.56
CHFJPY	25.15	25.15	NOKJPY	46.69	46.96
CHFSGD	43.79	43.62	NZDCAD	31.87	29.79
EURAUD	16.07	15.95	NZDCHF	49.10	37.22
EURCAD	19.85	19.66	NZDJPY	28.17	23.39
EURCHF	49.55	45.21	NZDUSD	31.58	28.63
EURGBP	35.44	35.31	USDCAD	31.44	31.23
EURJPY	30.43	29.22	USDCHF	33.04	32.95
EURNZD	31.84	25.44	USDJPY	29.73	29.48
EURSGD	36.90	36.88	USDTRY	42.71	40.17
EURTRY	27.95	27.48	USDZAR	75.08	64.26
EURUSD	43.73	43.53			

## 1.6 Hybrid Models

Any of the presented models are not necessarily and strictly used in the presented form only. They can contribute to one another. For example, Medeiros [1] suggested a hybrid model by including to the basic Evans-Lyons model [5] additional variables representing a country-risk premium. Tests performed on the Brazilian foreign exchange market showed data, showed that the model had a good  $R^2$ , which was further improved by a GARCH estimation.

## 2 New Model for Forex Intraday Trading

As a conclusion of the performed study, a new model is to be presented. It takes into account every test performed all over the study as a building block towards an improved model for high frequency foreign exchange modeling.

### 2.1 Framework

Below is the summary of main the findings made and a description of the framework for the new model.

1. **Market participants.** While taking into account the activity of all market participants, the target "users" of the developed model are active traders aiming speculative profit on the foreign exchange market, as well as brokerage companies and market makers.
2. **Trading Mechanisms.** It is assumed that no regulatory restrictions apply to trading. Transactions are done electronically by traders via brokerage companies and market makers. The electronic transmission of information is assumed to be immediate. No additional delay is present in case of trading via an intermediary broker.
3. **Trade Instructions.** Traders use immediate execution and pending orders on any available currency pair. Price for immediate execution has to be specified, i.e. there are no requests on "best possible" price.
4. **Market Efficiency.** The market is not efficient in strong and semi-strong form. Interest rate parity does not hold at all times. Carry trades are not consistently profitable. Market showed to be efficient in a weak form on minute data.
5. **Liquidity.** No major market crashes are happening. Major and small news announcements are coming regularly. The market liquidity is high and does not change depending if regional equity trading sessions is being active or not. The market liquidity does not change around the news announcements.
6. **Volatility and Risk Premia.** Volatility is appropriately measured by the Expected Tail Loss, as well as by the R-ratio. It changes over time.

Traders are risk averse. No evidence of hot potato trading was found. Volatility was found to be increased during the first and the last 30 minutes of either regional equity trading session. Foreign Exchange Market does not display fractal properties. Trading short term on higher frequency information is generally more risk than trading over long term.

7. **Trading Costs.** Trading costs are not negligible and are supposed to be incorporated in the spread. No other fee are applied.
8. **Technical Analysis.** As the market is shown to be efficient in a weak form, the technical analysis cannot provide consistently accurate prediction.
9. **Fundamental Analysis and Trading Psychology.** Fundamental analysis is susceptible to provide correct predictions. Certain industries are susceptible to move the currency exchange rates. Market prices are influenced by human behavior, in particular the overreactions.
10. **Applicability of Equity Market Models to the Foreign Exchange Market.** Equity models are generally not applicable to forex market as is, but can provide ideas and econometric tools.
11. **Macro Based Models.** Models based on macro fundamentals alone fail to explain the forex intraday market movements. The simple intuition behind is that macro fundamentals do not generally change during the day, while prices are moving permanently.
12. **Models for Forex Market Microstructure.** Models for market microstructure, in particular the model of orders flow, most appropriately explain the market behavior, comparing to other studied models:
  - Introduction of previous values of price changes (adding an  $AR(n)$  part) improves the coefficient of determination.
  - Model is appropriate for the out-of-sample forecast.
  - Predicting power of the order flow does not change depending on the financial size of traders generating this order flow.
  - $\alpha$ -stable distribution is appropriate to model the behavior of residuals.
  - Ceteris paribus, the quality of the model decreases as the data frequency increases.
  - Addition of macro fundamentals to build a hybrid model improves the forecasting ability of a model.

## 2.2 New model

Taking into account the considerations above, we start the construction of the new model based on the basic order flow model. In order to account for volatility clustering, i.e. when periods of high volatility are usually followed by other periods of high volatility, GARCH approach is to be used.

Next, we have seen that the addition of a macro fundamental improves the quality of the model. We include the overnight interest rate, as it both reflects the a macro characteristic of the economy and a part of the trading costs.

The data chosen for modeling should usually match the trading time horizon. However, as it was shown, the forecasting power of a model decreases as the data frequency increases. We thus decide to include a lower frequency data into equation. From the market standpoint, this decision is confirmed by the practice, when traders usually require that data on several time frames agree on the expected direction of the market, before they engage in a trading transaction.

Technically, we observed that series showed heavy tails. To account for this data specification, we admit the residuals follow an  $\alpha$ -stable distribution ([9], [10], [11]).

Finally, for the best fit, instead of a simple linear regression, we a looking for a more complex relationship in a spline form ([?], [?]).

$$\Delta P_{t+1} = \alpha_0 + \sum_{j=0}^k \alpha_{1j} \Delta P_{t-j} + \sum_{j=0}^l \alpha_{2j} \Delta P_{t-j}^l + \sum_{j=0}^m \alpha_{3j} r_t + \alpha_4 i + \varepsilon_t, \quad (16)$$

and

$$\sigma_t^2 = \beta_0 + \sum_{j=1}^p \beta_{1j} \varepsilon_{t-j}^2 + \sum_{j=1}^q \beta_{2j} \sigma_{t-j}^2, \quad (17)$$

where

- $\Delta P_t$  price change in the moment  $t$ ,
- $\Delta P_t^l$  price change in the moment  $t$ , on a lower frequency time frame,
- $r_t$  order flow,
- $i$  interest rates,
- $\alpha_i$  - parameters.

The distribution of residuals, i.e. the unexplained part of the price changes, can be approximated by an  $\alpha$ -stable distribution. Due to extremely high market liquidity, the model will also stay valid for news announcements periods.

The purpose of the research is not only to develop a model, but also to make this model usable for everyday live trading. The order flow used in the model above is not known to most traders on the market such as speculators or hedged



in interest arbitrageurs. However the order flow is positively correlated with the market liquidity, which can on its turn be reflected by the number of price ticks arriving in each particular moment of time. The model suggested for this group of market participants is the following:

$$\Delta P_{t+1} = \alpha_0 + \sum_{j=0}^k \alpha_{1j} \Delta P_{t-j} + \sum_{j=0}^l \alpha_{2j} \Delta P_{t-j}^l + \sum_{j=0}^m \alpha_{3j} v_t + \alpha_4 i + \varepsilon_t, \quad (18)$$

and

$$\sigma_t^2 = \beta_0 + \sum_{j=1}^p \beta_{1j} \varepsilon_{t-j}^2 + \sum_{j=1}^q \beta_{2j} \sigma_{t-j}^2, \quad (19)$$

where  $v_t$  is the number of price ticks in the period  $t$

### 2.3 Evidence of the New Model

The new model is being estimated in three steps:

1. Apply the Multivariate Adaptive Regression Splines (MARS) to the first equation of the model.
2. Estimate the residuals using  $GARCH(p, q)$ .
3. Estimate the parameters of  $\alpha$ -stable distribution of residual errors using McCulloch or any other method.

The "market makers' model" is estimated on the minute data over the last three months from 1st March 2009 till 31st May 2009. As a lower frequency time frame, hourly data is selected. To demonstrate that the model keeps the  $R^2$  on the same level also for the out-of-sample data, tick-by-tick data between 1st and 15th of June 2009 is used. Same estimations are repeated for the "traders' model" with the same results in terms of quality of modeling.

There are three very positive results out of testing this model:

- The model for market makers having the private information about the order flow is as good as the model for traders observing the price volatility.
- This model outperforms all the previously tested models on the minute data.
- The out-of-sample performance of the model is as good as its in-sample performance.

One more observation about the model worth being mentioned here. As the equation was estimated, the MARS regression provides the analysis of impact of each independent variable on the dependent variable. Several typical charts describing this impact are presented on the figure 3.

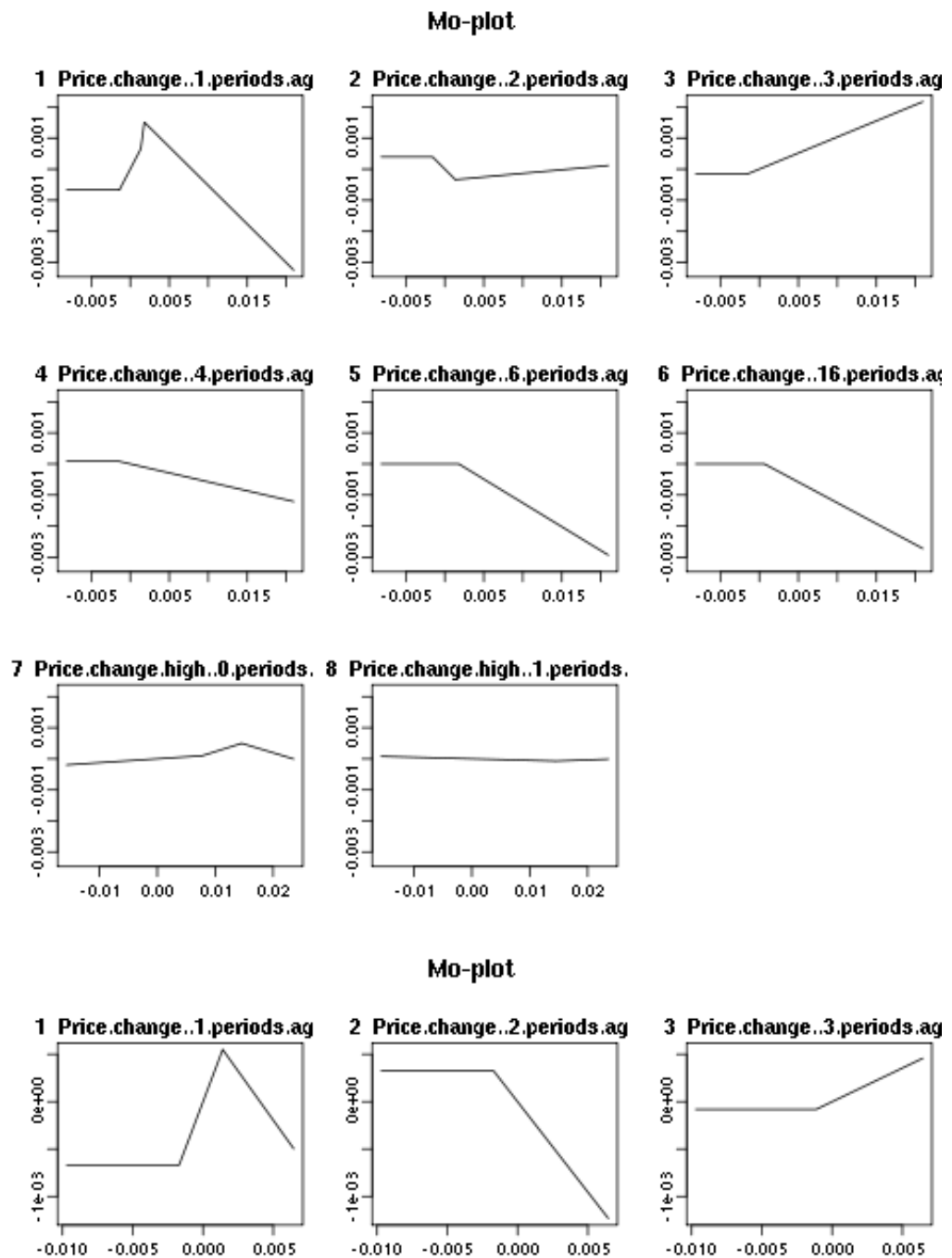


Figure 3: Influence of independent variables on the dependent variable in the new model for intraday trading.

Table 7:  $R^2$  of the estimated new "market makers" models, in-sample and out-of-sample data, in %.

Currency pair	$R^2$ , in-sample	$R^2$ , out-of-sample	Currency pair	$R^2$ , in-sample	$R^2$ , out-of-sample
AUDCAD	10.96	13.04	EURZAR	-	-
AUDCHF	13.48	16.97	GBPAUD	15.37	17.20
AUDJPY	20.08	20.52	GBPCAD	13.98	16.71
AUDNZD	7.59	7.81	GBPCHF	18.64	19.02
AUDUSD	16.53	17.80	GBPJPY	23.35	22.18
CADCHF	12.67	15.67	GBPNZD	-	-
CADJPY	18.28	20.59	GBPUSD	21.17	19.81
CHFJPY	17.29	18.34	NOKJPY	-	-
CHFSGD	9.48	11.97	NZDCAD	18.18	1.78
EURAUD	16.11	16.04	NZDCHF	10.39	8.69
EURCAD	14.20	15.36	NZDJPY	15.58	17.18
EURCHF	13.37	15.56	NZDUSD	10.14	11.98
EURGBP	14.83	15.32	USDCAD	14.15	18.81
EURJPY	22.48	21.95	USDCHF	15.88	22.19
EURNZD	12.10	14.23	USDJPY	15.79	8.66
EURSGD	13.70	15.82	USDTRY	-	-
EURTRY	-	-	USDZAR	64.26	
EURUSD	18.89	23.70			

As can be observed, the previous changes in price have the influence on the current change in price most of time only in their second part. In other words, increases in previous changes in price have more impact on the present change in price, than the decreases of them. This relationship has not been explored in details so far, but it is definitely another interesting research topic.

### 3 Concluding remarks

Step-by-step exploring and testing different aspects of the spot foreign exchange market, this research proposed a new model describing exchange rates, intended for intraday trading.

Two variants of this model were developed: one for market makers observing the order flow, one for traders who do not have this information. Both variants of the model have the same modeling quality, which is as good in-sample as out-of-sample. The  $R^2$  of this model is higher than the  $R^2$  of any other model tested here on the minute data.

Finally, some open questions provide room and ideas for further researched. In particular, the predicting power of the model can be further improved. Also, the observed asymmetric influence in positive and negative changes in previous observed price innovations is to be explored further.

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# FIXED POINT RESULTS FOR MAPPINGS SATISFYING A GENERAL CONTRACTIVE CONDITION OF OPERATOR TYPE IN DISLOCATED FUZZY QUASI-METRIC SPACES

CIHANGIR ALACA

**ABSTRACT.** In this paper, we define the notion of dislocated fuzzy quasi-metric spaces with the help of Hitzler and Seda [18] in the sense of Kramosil and Michalek [19] and also George and Veeramani [14]. Further, we give some fixed point results for coincidentally bi-commuting mappings satisfying a general contractive condition of operator type in dislocated fuzzy quasi-metric spaces.

## 1. INTRODUCTION

Zadeh's introduction [29] of the notion of fuzzy sets laid down the foundation of fuzzy mathematics. Many authors have introduced the fundamental concepts of fuzzy metric space and its fixed point theorems in different ways [1, 4, 10, 11, 12, 14, 15, 16, 19, 21, 22, 25, 26]. Furthermore, many authors obtained different fixed point results for mappings satisfying a general contractive condition of operator type and integral type in metric spaces [2, 3, 5, 6, 8, 9, 23, 24, 28].

Recently, the following definition of dislocated metric space and its fundamental properties was given by Hitzler and Seda [18].

**Definition 1.** Let  $X$  be a set and let  $\varrho : X \times X \rightarrow \mathbb{R}_0^+$  be a function, called a distance function. Consider the following conditions:

(D-i) For all  $x \in X$ ,  $\varrho(x, x) = 0$ , (D-ii) For all  $x, y \in X$ , if  $\varrho(x, y) = 0$ , then  $x = y$ , (D-iii) For all  $x, y \in X$ ,  $\varrho(x, y) = \varrho(y, x)$ , (D-iv) For all  $x, y, z \in X$ ,  $\varrho(x, y) \leq \varrho(x, z) + \varrho(z, y)$ , (D-v) For all  $x, y, z \in X$ ,  $\varrho(x, y) \leq \max \{ \varrho(x, z), \varrho(z, y) \}$ . If  $\varrho$  satisfies conditions (D-i) to (D-iv), then it is called a metric. If it satisfies conditions (D-i), (D-iii) and (D-iv), it is called a pseudo-metric. If it satisfies (D-ii), (D-iii) and (D-iv), we will call it a dislocated metric (or simply d-metric). If a (pseudo-, d-) metric satisfies the strong triangle inequality (D-iv'), then it is called a (pseudo-, d-) ultrametric.

The study of partial metric spaces and generalized ultrametric spaces have applications in theoretical computer science had been studied by Matthews [20]. Hitzler and Seda [18] introduced the concept of dislocated metric space as a generalization of metrics where self-distances need not be zero. They also proved a generalized version of Banach contraction mapping principle which was applied to obtain fixed point semantics for logic programs. Zeyada et al. [30] gave fixed point theorems for

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a multivalued function in complete dislocated metric spaces and complete partial metric space. Later afterwards, George and Khan [13] introduced the concept of dislocated fuzzy metric space and studied the fuzzy topology associated with it.

In the present paper, we define the notion of dislocated fuzzy quasi-metric spaces with the help of Hitzler and Seda [18] in the sense of Kramosil and Michalek [19] and also George and Veeramani [14]. Further, we give some fixed point results for coincidentally bi-commuting mappings satisfying a general contractive condition of operator type in dislocated fuzzy quasi-metric spaces.

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**Definition 2** ([27]). A binary operation  $\star : [0, 1] \times [0, 1] \rightarrow [0, 1]$  is continuous  $t$ -norm if  $([0, 1], \star)$  is a topological monoid with unit 1 such that  $a \star b \leq c \star d$  whenever  $a \leq c$  and  $b \leq d$ , and  $a, b, c, d \in [0, 1]$ .

Now, we define the notion of dislocated fuzzy quasi-metric spaces in the sense of Kramosil and Michalek [19].

**Definition 3.** A 3-tuple  $(X, M, \star)$  is said to be a dislocated fuzzy quasi-metric spaces in the sense of Kramosil and Michalek (in shortly  $D_{KM}$ -FqM-spaces) if  $X$  is an arbitrary set,  $\star$  is a continuous  $t$ -norm and  $M$  is a fuzzy set on  $X^2 \times [0, \infty)$  satisfying the following conditions: For all  $x, y, z \in X$  and  $s, t \in [0, \infty)$ ,

- (D-FqM-1)  $M(x, y, 0) = 0$ ,
- (D-FqM-2)  $M(x, y, t) = M(y, x, t) = 1$ , then  $x = y$ ,
- (D-FqM-3)  $M(x, y, t) \star M(y, z, s) \leq M(x, z, t + s)$ ,
- (D-FqM-4)  $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$  is left continuous,
- (D-FqM-5)  $\lim_{t \rightarrow \infty} M(x, y, t) = 1$ .

**Remark 1.** If we add the condition  $M(x, y, t) = M(y, x, t)$  to Definition 3, then we obtain the definition of  $D_{KM}$ -FM-spaces given by George and Khan [13].

Now, we define the notion of dislocated fuzzy quasi-metric spaces in the sense of George and Veeramani [14].

**Definition 4.** A 3-tuple  $(X, M, \star)$  is said to be a  $D$ -FqM-spaces in the sense of George and Veeramani (in shortly  $D_{GV}$ -FqM-spaces) if  $X$  is an arbitrary set,  $\star$  is a continuous  $t$ -norm and  $M$  is a fuzzy set in  $X^2 \times (0, \infty)$  satisfying the following conditions: For all  $x, y, z \in X$  and  $s, t \in (0, \infty)$ ,

- (D-FqM-1)  $M(x, y, t) > 0$ ,
- (D-FqM-2)  $M(x, y, t) = M(y, x, t) = 1$ , then  $x = y$ ,
- (D-FqM-3)  $M(x, y, t) \star M(y, z, s) \leq M(x, z, t + s)$ ,
- (D-FqM-4)  $M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$  is continuous.

**Remark 2.** Given a  $D$ -FqM-space  $(X, M, \star)$  we define the open ball  $B_M(x, r, t)$ , for  $x \in X$ ,  $0 < r < 1$ , and  $t > 0$ , as the set  $B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$ . Obviously,  $x \in B_M(x, r, t)$ . For each  $x \in X$ ,  $0 < r_1 < r_2 < 1$  and  $0 < t_1 < t_2 < 1$ , we have  $B_M(x, r_1, t_1) \subset B_M(x, r_2, t_2)$ . Consequently, we may define a topology  $\tau_M$  on  $X$  as

$$\tau_M = \{A \subseteq X : \text{for each } x \in A \text{ there are } r \in (0, 1), t > 0, \text{ with } B_M(x, r, t) \subseteq A\}.$$

Moreover, for each  $x \in X$  the collection of open balls  $\{B_M(x, \frac{1}{n}, \frac{1}{n}) : n = 2, 3, \dots\}$ , is a local base at  $x$  with respect to  $\tau_M$ . The topology  $\tau_M$  is called the topology generated

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by  $D$ -FqM-space  $(X, M, \star)$ . It is clear that each open ball  $B_M(x, r, t)$  is an open set for the topology  $\tau_M$ .

**Remark 3.** If  $(M, \star)$  is  $D_{GV}$ -FqM on  $X$ , then  $(M^{-1}, \star)$  is also a  $D_{GV}$ -FqM on  $X$ , where  $M^{-1}$  is the fuzzy set  $X^2 \times (0, \infty)$  defined by  $M^{-1}(x, y, t) = M(y, x, t)$ . Moreover, if we denote by  $M^\dagger$  the fuzzy set in  $X^2 \times (0, \infty)$  given by  $M^\dagger(x, y, t) = \min \{M(x, y, t), M^{-1}(x, y, t)\}$ , then  $(M^\dagger, \star)$  is a  $D_{GV}$ -FqM on  $X$ . We can give similar this case for  $D_{KM}$ -FqM-spaces.

**Example 1.** Let  $(X, d)$  be a dislocated (quasi-)metric space, let  $\star$  be any continuous  $t$ -norm and let  $M_d$  be the function defined on  $X^2 \times (0, \infty)$  by  $M_d = \frac{t}{t+d(x, y)}$ . Then  $(X, M_d, \star)$  is a  $D_{GV}$ -FqM-space called standard  $D_{GV}$ -FqM and  $(M_d, \star)$  is the  $D$ -FqM induced by  $d$ . Furthermore, it is easy to check that  $(M_d)^{-1} = M_{d^{-1}}$  and  $(M_d)^\dagger = M_{d^s}$ , and the topology  $\tau_d$ , generated by  $d$ , coincides with the topology  $\tau_{M_d}$  generated by the induced  $D_{GV}$ -FqM  $(M, \star)$ .

As in Grabiec [15], the following definition can be given.

**Definition 5.** Let  $(X, M, \star)$  be a  $D_{KM}$ -FqM-space. A sequence  $\{x_n\}$  in  $X$  is said to be **(i)** a bi-convergent to a point  $x \in X$  (denoted by  $\lim_{n \rightarrow \infty} x_n = x$ ) if  $\lim_{n \rightarrow \infty} M^\dagger(x_n, x, t) = 1$ , for all  $t > 0$ , **(ii)** a bi-Cauchy sequence if  $\lim_{n \rightarrow \infty} M^\dagger(x_{n+p}, x_n, t) = 1$ , for all  $t > 0$ ,  $p > 0$ , **(iii)** a bi-complete  $D_{KM}$ -FqM-space in which every bi-Cauchy sequence converges to a point in it.

As in George and Veeramani [14], the following definition can be given.

**Definition 6.** **(i)** A sequence  $\{x_n\}$  in a  $D_{GV}$ -FqM-space  $(X, M, \star)$  bi-converges to  $x \in X$  iff, for any  $\varepsilon \in (0, 1)$ ,  $t > 0$ , there exists  $n_0 \in \mathbb{N}$  such that  $M^\dagger(x_n, x, t) > 1 - \varepsilon$  for all  $n \geq n_0$ . **(ii)** A sequence  $\{x_n\}$  in a  $D_{GV}$ -FqM-space  $(X, M, \star)$  is a bi-Cauchy sequence iff for each  $\varepsilon \in (0, 1)$ ,  $t > 0$  there exists  $n_0 \in \mathbb{N}$  such that  $M^\dagger(x_n, x_m, t) > 1 - \varepsilon$  for all  $n, m \geq n_0$ . **(iii)** A  $D_{GV}$ -FqM-space  $(X, M, \star)$  in which every bi-Cauchy sequence in  $X$  is called bi-complete  $D_{GV}$ -FqM-space.

**Definition 7.** Let  $(X, M, \star)$  be a  $D$ -FqM-space,  $f$  and  $g$  be self maps of  $X$ .  $f$  and  $g$  are said to be bi-commute at  $x \in X$ , iff  $M^\dagger(fgx, gfx, t) = 1$  for all  $t \in [0, \infty)$ . If  $f$  and  $g$  bi-commute at all  $x \in X$ , then we say that  $f$  and  $g$  are bi-commuting on  $X$ .

**Definition 8.** Mappings  $f$  and  $g$  are said to be coincidentally bi-commuting iff they bi-commute at all the coincidence points of  $f$  and  $g$ .

**Remark 4.** There exists mappings  $f$  and  $g$  which bi-commutes at some coincidence points but do not bi-commute at all coincidence points of  $f$  and  $g$ .

## 3. FIXED POINT RESULTS FOR DISLOCATED FUZZY QUASI-METRIC SPACES

Now in this section, we prove a fuzzy version of Banach contraction mapping principle for coincidentally bi-commuting mappings satisfying a general contractive condition of operator type in dislocated fuzzy quasi-metric spaces.

Similar results for the following concept of  $O(\varphi; \cdot)$  and its similar examples was given by Altun and Turkoglu [7].

Let  $\Phi([0, \infty))$  be class of all function  $\varphi : [0, \infty) \rightarrow [0, \infty]$  and let  $\Xi$  be class of all operators

$$O(\bullet; \cdot) : \Phi([0, \infty)) \rightarrow \Phi([0, \infty)), \quad \varphi \rightarrow O(\varphi; \cdot)$$

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satisfying the following conditions:

- (i)  $O(\varphi; t) > 0$  for  $t > 0$  and  $O(\varphi; 0) = 0$ ,
- (ii)  $O(\varphi; t) \leq O(\varphi; s)$  for  $t \leq s$ ,
- (iii)  $\lim_{n \rightarrow \infty} O(\varphi; t_n) = O\left(\varphi; \lim_{n \rightarrow \infty} t_n\right)$ ,
- (iv)  $O(\varphi; \max\{t, s\}) = \max\{O(\varphi; t), O(\varphi; s)\}$  for some  $\varphi \in \Phi([0, 1])$ .

**Example 2.** If  $\varphi : [0, \infty) \rightarrow [0, \infty]$  is a Lebesgue integrable mapping which is finite integral on each compact subset of  $[0, \infty)$ , non-negative and such that for each  $t > 0$ ,  $\int_0^t \varphi(s) ds > 0$ , then the operator defined by

$$O(\varphi; t) = \int_0^t \varphi(s) ds$$

satisfies the conditions (i)-(iv).

**Example 3.** If  $\varphi : [0, \infty) \rightarrow [0, \infty)$  non-decreasing, continuous function such that  $\varphi(0) = 0$  and  $\varphi(t) > 0$  for  $t > 0$ , then the operator defined by

$$O(\varphi; t) = \frac{\varphi(t)}{1 + \varphi(t)}$$

satisfies the conditions (i)-(iv).

**Example 4.** If  $\varphi : [0, \infty) \rightarrow [0, \infty)$  non-decreasing, continuous function such that  $\varphi(0) = 0$  and  $\varphi(t) > 0$  for  $t > 0$ , then the operator defined by

$$O(\varphi; t) = \frac{\varphi(t)}{1 + \ln(1 + \varphi(t))}$$

satisfies the conditions (i)-(iv).

**Theorem 1.** Let  $(X, M^\dagger, \star)$  be a D-FqM-space and let  $f, g : X \rightarrow X$  be mappings that satisfy the following conditions:

- (4.1)  $f(X) \subset g(X)$ ,
- (4.2) one of  $f(X)$  or  $g(X)$  is bi-complete,
- (4.3)

$$O(\varphi; 1 - M^\dagger(fx, fy, t)) \leq \alpha O(\varphi; 1 - M^\dagger(gx, gy, t))$$

where  $O(\bullet; \cdot) \in \Xi$ . Then  $f$  and  $g$  have a coincidence point. Further if  $f$  and  $g$  bi-commute at some coincidence point, then  $f$  and  $g$  have a unique common fixed point.

*Proof.* Let  $x_0 \in X$ . Since  $f(X) \subset g(X)$ , choose  $x_1 \in X$  such that  $y_1 = fx_0 = gx_1$ . By induction we can form the sequence  $\{y_n\}$  such that  $y_{n+1} = fx_n = gx_{n+1}$ ,  $n = 0, 1, 2, \dots$  and  $y_0 = gx_0$ . For  $0 < \alpha < 1$  and  $t \in [0, \infty)$  we have

$$\begin{aligned} O(\varphi; 1 - M^\dagger(y_1, y_2, t)) &= O(\varphi; 1 - M^\dagger(fx_0, fx_1, t)) \\ &\leq \alpha O(\varphi; 1 - M^\dagger(gx_0, gx_1, t)) \\ &= O(\varphi; 1 - M^\dagger(y_0, y_1, t)) \end{aligned}$$

and

$$\begin{aligned} O(\varphi; 1 - M^\dagger(y_2, y_3, t)) &= O(\varphi; 1 - M^\dagger(fx_1, fx_2, t)) \\ &\leq \alpha O(\varphi; 1 - M^\dagger(gx_1, gx_2, t)) \\ &= O(\varphi; 1 - M^\dagger(y_1, y_2, t)). \end{aligned}$$



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Thus we have

$$O(\varphi; 1 - M^\dagger(y_2, y_3, t)) \leq \alpha^2 O(\varphi; 1 - M^\dagger(y_0, y_1, t)).$$

Proceeding this way and from (4.3), we have

$$(4.4) \quad O(\varphi; 1 - M^\dagger(y_n, y_{n+1}, t)) \leq \alpha^n O(\varphi; 1 - M^\dagger(y_0, y_1, t)).$$

Taking the limit of (4.4), as  $n \rightarrow \infty$ , gives

$$\lim_{n \rightarrow \infty} O(\varphi; 1 - M^\dagger(y_n, y_{n+1}, t)) = 0,$$

which, from (i), implies that

$$\lim_{n \rightarrow \infty} (1 - M^\dagger(y_n, y_{n+1}, t)) = 0.$$

That is, we have  $\lim_{n \rightarrow \infty} M^\dagger(y_n, y_{n+1}, t) = 1$ . Now we show that  $\{y_n\}$  is a bi-Cauchy sequence. Suppose that it is not. Then there exists an  $\varepsilon > 0$  and subsequences  $\{m(v)\}$  and  $\{n(v)\}$  such that  $m(v) < n(v) < m(v+1)$  with

$$(4.5) \quad 1 - M^\dagger(y_{m(v)}, y_{n(v)}, t_1 + t_2) \geq \varepsilon, \quad 1 - M^\dagger(y_{m(v)}, y_{n(v)-1}, t_1 + t_2) < \varepsilon$$

Since  $\lim_{n \rightarrow \infty} M^\dagger(y_n, y_{n+1}, t) = 1$ , we have

$$\lim_{v \rightarrow \infty} M^\dagger(y_{n(v)-1}, y_{n(v)}, t) = \lim_{v \rightarrow \infty} M^\dagger(y_{m(v)-1}, y_{m(v)}, t) = 1.$$

Using (D-FqM-3), we have

$$\begin{aligned} M^\dagger(y_{m(v)-1}, y_{n(v)-1}, t_1 + t_2) &\geq M^\dagger(y_{m(v)-1}, y_{m(v)}, t_1) \star M^\dagger(y_{m(v)}, y_{n(v)-1}, t_2) \\ &> M^\dagger(y_{m(v)-1}, y_{m(v)}, t_1) \star 1 - \varepsilon \end{aligned}$$

or  $1 - M^\dagger(y_{m(v)-1}, y_{n(v)-1}, t_1 + t_2) < \varepsilon$  as  $v \rightarrow \infty$ . Therefore, from (4.3), we get

$$\begin{aligned} &\alpha O(\varphi; 1 - M^\dagger(y_{m(v)-1}, y_{n(v)-1}, t_1 + t_2)) \\ &\geq O(\varphi; 1 - M^\dagger(y_{m(v)}, y_{n(v)}, t_1 + t_2)) \\ &\geq O(\varphi; \varepsilon) \end{aligned}$$

which implies  $\alpha O(\varphi; \varepsilon) \geq O(\varphi; \varepsilon)$  as  $v \rightarrow \infty$  a contradiction, since  $0 < \alpha < 1$  and  $\varepsilon > 0$ . Therefore,  $\{y_n\}$  is a bi-Cauchy sequence in  $X$ . From (4.2), since  $g(X)$  is bi-complete. Then there exists  $u \in g(X)$  such that  $\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_{n+1} = u = \lim_{n \rightarrow \infty} y_{n+1}$ . Since  $u \in g(X)$  therefore, there exists a point  $p \in X$  such that  $gp = u$ .

Now we show that  $fp = gp = u$ . If possible  $fp \neq gp$ , by inequality (4.3), we have

$$O(\varphi; 1 - M^\dagger(fp, gp, t)) = \alpha O(\varphi; 1 - M^\dagger(fp, f x_n, t))$$

for each  $x, y \in X$ ,  $0 \leq \alpha < 1$ . Letting  $n \rightarrow \infty$ , we have

$$\begin{aligned} O(\varphi; 1 - M^\dagger(fp, gp, t)) &= O\left(\varphi; 1 - \lim_{n \rightarrow \infty} M^\dagger(fp, f x_n, t)\right) \\ &\leq \alpha O\left(\varphi; 1 - \lim_{n \rightarrow \infty} M^\dagger(gp, g x_n, t)\right) \\ &= \alpha O\left(\varphi; 1 - \lim_{n \rightarrow \infty} M^\dagger(u, g x_n, t)\right) \end{aligned}$$

Proceeding this way from (4.1) and (4.3), we have

$$(4.6) \quad O(\varphi; 1 - M^\dagger(fp, gp, t)) \leq \alpha^n O\left(\varphi; 1 - \lim_{n \rightarrow \infty} M^\dagger(u, g x_n, t)\right)$$

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Taking the limit of (4.6), as  $n \rightarrow \infty$ , gives

$$\lim_{n \rightarrow \infty} O(\varphi; 1 - M^\dagger(fp, gp, t)) = 0,$$

which, from (i), implies that

$$\lim_{n \rightarrow \infty} (1 - M^\dagger(fp, gp, t)) = 0.$$

That is, we have  $\lim_{n \rightarrow \infty} M^\dagger(fp, gp, t) = 1$ . Therefore from (D-FqM-2), we have  $fp = gp$  i.e.  $f$  and  $g$  bi-commute at some coincidence point say  $z$ , i.e.  $M^\dagger(fgz, gfz, t) = 1$   $\forall t > 0 \Rightarrow fgz = gfz$  by (D-FqM-2). Let  $fz = gz = k$ . Then we get

$$\begin{aligned} O(\varphi; 1 - M^\dagger(fk, k, t)) &\leq O(\varphi; 1 - M^\dagger(fk, fz, t)) \\ &\leq O(\varphi; 1 - M^\dagger(gk, gz, t)) = O(\varphi; 1 - M^\dagger(gfz, fz, t)) \\ &= O(\varphi; 1 - M^\dagger(fgz, fz, t)) \leq \alpha O(\varphi; 1 - M^\dagger(gk, gz, t)) \\ &\dots \\ &\leq \alpha^n O(\varphi; 1 - M^\dagger(gk, gz, t)). \end{aligned}$$

Then we get

$$(4.7) \quad O(\varphi; 1 - M^\dagger(fk, k, t)) \leq \alpha^n O(\varphi; 1 - M^\dagger(gk, gz, t)).$$

Taking the limit of (4.7), as  $n \rightarrow \infty$ , gives

$$\lim_{n \rightarrow \infty} O(\varphi; 1 - M^\dagger(fk, k, t)) = 0,$$

which, from (i), implies that

$$\lim_{n \rightarrow \infty} (1 - M^\dagger(fk, k, t)) = 0.$$

That is, we have  $\lim_{n \rightarrow \infty} M^\dagger(fk, k, t) = 1$ . Therefore from (D-FqM-2), we have  $fk = k$ . Similarly we can show that  $gk = k$ . That is  $k$  is a common fixed point of  $f$  and  $g$ . A similarly proof follows if  $f(X)$  is bi-complete.

**Uniqueness:** Suppose  $w$  is another common fixed point of  $f$  and  $g$ . Then we have

$$\begin{aligned} O(\varphi; 1 - M^\dagger(k, w, t)) &= O(\varphi; 1 - M^\dagger(fk, fw, t)) \\ &\leq \alpha O(\varphi; 1 - M^\dagger(gk, gw, t)) \\ &\leq \alpha O(\varphi; 1 - M^\dagger(k, w, t)) \\ &= \alpha O(\varphi; 1 - M^\dagger(fk, fw, t)) \\ &\leq \alpha^2 O(\varphi; 1 - M^\dagger(gk, gw, t)) \\ &= \alpha^2 O(\varphi; 1 - M^\dagger(k, w, t)) \\ &\dots \\ &\leq \alpha^n O(\varphi; 1 - M^\dagger(k, w, t)). \end{aligned}$$

Then we get

$$(4.8) \quad O(\varphi; 1 - M^\dagger(k, w, t)) \leq \alpha^n O(\varphi; 1 - M^\dagger(k, w, t)).$$

Taking the limit of (4.8), as  $n \rightarrow \infty$ , gives

$$\lim_{n \rightarrow \infty} O(\varphi; 1 - M^\dagger(k, w, t)) = 0,$$

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which, from (i), implies that

$$\lim_{n \rightarrow \infty} (1 - M^\dagger(k, w, t)) = 0.$$

That is, we have  $\lim_{n \rightarrow \infty} M^\dagger(k, w, t) = 1$ . Therefore from (D-FqM-2), we have  $k = w$ . Hence the common fixed point is unique.  $\square$

In Theorem 1, if we take  $g = I_X$ , the identity mapping on  $X$ , then we have the following.

**Theorem 2.** *Let  $(X, M^\dagger, \star)$  be a D-FqM-space and let  $f : X \rightarrow X$  be mappings that satisfy the following conditions:*

- (i)  $f(X)$  is bi-complete,
- (ii) For each  $x, y \in X$ ,  $0 \leq \alpha < 1$ ,  $t \in [0, \infty)$

$$O(\varphi; 1 - M^\dagger(fx, fy, t)) \leq \alpha O(\varphi; 1 - M^\dagger(x, y, t))$$

where  $O(\bullet; \cdot) \in \Xi$ . Then  $f$  has a unique fixed point.

**Remark 5.** *It is clear that Theorem 1 is a generalization of Theorem 3.1 in [13].*

**Remark 6.** *We can have new result, if we combine Theorem 1 and some examples for  $O(f; \cdot)$ .*

**Remark 7.** *If we combine Example 2 and Theorem 1, then we obtain a new generalized fuzzy version of Banach contraction mapping principle satisfying a general contractive condition of operator type or integral type in D-FqM-spaces.*

**Conclusion.** *The notion of dislocated metric is useful in the context of electronic engineering (see [17]). In this work we prove a common fixed point for coincidentally bi-commuting mappings satisfying a general contractive condition of operator type in D-FqM-spaces. A few applications of dislocated metrics and in particular of the generalized Banach contraction mapping principle, are known in Theoretical Computer Science, it is at this stage unclear, with investigating whether or not other applications can be found and where else in Fuzzy Mathematics these spaces appear. The scientists who went to study in this area can investigate the results of the fixed point theory for another types of contraction mappings.*

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# A note on a certain subclass of analytic functions defined by multiplier transformation

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## Abstract

In the present paper we define a new operator, by means of convolution product between Ruscheweyh operator and the multiplier transformation  $I(m, \lambda, l)$ . For functions  $f$  belonging to the class  $A$  we define the differential operator  $IR_{\lambda,l}^m : A \rightarrow A$ ,  $IR_{\lambda,l}^m f(z) := (I(m, \lambda, l) * R^m) f(z)$  where  $A_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots, z \in U\}$  is the class of normalized analytic functions with  $A_1 = A$ . We study certain differential subordinations regarding the operator  $IR_{\lambda,l}^m$ .

**Keywords:** differential subordination, convex function, best dominant, differential operator, convolution product.

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## 1 Introduction

Denote by  $U$  the unit disc of the complex plane,  $U = \{z \in \mathbb{C} : |z| < 1\}$  and  $\mathcal{H}(U)$  the space of holomorphic functions in  $U$ .

Let

$$A(p, n) = \{f \in \mathcal{H}(U) : f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, z \in U\},$$

with  $A(1, n) = A_n$  and  $A(1, 1) = A_1 = A$ , where  $p, n \in \mathbb{N}$ .

Denote by

$$K = \left\{ f \in A : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, z \in U \right\}$$

the class of normalized convex functions in  $U$ .

If  $f$  and  $g$  are analytic functions in  $U$ , we say that  $f$  is subordinate to  $g$ , written  $f \prec g$ , if there is a function  $w$  analytic in  $U$ , with  $w(0) = 0$ ,  $|w(z)| < 1$ , for all  $z \in U$  such that  $f(z) = g(w(z))$  for all  $z \in U$ . If  $g$  is univalent, then  $f \prec g$  if and only if  $f(0) = g(0)$  and  $f(U) \subseteq g(U)$ .

Let  $\psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$  and  $h$  an univalent function in  $U$ . If  $p$  is analytic in  $U$  and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad \text{for } z \in U, \quad (1)$$

then  $p$  is called a solution of the differential subordination. The univalent function  $q$  is called a dominant of the solutions of the differential subordination, or more simply a dominant, if  $p \prec q$  for all  $p$  satisfying (1).

A dominant  $\tilde{q}$  that satisfies  $\tilde{q} \prec q$  for all dominants  $q$  of (1) is said to be the best dominant of (1). The best dominant is unique up to a rotation of  $U$ .

**Definition 1** [4] For  $f \in A(p, n)$ ,  $p, n \in \mathbb{N}$ ,  $m \in \mathbb{N} \cup \{0\}$ ,  $\lambda, l \geq 0$ , the operator  $I_p(m, \lambda, l)f(z)$  is defined by the following infinite series

$$I_p(m, \lambda, l)f(z) := z^p + \sum_{j=p+n}^{\infty} \left( \frac{p + \lambda(j-1) + l}{p+l} \right)^m a_j z^j.$$

**Remark 1** It follows from the above definition that

$$I_p(0, \lambda, l)f(z) = f(z),$$

$$(p+l)I_p(m+1, \lambda, l)f(z) = [p(1-\lambda) + l]I_p(m, \lambda, l)f(z) + \lambda z(I_p(m, \lambda, l)f(z))', \quad \text{for } z \in U.$$

**Remark 2** If  $p = 1$ ,  $n = 1$ , we have  $A(1, 1) = A_1 = A$ ,  $I_1(m, \lambda, l)f(z) = I(m, \lambda, l)$  and

$$(l+1)I(m+1, \lambda, l)f(z) = [l+1-\lambda]I(m, \lambda, l)f(z) + \lambda z(I(m, \lambda, l)f(z))', \quad \text{for } z \in U.$$

**Remark 3** If  $f \in A$ ,  $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ , then  $I(m, \lambda, l)f(z) = z + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m a_j z^j$ , for  $z \in U$ .

**Remark 4** For  $l = 0$ ,  $\lambda \geq 0$ , the operator  $D_{\lambda}^m = I(m, \lambda, 0)$  was introduced and studied by Al-Oboudi, which reduced to the Sal agean differential operator  $S^m = I(m, 1, 0)$  for  $\lambda = 1$ .

**Definition 2** (Ruscheweyh [6]) For  $f \in A$ ,  $m \in \mathbb{N}$  the operator  $R^m$  is defined by  $R^m : A \rightarrow A$ ,

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z) \\ &\dots \\ (m+1)R^{m+1}f(z) &= z(R^m f(z))' + mR^m f(z), \quad \text{for } z \in U. \end{aligned}$$

**Remark 5** If  $f \in A$ ,  $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ , then  $R^m f(z) = z + \sum_{j=2}^{\infty} C_{m+j-1}^m a_j z^j$ , for  $z \in U$ .

**Lemma 3** (Miller and Mocanu [5]) Let  $g$  be a convex function in  $U$  and let

$$h(z) = g(z) + n\alpha z g'(z), \quad \text{for } z \in U,$$

where  $\alpha > 0$  and  $n$  is a positive integer.

If

$$p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \dots, \quad \text{for } z \in U$$

is holomorphic in  $U$  and

$$p(z) + \alpha z p'(z) \prec h(z), \quad \text{for } z \in U$$

then

$$p(z) \prec g(z)$$

and this result is sharp.

## 2 Main Results

**Definition 4** Let  $m, \lambda, l \in \mathbb{N}$ . Denote by  $IR_{\lambda, l}^m$  the operator given by the Hadamard product (the convolution product) of the operator  $I(m, \lambda, l)$  and the Ruscheweyh operator  $R^m$ ,  $IR_{\lambda, l}^m : A \rightarrow A$ ,

$$IR_{\lambda, l}^m f(z) = (I(m, \lambda, l) * R^m) f(z).$$

**Remark 6** If  $f \in A$ ,  $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ , then  $IR_{\lambda, l}^m f(z) = z + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m a_j^2 z^j$ , for  $z \in U$ .

**Remark 7** For  $l = 0$ ,  $\lambda \geq 0$ , we obtain the Hadamard product  $DR_{\lambda}^n$  [2] of the generalized Sălăgean operator  $D_{\lambda}^n$  and Ruscheweyh operator  $R^n$ .

For  $l = 0$  and  $\lambda = 1$ , we obtain the Hadamard product  $SR^n$  [1] of the Sălăgean operator  $S^n$  and Ruscheweyh operator  $R^n$ .

**Theorem 5** Let  $g$  be a convex function such that  $g(0) = 1$  and let  $h$  be the function  $h(z) = g(z) + zg'(z)$ , for  $z \in U$ . If  $m, \lambda, l \in \mathbb{N}$ ,  $f \in A$  and the differential subordination

$$\begin{aligned} & \frac{1}{z} \left( \frac{m+1}{\lambda} IR_{\lambda,l}^{m+1} f(z) - \frac{m-2}{\lambda} IR_{\lambda,l}^m f(z) \right) + \frac{\lambda(m-1)-(l+1)}{\lambda(l+1)} (IR_{\lambda,l}^m f(z))' + \\ & \left( 1 - \frac{m-1}{l+1} - \frac{2}{\lambda} \right) - \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)} \int_0^z \frac{IR_{\lambda,l}^m f(t) - t}{t^2} dt \prec h(z), \quad \text{for } z \in U \end{aligned} \quad (2)$$

holds, then

$$(IR_{\lambda,l}^m f(z))' \prec g(z), \quad \text{for } z \in U$$

and this result is sharp.

**Proof.** With notation  $p(z) = (IR_{\lambda,l}^m f(z))' = 1 + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m j a_j^2 z^{j-1}$  and  $p(0) = 1$ , we obtain for  $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ ,

$$\begin{aligned} p(z) + zp'(z) &= 1 + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m j a_j^2 z^{j-1} + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m j(j-1) a_j^2 z^{j-1} = \\ &= 1 + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m j^2 a_j^2 z^{j-1} = \\ &= \frac{1}{z} \left( z + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^{m+1} C_{m+j}^{m+1} \frac{m+1}{\lambda} a_j^2 z^j - \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m \frac{\lambda(m-1)-(l+1)}{\lambda(l+1)} j a_j^2 z^j - \right. \\ & \quad \left. \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m \frac{m-2}{\lambda} a_j^2 z^j - \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m \frac{1}{j-1} \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)} a_j^2 z^j \right) = \\ &= \frac{1}{z} \left[ \frac{m+1}{\lambda} \left( z + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^{m+1} C_{m+j}^{m+1} a_j^2 z^j \right) - \frac{m-2}{\lambda} \left( z + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m a_j^2 z^j \right) \right] + \\ & \quad \left( 1 - \frac{m+1}{\lambda} - \frac{m-2}{\lambda} \right) + \left( 1 + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m a_j^2 j z^{j-1} \right) \frac{\lambda(m-1)-(l+1)}{\lambda(l+1)} - \\ & \quad \frac{\lambda(m-1)-(l+1)}{\lambda(l+1)} - \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m \frac{1}{j-1} \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)} a_j^2 z^{j-1} = \\ &= \frac{1}{z} \left( \frac{m+1}{\lambda} IR_{\lambda,l}^{m+1} f(z) - \frac{m-2}{\lambda} IR_{\lambda,l}^m f(z) \right) + \frac{\lambda(m-1)-(l+1)}{\lambda(l+1)} (IR_{\lambda,l}^m f(z))' + \\ & \quad \frac{\lambda l - \lambda m + 2\lambda - 2l - 2}{\lambda(l+1)} - \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)} \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m \frac{1}{j-1} a_j^2 z^{j-1} = \\ &= \frac{1}{z} \left( \frac{m+1}{\lambda} IR_{\lambda,l}^{m+1} f(z) - \frac{m-2}{\lambda} IR_{\lambda,l}^m f(z) \right) + \frac{\lambda(m-1)-(l+1)}{\lambda(l+1)} (IR_{\lambda,l}^m f(z))' + \left( 1 - \frac{m-1}{l+1} - \frac{2}{\lambda} \right) - \\ & \quad \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)} \int_0^z \frac{IR_{\lambda,l}^m f(t) - t}{t^2} dt. \end{aligned}$$

We have  $p(z) + zp'(z) \prec h(z) = g(z) + zg'(z)$ , for  $z \in U$ . By using Lemma 3 we obtain  $p(z) \prec g(z)$ , for  $z \in U$ , i.e.  $(IR_{\lambda,l}^m f(z))' \prec g(z)$ , for  $z \in U$  and this result is sharp. ■

**Corollary 6** ([2]) Let  $g$  be a convex function such that  $g(0) = 1$  and let  $h$  be the function  $h(z) = g(z) + zg'(z)$ , for  $z \in U$ . If  $\lambda \geq 0$ ,  $n \in \mathbb{N}$ ,  $f \in A$  and the differential subordination

$$\frac{n+1}{\lambda z} DR_{\lambda}^{n+1} f(z) - \frac{n(1-\lambda)}{\lambda z} DR_{\lambda}^n f(z) - \left( n-1 + \frac{1}{\lambda} \right) (DR_{\lambda}^n f(z))' \prec h(z), \quad \text{for } z \in U \quad (3)$$

holds, then

$$(DR_{\lambda}^n f(z))' \prec g(z), \quad \text{for } z \in U$$

and this result is sharp.

**Corollary 7** ([1]) Let  $g$  be a convex function such that  $g(0) = 1$  and let  $h$  be the function  $h(z) = g(z) + zg'(z)$ , for  $z \in U$ . If  $n \in \mathbb{N}$ ,  $f \in A$  and the differential subordination

$$\frac{1}{z}SR^{n+1}f(z) + \frac{n}{n+1}z(SR^n f(z))'' \prec h(z), \quad \text{for } z \in U \quad (4)$$

holds, then

$$(SR^n f(z))' \prec g(z), \quad \text{for } z \in U$$

and this result is sharp.

**Theorem 8** Let  $g$  be a convex function,  $g(0) = 1$  and let  $h$  be the function  $h(z) = g(z) + zg'(z)$ , for  $z \in U$ . If  $m, \lambda, l \in \mathbb{N}$ ,  $f \in A$  and verifies the differential subordination

$$(IR_{\lambda,l}^m f(z))' \prec h(z), \quad \text{for } z \in U, \quad (5)$$

then

$$\frac{IR_{\lambda,l}^m f(z)}{z} \prec g(z), \quad \text{for } z \in U$$

and this result is sharp.

**Proof.** For  $f \in A$ ,  $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$  we have  $IR_{\lambda,l}^m f(z) = z + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m a_j^2 z^j$ , for  $z \in U$ .

$$\text{Consider } p(z) = \frac{IR_{\lambda,l}^m f(z)}{z} = \frac{z + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m a_j^2 z^j}{z} = 1 + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m a_j^2 z^{j-1}.$$

$$\text{We have } p(z) + zp'(z) = \left( IR_{\lambda,l}^m f(z) \right)', \quad \text{for } z \in U.$$

Then  $\left( IR_{\lambda,l}^m f(z) \right)' \prec h(z)$ , for  $z \in U$  becomes  $p(z) + zp'(z) \prec h(z) = g(z) + zg'(z)$ , for  $z \in U$ . By using Lemma 3 we obtain  $p(z) \prec g(z)$ , for  $z \in U$ , i.e.  $\frac{IR_{\lambda,l}^m f(z)}{z} \prec g(z)$ , for  $z \in U$ . ■

**Corollary 9** ([2]) Let  $g$  be a convex function,  $g(0) = 1$  and let  $h$  be the function  $h(z) = g(z) + zg'(z)$ , for  $z \in U$ . If  $n \in \mathbb{N}$ ,  $f \in A$  and verifies the differential subordination

$$(DR_{\lambda}^n f(z))' \prec h(z), \quad \text{for } z \in U, \quad (6)$$

then

$$\frac{DR_{\lambda}^n f(z)}{z} \prec g(z), \quad \text{for } z \in U$$

and this result is sharp.

**Corollary 10** ([1]) Let  $g$  be a convex function,  $g(0) = 1$  and let  $h$  be the function  $h(z) = g(z) + zg'(z)$ , for  $z \in U$ . If  $n \in \mathbb{N}$ ,  $f \in A$  and verifies the differential subordination

$$(SR^n f(z))' \prec h(z), \quad \text{for } z \in U, \quad (7)$$

then

$$\frac{SR^n f(z)}{z} \prec g(z), \quad \text{for } z \in U$$

and this result is sharp.

**Theorem 11** Let  $g$  be a convex function such that  $g(0) = 1$  and let  $h$  be the function  $h(z) = g(z) + zg'(z)$ , for  $z \in U$ . If  $m, \lambda, l \in \mathbb{N}$ ,  $f \in A$  and verifies the differential subordination

$$\left( \frac{zIR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^m f(z)} \right)' \prec h(z), \quad \text{for } z \in U, \quad (8)$$

then

$$\frac{IR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^m f(z)} \prec g(z), \quad \text{for } z \in U$$

and this result is sharp.



**Proof.** For  $f \in A$ ,  $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$  we have  $IR_{\lambda,l}^m f(z) = z + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m a_j^2 z^j$ , for  $z \in U$ .

$$\text{Consider } p(z) = \frac{IR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^m f(z)} = \frac{z + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^{m+1} C_{m+j}^{m+1} a_j^2 z^j}{z + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m a_j^2 z^j} = \frac{1 + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^{m+1} C_{m+j}^{m+1} a_j^2 z^{j-1}}{1 + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m C_{m+j-1}^m a_j^2 z^{j-1}}.$$

$$\text{We have } p'(z) = \frac{(IR_{\lambda,l}^{m+1} f(z))'}{IR_{\lambda,l}^m f(z)} - p(z) \cdot \frac{(IR_{\lambda,l}^m f(z))'}{IR_{\lambda,l}^m f(z)}.$$

$$\text{Then } p(z) + zp'(z) = \left( \frac{z IR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^m f(z)} \right)'.$$

Relation (8) becomes  $p(z) + zp'(z) \prec h(z) = g(z) + zg'(z)$ , for  $z \in U$  and by using Lemma 3 we obtain  $p(z) \prec g(z)$ , for  $z \in U$ , i.e.  $\frac{IR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^m f(z)} \prec g(z)$ , for  $z \in U$ . ■

**Corollary 12** ([2]) Let  $g$  be a convex function such that  $g(0) = 1$  and let  $h$  be the function  $h(z) = g(z) + zg'(z)$ , for  $z \in U$ . If  $n \in \mathbb{N}$ ,  $f \in A$  and verifies the differential subordination

$$\left( \frac{z DR_{\lambda}^{n+1} f(z)}{DR_{\lambda}^n f(z)} \right)' \prec h(z), \quad \text{for } z \in U, \quad (9)$$

then

$$\frac{DR_{\lambda}^{n+1} f(z)}{DR_{\lambda}^n f(z)} \prec g(z), \quad \text{for } z \in U$$

and this result is sharp.

**Corollary 13** ([1]) Let  $g$  be a convex function such that  $g(0) = 1$  and let  $h$  be the function  $h(z) = g(z) + zg'(z)$ , for  $z \in U$ . If  $n \in \mathbb{N}$ ,  $f \in A$  and verifies the differential subordination

$$\left( \frac{z SR_{\lambda}^{n+1} f(z)}{SR_{\lambda}^n f(z)} \right)' \prec h(z), \quad \text{for } z \in U, \quad (10)$$

then

$$\frac{SR_{\lambda}^{n+1} f(z)}{SR_{\lambda}^n f(z)} \prec g(z), \quad \text{for } z \in U$$

and this result is sharp.

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# **Journal of Computational Analysis and Applications**

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# A special comprehensive class of analytic functions defined by multiplier transformation

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## Abstract

For functions belonging to the class  $S_m(\delta, \alpha)$ ,  $\delta \in [0, 1)$ ,  $\alpha \geq 0$  and  $m \in \mathbb{N}$ , of normalized analytic functions in the open unit disc  $U$ , which are investigated in this paper, the author derives several interesting differential subordination results. These subordinations are established by means of a special case of the multiplier transformations  $I_p(m, \lambda, l)f(z)$  namely

$$I_p(m, \lambda, l)f(z) := z^p + \sum_{j=p+n}^{\infty} \left( \frac{p + \lambda(j-1) + l}{p+l} \right)^m a_j z^j,$$

where  $p, n \in \mathbb{N}$ ,  $m \in \mathbb{N} \cup \{0\}$ ,  $\lambda, l \geq 0$  and  $f \in A(p, n)$ ,

$$A(p, n) = \{f \in \mathcal{H}(U) : f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, \quad z \in U\}.$$

A number of interesting consequences of some of these subordination results are discussed. Relevant connections of some of the new results obtained in this paper with those in earlier works are also provided.

**Keywords:** differential subordination, convex function, best dominant, differential operator.

**2000 Mathematical Subject Classification:** 30C45, 30A20, 34A40.

## 1 Introduction

Denote by  $U$  the unit disc of the complex plane,  $U = \{z \in \mathbb{C} : |z| < 1\}$  and  $\mathcal{H}(U)$  the space of holomorphic functions in  $U$ .

Let

$$A(p, n) = \{f \in \mathcal{H}(U) : f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, \quad z \in U\},$$

with  $A(1, n) = A_n$ ,  $A(1, 1) = A_1 = A$  and

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, \quad z \in U\},$$

where  $p, n \in \mathbb{N}$ ,  $a \in \mathbb{C}$ .

Denote by

$$K = \left\{ f \in A : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, \quad z \in U \right\}$$

the class of normalized convex functions in  $U$ .

If  $f$  and  $g$  are analytic functions in  $U$ , we say that  $f$  is subordinate to  $g$ , written  $f \prec g$ , if there is a function  $w$  analytic in  $U$ , with  $w(0) = 0$ ,  $|w(z)| < 1$ , for all  $z \in U$  such that  $f(z) = g(w(z))$  for all  $z \in U$ . If  $g$  is univalent, then  $f \prec g$  if and only if  $f(0) = g(0)$  and  $f(U) \subseteq g(U)$ .

Let  $\psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$  and  $h$  an univalent function in  $U$ . If  $p$  is analytic in  $U$  and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad \text{for } z \in U, \quad (1)$$

then  $p$  is called a solution of the differential subordination. The univalent function  $q$  is called a dominant of the solutions of the differential subordination, or more simply a dominant, if  $p \prec q$  for all  $p$  satisfying (1).

A dominant  $\tilde{q}$  that satisfies  $\tilde{q} \prec q$  for all dominants  $q$  of (1) is said to be the best dominant of (1). The best dominant is unique up to a rotation of  $U$ .

**Definition 1.1.** [4] For  $f \in A(p, n)$ ,  $p, n \in \mathbb{N}$ ,  $m \in \mathbb{N} \cup \{0\}$ ,  $\lambda, l \geq 0$ , the operator  $I_p(m, \lambda, l)f(z)$  is defined by the following infinite series

$$I_p(m, \lambda, l)f(z) := z^p + \sum_{j=p+n}^{\infty} \left( \frac{p + \lambda(j-1) + l}{p+l} \right)^m a_j z^j.$$

**Remark 1.2.** It follows from the above definition that

$$I_p(0, \lambda, l)f(z) = f(z),$$

$$(p+l)I_p(m+1, \lambda, l)f(z) = [p(1-\lambda) + l]I_p(m, \lambda, l)f(z) + \lambda z(I_p(m, \lambda, l)f(z))',$$

for  $z \in U$ .

**Remark 1.3.** If  $p = 1$ ,  $n = 1$ , we have  $A(1, 1) = A_1 = A$ ,  $I_1(m, \lambda, l)f(z) = I(m, \lambda, l)$  and

$$(l+1)I(m+1, \lambda, l)f(z) = [l+1-\lambda]I(m, \lambda, l)f(z) + \lambda z(I(m, \lambda, l)f(z))',$$

for  $z \in U$ .

**Remark 1.4.** If  $f \in A$ ,  $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ , then

$$I(m, \lambda, l)f(z) = z + \sum_{j=2}^{\infty} \left( \frac{1 + \lambda(j-1) + l}{l+1} \right)^m a_j z^j,$$

for  $z \in U$ .

**Remark 1.5.** For  $l = 0$ ,  $\lambda \geq 0$ , the operator  $D_{\lambda}^m = I(m, \lambda, 0)$  was introduced and studied by Al-Oboudi, which reduced to the Sălăgean differential operator  $S^m = I(m, 1, 0)$  for  $\lambda = 1$ .

**Lemma 1.6.** (Hallenbeck and Ruscheweyh [5, Th. 3.1.6, p. 71]) Let  $h$  be a convex function with  $h(0) = a$ , and let  $\gamma \in \mathbb{C}^*$  be a complex number with  $\operatorname{Re} \gamma \geq 0$ . If  $p \in \mathcal{H}[a, n]$  and

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z), \quad \text{for } z \in U,$$

then

$$p(z) \prec g(z) \prec h(z), \quad \text{for } z \in U,$$

where

$$g(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1} dt, \quad \text{for } z \in U.$$

**Lemma 1.7.** (Miller and Mocanu [6]) Let  $g$  be a convex function in  $U$  and let

$$h(z) = g(z) + n\alpha zg'(z), \quad \text{for } z \in U,$$

where  $\alpha > 0$  and  $n$  is a positive integer.

If

$$p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \dots, \quad \text{for } z \in U$$

is holomorphic in  $U$  and

$$p(z) + \alpha zp'(z) \prec h(z), \quad \text{for } z \in U,$$

then

$$p(z) \prec g(z)$$

and this result is sharp.



## 2 Main results

**Definition 2.1.** Let  $\delta \in [0, 1)$ ,  $\alpha \geq 0$  and  $m \in \mathbb{N}$ . A function  $f \in A$  is said to be in the class  $S_m(\delta, \alpha)$  if it satisfies the inequality

$$\operatorname{Re}(I(m, \lambda, l)f(z))' > \delta, \quad \text{for } z \in U. \quad (2)$$

**Theorem 2.2.** The set  $S_m(\delta, \alpha)$  is convex.

*Proof.* Let the functions

$$f_j(z) = z + \sum_{j=2}^{\infty} a_{jk} z^j, \quad \text{for } k = 1, 2, \quad z \in U,$$

be in the class  $S_m(\delta, \alpha)$ . It is sufficient to show that the function

$$h(z) = \eta_1 f_1(z) + \eta_2 f_2(z)$$

is in the class  $S_m(\delta, \alpha)$ , with  $\eta_1$  and  $\eta_2$  nonnegative such that  $\eta_1 + \eta_2 = 1$ .

Since

$$h(z) = z + \sum_{j=2}^{\infty} (\eta_1 a_{j1} + \eta_2 a_{j2}) z^j, \quad \text{for } z \in U,$$

then

$$I(m, \lambda, l)h(z) = z + \sum_{j=2}^{\infty} \left( \frac{1 + \lambda(j-1) + l}{l+1} \right)^m (\eta_1 a_{j1} + \eta_2 a_{j2}) z^j, \quad \text{for } z \in U. \quad (3)$$

Differentiating (3) we obtain

$$(I(m, \lambda, l)h(z))' = 1 + \sum_{j=2}^{\infty} \left( \frac{1 + \lambda(j-1) + l}{l+1} \right)^m (\eta_1 a_{j1} + \eta_2 a_{j2}) j z^{j-1}, \quad \text{for } z \in U.$$

Hence

$$\begin{aligned} \operatorname{Re}(I(m, \lambda, l)h(z))' &= 1 + \operatorname{Re} \left( \eta_1 \sum_{j=2}^{\infty} j \left( \frac{1 + \lambda(j-1) + l}{l+1} \right)^m a_{j1} z^{j-1} \right) \\ &\quad + \operatorname{Re} \left( \eta_2 \sum_{j=2}^{\infty} j \left( \frac{1 + \lambda(j-1) + l}{l+1} \right)^m a_{j2} z^{j-1} \right). \end{aligned} \quad (4)$$

Taking into account that  $f_1, f_2 \in S_m(\delta, \alpha)$  we deduce

$$\operatorname{Re} \left( \eta_k \sum_{j=2}^{\infty} j \left( \frac{1 + \lambda(j-1) + l}{l+1} \right)^m a_{jk} z^{j-1} \right) > \eta_k (\delta - 1), \quad \text{for } k = 1, 2. \quad (5)$$

Using (5) we get from (4)

$$\operatorname{Re}(I(m, \lambda, l)h(z))' > 1 + \eta_1 (\delta - 1) + \eta_2 (\delta - 1), \quad \text{for } z \in U,$$

that is

$$\operatorname{Re}(I(m, \lambda, l)h(z))' > \delta, \quad \text{for } z \in U,$$

which is equivalent that  $S_m(\delta, \alpha)$  is convex. ■

**Theorem 2.3.** Let  $g$  be a convex function in  $U$  and let

$$h(z) = g(z) + \frac{1}{c+2} z g'(z), \quad \text{where } z \in U, c > 0. \quad (6)$$

If  $f \in S_m(\delta, \alpha)$  and  $F(z) = I_c(f)(z)$ , where

$$F(z) = I_c(f)(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t) dt, \quad \text{for } z \in U, \quad (7)$$

then

$$(I(m, \lambda, l) f(z))' \prec h(z), \quad \text{for } z \in U, \quad (8)$$

implies

$$(I(m, \lambda, l) F(z))' \prec g(z), \quad \text{for } z \in U$$

and this result is sharp.

*Proof.* We deduce from (7)

$$z^{c+1} F(z) = (c+2) \int_0^z t^c f(t) dt. \quad (9)$$

Differentiating (9), with respect to  $z$ , we obtain

$$(c+1) F(z) + z F'(z) = (c+2) f(z) \quad (10)$$

and

$$(c+1) I(m, \lambda, l) F(z) + z (I(m, \lambda, l) F(z))' = (c+2) I(m, \lambda, l) f(z), \quad \text{for } z \in U. \quad (11)$$

Differentiating (11) we have

$$(I(m, \lambda, l) F(z))' + \frac{1}{c+2} z (I(m, \lambda, l) F(z))'' = (I(m, \lambda, l) f(z))', \quad \text{for } z \in U. \quad (12)$$

Using (12), the differential subordination (8) becomes

$$(I(m, \lambda, l) F(z))' + \frac{1}{c+2} z (I(m, \lambda, l) F(z))'' \prec g(z) + \frac{1}{c+2} z g'(z). \quad (13)$$

If we denote

$$p(z) = (I(m, \lambda, l) F(z))', \quad (14)$$

then  $p \in \mathcal{H}[1, 1]$ .

Replacing (14) in (13) we obtain

$$p(z) + \frac{1}{c+2} z p'(z) \prec g(z) + \frac{1}{c+2} z g'(z), \quad \text{for } z \in U.$$

Using Lemma 1.7 we have

$$p(z) \prec g(z),$$

that is

$$(I(m, \lambda, l) F(z))' \prec g(z), \quad \text{for } z \in U$$

and  $g$  is the best dominant. ■

**Example 2.4.** If  $f \in S_1(1, \frac{1}{2})$  then

$$f'(z) + z f''(z) \prec \frac{3-2z}{3(1-z)^2},$$

implies

$$F'(z) + z F''(z) \prec \frac{1}{1-z},$$

where  $F(z) = \frac{3}{z^2} \int_0^z t f(t) dt$ .

**Theorem 2.5.** *Let*

$$h(z) = \frac{1 + (2\delta - 1)z}{1 + z}, \quad (15)$$

*with  $\delta \in [0, 1)$  and  $c > 0$ .*

*If  $\alpha \geq 0$ ,  $m \in \mathbb{N}$  and  $I_c$  is given by (7) then*

$$I_c[S_m(\delta, \alpha)] \subset S_m(\delta^*, \alpha), \quad (16)$$

*where*

$$\delta^* = 2\delta - 1 + (c + 2)(2 - 2\delta)\beta(c) \quad (17)$$

*and*

$$\beta(x) = \int_0^1 \frac{t^{x+1}}{t+1} dt. \quad (18)$$

*Proof.* If we consider the function  $h$  given in (15) then  $h$  is convex and using the same steps as in the proof of Theorem 2.3 we get from the hypothesis of Theorem 2.5 that

$$p(z) + \frac{1}{c+2}zp'(z) \prec h(z),$$

where  $p(z)$  is defined in (14).

Using Lemma 1.6 we deduce that

$$p(z) \prec g(z) \prec h(z),$$

that is

$$(I(m, \lambda, l)F(z))' \prec g(z) \prec h(z),$$

where

$$\begin{aligned} g(z) &= \frac{c+2}{z^{c+2}} \int_0^z t^{c+1} \frac{1 + (2\delta - 1)t}{1 + t} dt = \\ &= 2\delta - 1 + \frac{(c+2)(2-2\delta)}{z^{c+2}} \int_0^z \frac{t^{c+1}}{t+1} dt. \end{aligned}$$

Since  $g$  is convex and  $g(U)$  is symmetric with respect to the real axis, we deduce

$$\begin{aligned} \operatorname{Re}(I(m, \lambda, l)F(z))' &\geq \min_{|z|=1} \operatorname{Re} g(z) = \operatorname{Re} g(1) = \delta^* = \\ &= 2\delta - 1 + (c+2)(2-2\delta)\beta(c), \end{aligned} \quad (19)$$

where  $\beta$  is given in (18).

From (19) we deduce inclusion (16). ■

**Theorem 2.6.** *Let  $g$  be a convex function,  $g(0) = 1$  and let  $h$  be the function*

$$h(z) = g(z) + zg'(z), \quad \text{for } z \in U.$$

*If  $\alpha \geq 0$ ,  $m \in \mathbb{N}$ ,  $f \in A$  and verifies the differential subordination*

$$(I(m, \lambda, l)f(z))' \prec h(z), \quad \text{for } z \in U, \quad (20)$$

*then*

$$\frac{I(m, \lambda, l)f(z)}{z} \prec g(z), \quad \text{for } z \in U$$

*and this result is sharp.*

*Proof.* Consider

$$\begin{aligned} p(z) &= \frac{I(m, \lambda, l) f(z)}{z} = \frac{z + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m a_j z^j}{z} \\ &= 1 + p_1 z + p_2 z^2 + \dots, \quad \text{for } z \in U. \end{aligned}$$

We deduce that  $p \in \mathcal{H}[1, 1]$ .

Let

$$I(m, \lambda, l) f(z) = zp(z), \quad \text{for } z \in U.$$

Differentiating we obtain

$$(I(m, \lambda, l) f(z))' = p(z) + zp'(z), \quad \text{for } z \in U.$$

Then (20) becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + zg'(z), \quad \text{for } z \in U.$$

By using Lemma 1.7, we have

$$p(z) \prec g(z), \quad \text{for } z \in U,$$

i.e.

$$\frac{I(m, \lambda, l) f(z)}{z} \prec g(z), \quad \text{for } z \in U.$$

■

**Theorem 2.7.** Let  $h$  be an holomorphic function which verifies the inequality

$$\operatorname{Re} \left( 1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2}, \quad \text{for } z \in U \text{ and } h(0) = 1.$$

If  $\alpha \geq 0$ ,  $m \in \mathbb{N}$ ,  $f \in A$  and verifies the differential subordination

$$(I(m, \lambda, l) f(z))' \prec h(z), \quad \text{for } z \in U, \tag{21}$$

then

$$\frac{I(m, \lambda, l) f(z)}{z} \prec q(z), \quad \text{for } z \in U,$$

where

$$q(z) = \frac{1}{z} \int_0^z h(t) dt.$$

The function  $q$  is convex and it is the best dominant.

*Proof.* Let

$$\begin{aligned} p(z) &= \frac{I(m, \lambda, l) f(z)}{z} = \frac{z + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m a_j z^j}{z} \\ &= 1 + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m a_j z^{j-1} = 1 + z + \sum_{j=2}^{\infty} p_j z^{j-1}, \end{aligned}$$

where  $z \in U$ ,  $p \in \mathcal{H}[1, 1]$ .

Differentiating, we obtain

$$(I(m, \lambda, l) f(z))' = p(z) + zp'(z), \quad \text{for } z \in U$$

and (21) becomes

$$p(z) + zp'(z) \prec h(z), \quad \text{for } z \in U.$$

Using Lemma 1.6, we have

$$p(z) \prec q(z) = \frac{1}{z} \int_0^z h(t) dt, \quad \text{for } z \in U,$$

i.e.

$$\frac{I(m, \lambda, l) f(z)}{z} \prec q(z) = \frac{1}{z} \int_0^z h(t) dt, \quad \text{for } z \in U$$

and  $q$  is the best dominant. ■

**Theorem 2.8.** Let  $g$  be a convex function such that  $g(0) = 1$  and let  $h$  be the function  $h(z) = g(z) + zg'(z)$ , for  $z \in U$ .

If  $\alpha \geq 0$ ,  $m \in \mathbb{N}$ ,  $f \in A$  and the differential subordination

$$\left( \frac{z I(m+1, \lambda, l) f(z)}{I(m, \lambda, l) f(z)} \right)' \prec h(z), \quad \text{for } z \in U \quad (22)$$

holds, then

$$\frac{I(m+1, \lambda, l) f(z)}{I(m, \lambda, l) f(z)} \prec g(z), \quad \text{for } z \in U$$

and this result is sharp.

*Proof.* Consider

$$p(z) = \frac{I(m+1, \lambda, l) f(z)}{I(m, \lambda, l) f(z)} = \frac{z + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^{m+1} a_j z^j}{z + \sum_{j=2}^{\infty} \left( \frac{1+\lambda(j-1)+l}{l+1} \right)^m a_j z^j}.$$

We have  $p'(z) = \frac{(I(m+1, \lambda, l) f(z))'}{I(m, \lambda, l) f(z)} - p(z) \cdot \frac{(I(m, \lambda, l) f(z))'}{I(m, \lambda, l) f(z)}$  and we obtain

$$p(z) + z \cdot p'(z) = \left( \frac{z I(m+1, \lambda, l) f(z)}{I(m, \lambda, l) f(z)} \right)'.$$

Relation (22) becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + zg'(z), \quad \text{for } z \in U.$$

By using Lemma 1.7, we have

$$p(z) \prec g(z), \quad \text{for } z \in U,$$

i.e.

$$\frac{I(m+1, \lambda, l) f(z)}{I(m, \lambda, l) f(z)} \prec g(z), \quad \text{for } z \in U.$$

■

**Theorem 2.9.** Let  $g$  be a convex function such that  $g(0) = 1$  and let  $h$  be the function

$$h(z) = g(z) + zg'(z), \quad \text{for } z \in U.$$

If  $\alpha \geq 0$ ,  $m \in \mathbb{N}$ ,  $f \in A$  and the differential subordination

$$\frac{l+1}{\lambda} I(m+1, \lambda, l) f(z) + \left( 2 - \frac{l+1}{\lambda} \right) I(m, \lambda, l) f(z) \prec h(z), \quad \text{for } z \in U \quad (23)$$

holds, then

$$[I(m, \lambda, l) f(z)]' \prec g(z), \quad \text{for } z \in U.$$

This result is sharp.

*Proof.* Let

$$\begin{aligned} p(z) &= (I(m, \lambda, l) f(z))' \\ &= 1 + \sum_{j=2}^{\infty} \left( \frac{1 + \lambda(j-1) + l}{l+1} \right)^m j a_j z^{j-1} = 1 + p_1 z + p_2 z^2 + \dots \end{aligned} \quad (24)$$

We deduce that  $p \in \mathcal{H}[1, 1]$ .

$$\begin{aligned} \text{We obtain } p(z) + z \cdot p'(z) &= I(m, \lambda, l) f(z) + z (I(m, \lambda, l) f(z))' = \\ &= I(m, \lambda, l) f(z) + \frac{(l+1)I(m+1, \lambda, l) f(z) - (l+1-\lambda)I(m, \lambda, l) f(z)}{\lambda} = \\ &= \frac{l+1}{\lambda} I(m+1, \lambda, l) f(z) + \left(2 - \frac{l+1}{\lambda}\right) I(m, \lambda, l) f(z) \end{aligned}$$

Using the notation in (24), the differential subordination becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + zg'(z).$$

By using Lemma 1.7, we have

$$p(z) \prec g(z), \quad \text{for } z \in U,$$

i.e.

$$(I(m, \lambda, l) f(z))' \prec g(z), \quad \text{for } z \in U$$

and this result is sharp. ■

**Example 2.10.** If  $n = 1$ ,  $\alpha = 1$ ,  $f \in A$ , we deduce that

$$f'(z) + 3zf''(z) + z^2f'''(z) \prec g(z) + zg'(z),$$

which yields that

$$f'(z) + zf''(z) \prec g(z), \quad \text{for } z \in U.$$

**Theorem 2.11.** Let

$$h(z) = \frac{1 + (2\beta - 1)z}{1 + z},$$

a convex function in  $U$ ,  $0 \leq \beta < 1$ .

If  $\alpha \geq 0$ ,  $m \in \mathbb{N}$ ,  $f \in A$  and verifies the differential subordination

$$\frac{l+1}{\lambda} I(m+1, \lambda, l) f(z) + \left(2 - \frac{l+1}{\lambda}\right) I(m, \lambda, l) f(z) \prec h(z), \quad \text{for } z \in U, \quad (25)$$

then

$$[I(m, \lambda, l) f(z)]' \prec q(z), \quad \text{for } z \in U,$$

where  $q$  is given by

$$q(z) = 2\beta - 1 + 2(1 - \beta) \frac{\ln(1+z)}{z}, \quad \text{for } z \in U.$$

The function  $q$  is convex and it is the best dominant.

*Proof.* Following the same steps as in the proof of Theorem 2.9 and considering  $p(z) = (I(m, \lambda, l) f(z))'$ , the differential subordination (25) becomes

$$p(z) + zp'(z) \prec h(z) = \frac{1 + (2\beta - 1)z}{1 + z}, \quad \text{for } z \in U.$$

By using Lemma 1.6 for  $\gamma = 1$  and  $n = 1$ , we have  $p(z) \prec q(z)$ , i.e.,

$$\begin{aligned} (I(m, \lambda, l) f(z))' &\prec q(z) = \frac{1}{z} \int_0^z h(t) dt = \frac{1}{z} \int_0^z \frac{1 + (2\beta - 1)t}{1 + t} dt \\ &= \frac{1}{z} \int_0^z \left( 2\beta - 1 + \frac{2(1 - \beta)}{1 + t} \right) dt = 2\beta - 1 + 2(1 - \beta) \frac{1}{z} \ln(z + 1), \end{aligned}$$

for  $z \in U$ . ■

**Theorem 2.12.** Let  $h$  be an holomorphic function which verifies the inequality

$$\operatorname{Re} \left[ 1 + \frac{zh''(z)}{h'(z)} \right] > -\frac{1}{2}, \text{ for } z \in U \text{ and } h(0) = 1.$$

If  $\alpha \geq 0$ ,  $m \in \mathbb{N}$ ,  $f \in A$  and satisfies the differential subordination

$$\frac{l+1}{\lambda} I(m+1, \lambda, l) f(z) + \left( 2 - \frac{l+1}{\lambda} \right) I(m, \lambda, l) f(z) \prec h(z), \quad \text{for } z \in U, \quad (26)$$

then

$$[I(m, \lambda, l) f(z)]' \prec q(z), \quad \text{for } z \in U,$$

where  $q$  is given by

$$q(z) = \frac{1}{z} \int_0^z h(t) dt.$$

The function  $q$  is convex and it is the best dominant.

*Proof.* Using the properties of operator  $I(m, \lambda, l)$  and considering  $p(z) = (I(m, \lambda, l) f(z))'$ , we obtain

$$p(z) + zp'(z) = \frac{l+1}{\lambda} I(m+1, \lambda, l) f(z) + \left( 2 - \frac{l+1}{\lambda} \right) I(m, \lambda, l) f(z), \quad \text{for } z \in U.$$

Then (26) becomes

$$p(z) + zp'(z) \prec h(z), \quad \text{for } z \in U.$$

Since  $p \in \mathcal{H}[1, 1]$ , using Lemma 1.6, we deduce

$$p(z) \prec q(z), \quad \text{for } z \in U,$$

where

$$q(z) = \frac{1}{z} \int_0^z h(t) dt, \quad \text{for } z \in U,$$

i.e.

$$(I(m, \lambda, l) f(z))' \prec q(z) = \frac{1}{z} \int_0^z h(t) dt, \quad \text{for } z \in U,$$

and  $q$  is the best dominant. ■

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# Approximation by a nonlinear Cardaliaguet-Euvrard neural network operator of max-product kind

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## Abstract

The aim of this note is that by using the so-called max-product method, to associate to the Cardaliaguet-Euvrard linear operator, a nonlinear neural network operator for which a Jackson-type approximation order is obtained. In some classes of functions, the order of approximation is essentially better than the order of approximation of the linear operator.

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**Keywords:** neural networks, nonlinear Cardaliaguet-Euvrard operator, degree of approximation.

## 1 Introduction

Based on the Open Problem 5.5.4, pp. 324-326 in Gal [7], in a series of recent papers submitted for publication we have introduced and studied the so-called max-product operators attached to the Bernstein polynomials and to other linear Bernstein-type operators, like those of Favard-Szász-Mirakjan operators (truncated and nontruncated case), Baskakov operators (truncated and nontruncated case), Meyer-König and Zeller operators and Bleimann-Butzer-Hahn operators.

This idea applied, for example, to the linear Bernstein operators  $B_n(f)(x) = \sum_{k=0}^n p_{n,k}(x)f(k/n)$ , where  $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$ , works as follows. Writ-



ing in the equivalent form  $B_n(f)(x) = \frac{\sum_{k=0}^n p_{n,k}(x) f(k/n)}{\sum_{k=0}^n p_{n,k}(x)}$  and then replacing the sum operator  $\Sigma$  by the maximum operator  $\bigvee$ , one obtains the nonlinear Bernstein operator of max-product kind

$$B_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^n p_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^n p_{n,k}(x)},$$

where the notation  $\bigvee_{k=0}^n p_{n,k}(x)$  means  $\max\{p_{n,k}(x); k \in \{0, \dots, n\}\}$  and similarly for the numerator.

For this max-product operator nice approximation and shape preserving properties were found in e.g. Bede, Coroianu & Gal [5].

For example, it is proved that for some classes of functions (like those of concave functions), the order of approximation given by the max-product Bernstein operators, are essentially better than the approximation order of their linear counterparts.

The aim of the present paper is to use the same idea to the neural networks of Cardaliaguet-Euvrard-type introduced and studied in e.g. Cardaliaguet & Euvrard [6], Anastassiou [1]-[3], Zhang, Cao, & Xu [8] (see also the references cited there). We will obtain that in the class of Lipschitz functions with positive values, the new obtained nonlinear neural network operator has essentially better approximation property than its linear counterpart.

Thus, by following Cardaliaguet & Euvrard [6], for  $b : \mathbb{R} \rightarrow \mathbb{R}_+$  a centered bell-shaped function (that is, nondecreasing on  $(-\infty, 0]$ , nonincreasing on  $[0, +\infty)$ ), with compact support  $[-T, T]$ ,  $T > 0$  (that is  $b(x) > 0$  for all  $x \in (-T, T)$ ) and therefore such that  $I = \int_{-T}^T b(x) dx > 0$ , the Cardaliaguet-Euvrard neural network is defined by

$$C_{n,\alpha}(f)(x) = \sum_{k=-n^2}^{n^2} \frac{f(k/n)}{I \cdot n^{1-\alpha}} \cdot b\left(n^{1-\alpha} \left(x - \frac{k}{n}\right)\right),$$

where  $0 < \alpha < 1$ ,  $n \in \mathbb{N}$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and bounded or uniformly continuous on  $\mathbb{R}$

Denoting by  $CB(\mathbb{R})$  the space of all real-valued continuous and bounded functions on  $\mathbb{R}$  and  $CB_+(\mathbb{R}) = \{f : \mathbb{R} \rightarrow [0, \infty); f \in CB(\mathbb{R})\}$ , applying the max-product method as in the above case of Bernstein polynomials, the corresponding max-product Cardaliaguet-Euvrard network operator will be formally given by

$$C_{n,\alpha}^{(M)}(f)(x) = \frac{\bigvee_{k=-n^2}^{n^2} b\left[n^{1-\alpha} \left(x - \frac{k}{n}\right)\right] f\left(\frac{k}{n}\right)}{\bigvee_{k=-n^2}^{n^2} b\left[n^{1-\alpha} \left(x - \frac{k}{n}\right)\right]}, x \in \mathbb{R}, f \in CB_+(\mathbb{R}).$$

**Remark.** For any  $x \in \mathbb{R}$ , denoting  $J_{T,n}(x) = \{k \in \mathbb{Z}; -n^2 \leq k \leq n^2, n^{1-\alpha}(x - k/n) \in (-T, T)\}$ , then we can write as a well defined operator

$$C_{n,\alpha}^{(M)}(f)(x) = \frac{\bigvee_{k \in J_{T,n}(x)} b \left[ n^{1-\alpha} \left( x - \frac{k}{n} \right) \right] f \left( \frac{k}{n} \right)}{\bigvee_{k \in J_{T,n}(x)} b \left[ n^{1-\alpha} \left( x - \frac{k}{n} \right) \right]}, \quad x \in \mathbb{R}, n > \max\{T + |x|, T^{-1/\alpha}\}, \quad (1)$$

where  $J_{T,n}(x) \neq \emptyset$ , for all  $x \in \mathbb{R}$  and  $n > \max\{T + |x|, T^{-1/\alpha}\}$ . Indeed, we have

$$\bigvee_{k \in J_{T,n}(x)} b \left[ n^{1-\alpha} \left( x - \frac{k}{n} \right) \right] > 0, \quad \text{for all } x \in \mathbb{R} \text{ and } n > \max\{T + |x|, T^{-1/\alpha}\},$$

because by e.g. Anastassiou [1], relationships (2)-(4), pp. 238-239, if  $n \geq T + |x|$  then  $-n^2 \leq nx - Tn^\alpha \leq nx + Tn^\alpha \leq n^2$ , while  $n^{1-\alpha}|x - k/n| < T$  is equivalent to  $nx - Tn^\alpha < k < nx + Tn^\alpha$ . This implies that if  $(nx + Tn^\alpha) - (nx - Tn^\alpha) = 2Tn^\alpha > 2$  and  $n \geq T + |x|$ , then  $J_{T,n}(x) \neq \emptyset$ , which proves our assertion.

The plan of the paper goes as follows : in Section 2 we present some auxiliary results, in Section 3 we obtain the main approximation result, while in Section 4 we compare the approximation result in Section 3 with that for the corresponding linear neural Cardaliaguet-Euvrard network operator.

## 2 Auxiliary Results

**Remark.** From the consideration in the last Remark of Section 1, it is clear that  $C_{n,\alpha}^{(M)}(f)(x)$  is a well-defined function for all  $x \in \mathbb{R}$  and  $n > \max\{T + |x|, T^{-1/\alpha}\}$  and it is continuous on  $\mathbb{R}$  if  $b$  is continuous on  $\mathbb{R}$ .

In addition,  $C_{n,\alpha}^{(M)}(e_0)(x) = 1$ , where  $e_0(x) = 1$ , for all  $x \in \mathbb{R}$  and  $n > \max\{T + |x|, T^{-1/\alpha}\}$ .

In what follows we will see that for  $f \in CB_+(\mathbb{R})$ , the  $C_{n,\alpha}^{(M)}$  operator fulfils similar properties with those of the  $B_n^{(M)}(f)$  operator in Bede & Gal [4].

**Lemma 2.1.** *Let  $b(x)$  be a centered bell-shaped function, continuous and with compact support  $[-T, T]$ ,  $T > 0$ ,  $0 < \alpha < 1$  and  $C_{n,\alpha}^{(M)}$  be defined as in Section 1.*

(i) *If  $|f(x)| \leq c$  for all  $x \in \mathbb{R}$  then  $|C_{n,\alpha}^{(M)}(f)(x)| \leq c$ , for all  $x \in \mathbb{R}$  and  $n > \{T + |x|, T^{-1/\alpha}\}$  and  $C_{n,\alpha}^{(M)}(f)(x)$  is continuous at any point  $x \in \mathbb{R}$ , for all  $n > \max\{T + |x|, T^{-1/\alpha}\}$ ;*

(ii) *If  $f, g \in CB_+(\mathbb{R})$  satisfy  $f(x) \leq g(x)$  for all  $x \in \mathbb{R}$ , then  $C_{n,\alpha}^{(M)}(f)(x) \leq C_{n,\alpha}^{(M)}(g)(x)$  for all  $x \in \mathbb{R}$  and  $n > \max\{T + |x|, T^{-1/\alpha}\}$  ;*

(iii)  *$C_{n,\alpha}^{(M)}(f + g)(x) \leq C_{n,\alpha}^{(M)}(f)(x) + C_{n,\alpha}^{(M)}(g)(x)$  for all  $f, g \in CB_+(\mathbb{R})$ ,  $x \in \mathbb{R}$  and  $n > \max\{T + |x|, T^{-1/\alpha}\}$  ;*

(iv) *For all  $f, g \in CB_+(\mathbb{R})$ ,  $x \in \mathbb{R}$  and  $n > \max\{T + |x|, T^{-1/\alpha}\}$ , we have*

$$|C_{n,\alpha}^{(M)}(f)(x) - C_{n,\alpha}^{(M)}(g)(x)| \leq C_{n,\alpha}^{(M)}(|f - g|)(x);$$

(v)  $C_{n,\alpha}^{(M)}$  is positive homogenous, that is  $C_{n,\alpha}^{(M)}(\lambda f)(x) = \lambda C_{n,\alpha}^{(M)}(f)(x)$  for all  $\lambda \geq 0$ ,  $x \in \mathbb{R}$ ,  $n > \max\{T + |x|, T^{-1/\alpha}\}$  and  $f \in CB_+(\mathbb{R})$ .

**Proof.** (i) Immediate by the formula of definition for  $C_{n,\alpha}^{(M)}$  in (1).

(ii) Let  $f, g \in CB_+(\mathbb{R})$  be with  $f \leq g$  and fix  $x \in \mathbb{R}$ ,  $n > \max\{T + |x|, T^{-1/\alpha}\}$ . Since  $J_{T,n}(x)$  is independent of  $f$  and  $g$ , by (1) we immediately get the conclusion.

(iii) By (1) and by the sublinearity of  $\vee$ , it is immediate.

(iv) Let  $f, g \in CB_+(\mathbb{R})$ . We have  $f = f - g + g \leq |f - g| + g$ , which by (i) – (iii) successively implies  $C_{n,\alpha}^{(M)}(f)(x) \leq C_{n,\alpha}^{(M)}(|f - g|)(x) + C_{n,\alpha}^{(M)}(g)(x)$ , that is  $C_{n,\alpha}^{(M)}(f)(x) - C_{n,\alpha}^{(M)}(g)(x) \leq C_{n,\alpha}^{(M)}(|f - g|)(x)$ , for all  $x \in \mathbb{R}$  and  $n > \max\{T + |x|, T^{-1/\alpha}\}$ .

Writing now  $g = g - f + f \leq |f - g| + f$  and applying the above reasonings, it follows  $C_{n,\alpha}^{(M)}(g)(x) - C_{n,\alpha}^{(M)}(f)(x) \leq C_{n,\alpha}^{(M)}(|f - g|)(x)$ , which combined with the above inequality gives  $|C_{n,\alpha}^{(M)}(f)(x) - C_{n,\alpha}^{(M)}(g)(x)| \leq C_{n,\alpha}^{(M)}(|f - g|)(x)$ , for all  $x \in \mathbb{R}$  and  $n > \max\{T + |x|, T^{-1/\alpha}\}$ .

(v) By (1) it is immediate.  $\square$

**Remark.** By (1) it is easy to see that instead of (ii),  $C_{n,\alpha}^{(M)}$  satisfies the stronger condition

$$C_{n,\alpha}(f \vee g)(x) = C_{n,\alpha}(f)(x) \vee C_{n,\alpha}(g)(x),$$

for all  $f, g \in CB_+(\mathbb{R})$ ,  $x \in \mathbb{R}$ ,  $n > \max\{T + |x|, T^{-1/\alpha}\}$ .

**Corollary 2.2.** For all  $f \in CB_+(\mathbb{R})$ ,  $0 < \alpha < 1$ ,  $b(x)$  as in the statement of Lemma 2.1,  $x \in \mathbb{R}$  and  $n > \max\{T + |x|, T^{-1/\alpha}\}$ , we have

$$|f(x) - C_{n,\alpha}^{(M)}(f)(x)| \leq \left[ \frac{1}{\delta} C_{n,\alpha}^{(M)}(\Phi_x)(x) + 1 \right] \omega_1(f; \delta)_{\mathbb{R}},$$

where  $\delta > 0$ ,  $\Phi_x(u) = |x - u|$  for all  $x, u \in \mathbb{R}$ , and  $\omega_1(f; \delta)_{\mathbb{R}} = \max\{|f(x) - f(y)|; x, y \in \mathbb{R}, |x - y| \leq \delta\}$ .

**Proof.** Indeed, denoting  $e_0(x) = 1$ , from the identity valid for all  $x \in \mathbb{R}$  and  $n > \max\{T + |x|, T^{-1/\alpha}\}$ ,

$$C_{n,\alpha}^{(M)}(f)(x) - f(x) = [C_{n,\alpha}^{(M)}(f)(x) - f(x) \cdot C_{n,\alpha}^{(M)}(e_0)(x)] + f(x)[C_{n,\alpha}^{(M)}(e_0)(x) - 1],$$

by Lemma 2.1 it easily follows

$$|f(x) - C_{n,\alpha}^{(M)}(f)(x)| \leq$$

$$|C_{n,\alpha}^{(M)}(f(x))(x) - C_{n,\alpha}^{(M)}(f(u))(x)| + |f(x)| \cdot |C_{n,\alpha}^{(M)}(e_0)(x) - 1| \leq$$

$$C_{n,\alpha}^{(M)}(|f(u) - f(x)|)(x) + |f(x)| \cdot |C_{n,\alpha}^{(M)}(e_0)(x) - 1|.$$

Now, since for all  $u, x \in \mathbb{R}$  we have

$$|f(u) - f(x)| \leq \omega_1(f; |u - x|)_{\mathbb{R}} \leq \left[ \frac{1}{\delta} |u - x| + 1 \right] \omega_1(f; \delta)_{\mathbb{R}},$$

replacing above and taking into account that  $C_{n,\alpha}^{(M)}(e_0) = 1$ , we immediately obtain the estimate in the statement.  $\square$

**Remark.** Therefore, to get an approximation property for  $C_{n,\alpha}^{(M)}$ , it is enough to obtain a good estimate for

$$E_{n,\alpha}(x) = C_{n,\alpha}^{(M)}(\Phi_x)(x) = \frac{\bigvee_{k \in J_{T,n}(x)} b \left[ n^{1-\alpha} \left( x - \frac{k}{n} \right) \right] |x - k/n|}{\bigvee_{k \in J_{T,n}(x)} b \left[ n^{1-\alpha} \left( x - \frac{k}{n} \right) \right]},$$

for all  $x \in \mathbb{R}$  and  $n > \max\{T + |x|, T^{-1/\alpha}\}$ .

### 3 Approximation Results

In this section we obtain an approximation result for the operator  $C_{n,\alpha}^{(M)}(f)$ . For this purpose, first of all we need to calculate the denominators of  $C_{n,\alpha}^{(M)}(f)(x)$  and of  $E_{n,\alpha}(x)$ , that is we will exactly calculate the expression

$$\bigvee_{k \in J_{T,n}(x)} b \left[ n^{1-\alpha} \left( x - \frac{k}{n} \right) \right] = \bigvee_{k=-n^2}^{n^2} b \left[ n^{1-\alpha} \left( x - \frac{k}{n} \right) \right].$$

In this sense, we present the following.

**Lemma 3.1.** *Let  $b(x)$  be a centered bell-shaped function, continuous and with compact support  $[-T, T]$ ,  $T > 0$  and  $0 < \alpha < 1$ .*

*Then for any  $j \in \mathbb{Z}$  with  $-n^2 \leq j \leq n^2$ , all  $x \in [j/n, (j+1)/n]$  and  $n > \max\{T + |x|, T^{-1/\alpha}\}$ , we have*

$$\begin{aligned} & \bigvee_{k=-n^2}^{n^2} b \left[ n^{1-\alpha} \left( x - \frac{k}{n} \right) \right] = \\ & \max \left\{ b \left[ n^{1-\alpha} \left( x - \frac{j}{n} \right) \right], b \left[ n^{1-\alpha} \left( x - \frac{j+1}{n} \right) \right] \right\} > 0. \end{aligned}$$

**Proof.** Let  $j \in \mathbb{Z}$  with  $-n^2 \leq j \leq n^2$ ,  $x \in [j/n, (j+1)/n]$  and  $n > \max\{T + |x|, T^{-1/\alpha}\}$ . We can write

$$\begin{aligned} & \bigvee_{k=-n^2}^{n^2} b \left[ n^{1-\alpha} \left( x - \frac{k}{n} \right) \right] = \\ & \max \left\{ \bigvee_{k=-n^2}^j b \left[ n^{1-\alpha} \left( x - \frac{k}{n} \right) \right], \bigvee_{k=j+1}^{n^2} b \left[ n^{1-\alpha} \left( x - \frac{k}{n} \right) \right] \right\}. \end{aligned}$$

We observe that for  $k \in \{-n^2, \dots, j\}$  we have  $n^{1-\alpha}(x - k/n) \geq n^{1-\alpha}(x - j/n) \geq 0$  and since  $b$  is nonincreasing on  $[0, +\infty)$ , it easily follows that

$$\bigvee_{k=-n^2}^j b \left[ n^{1-\alpha} \left( x - \frac{k}{n} \right) \right] = b \left[ n^{1-\alpha} \left( x - \frac{j}{n} \right) \right].$$

Similarly, observing that for  $k \in \{j+1, \dots, n^2\}$  we have  $n^{1-\alpha}(x - k/n) \leq n^{1-\alpha}(x - (j+1)/n) \leq 0$ , since  $b(x)$  is nondecreasing on  $(-\infty, 0]$ , it easily follows that

$$\bigvee_{k=j+1}^{n^2} b \left[ n^{1-\alpha} \left( x - \frac{k}{n} \right) \right] = b \left[ n^{1-\alpha} \left( x - \frac{j+1}{n} \right) \right].$$

It remains to prove that for  $x \in [j/n, (j+1)/n]$  and  $n > \max\{T + |x|, T^{-1/\alpha}\}$  we have  $j, j+1 \in J_{T,n}(x)$ . Indeed, since  $x \in [j/n, (j+1)/n]$  is equivalent to  $j \leq nx \leq j+1$ , we evidently get  $j < nx + Tn^\alpha \leq n^2$ , for all  $n \geq T + |x|$  and  $j+1 \leq nx+1 < nx+Tn^\alpha \leq n^2$ , for all  $n > \max\{T + |x|, T^{-1/\alpha}\}$ . Also, because  $-n^2 \leq nx - Tn^\alpha \leq j+1 - Tn^\alpha < j < j+1$ , for all  $n > \max\{T + |x|, T^{-1/\alpha}\}$ , we get that  $j, j+1 \in J_{T,n}(x)$  for all  $n > \max\{T + |x|, T^{-1/\alpha}\}$ , which proves the lemma.  $\square$

**Remark.** The formula in the statement of Lema 3.1 is valid for all  $x \in [-n, +n]$  only. Indeed, since in Lemma 3.1 we suppose that  $n > |x| + T$ , it follows that we cannot have the complementary possibilities for  $x$ ,  $x \in (n, +\infty)$  or  $x \in (-\infty, -n)$ , because in both cases this would imply the contradiction  $|x| > n > |x| + T$ .

**Theorem 3.2.** Let  $b(x)$  be a centered bell-shaped function, continuous and with compact support  $[-T, T]$ ,  $T > 0$  and  $0 < \alpha < 1$ . In addition, suppose that the following requirements are fulfilled:

(i) There exist  $0 < m_1 \leq M_1 < \infty$  such that  $m_1(T - x) \leq b(x) \leq M_1(T - x)$  for all  $x \in [0, T]$ ;

(ii) There exist  $0 < m_2 \leq M_2 < \infty$  such that  $m_2(x + T) \leq b(x) \leq M_2(x + T)$  for all  $x \in [-T, 0]$ .

Then for all  $f \in CB_+(R)$ ,  $x \in \mathbb{R}$  and for all  $n \in \mathbb{N}$  satisfying  $n > \max\{T + |x|, (2/T)^{1/\alpha}\}$ , we have the estimate

$$|f(x) - C_{n,\alpha}^{(M)}(f)(x)| \leq c\omega_1(f; n^{\alpha-1})_{\mathbb{R}},$$

where

$$c = 2 \left( \max \left\{ \frac{TM_2}{2m_2}, \frac{TM_1}{2m_1} \right\} + 1 \right).$$

**Proof.** Let  $x \in \mathbb{R}$  and let  $j \in \mathbb{Z}$  with  $-n^2 \leq j \leq n^2 - 1$  such that  $x \in [j/n, (j+1)/n]$ . Also, let  $k_x \in J_{T,n}(x)$  be such that

$$\bigvee_{k \in J_{T,n}(x)} b \left[ n^{1-\alpha} \left( x - \frac{k}{n} \right) \right] |x - \frac{k}{n}| = b \left[ n^{1-\alpha} \left( x - \frac{k_x}{n} \right) \right] \left| x - \frac{k_x}{n} \right|.$$

It follows that

$$E_{n,\alpha}(x) = \frac{b \left[ n^{1-\alpha} \left( x - \frac{k_x}{n} \right) \right] |x - \frac{k_x}{n}|}{\bigvee_{k \in J_{T,n}(x)} b \left[ n^{1-\alpha} \left( x - \frac{k}{n} \right) \right]}.$$

Tacking into account Lemma 3.1 we immediately obtain

$$E_{n,\alpha}(x) = \min \left\{ \frac{b \left[ n^{1-\alpha} \left( x - \frac{k_x}{n} \right) \right] |x - \frac{k_x}{n}|}{b \left[ n^{1-\alpha} \left( x - \frac{j}{n} \right) \right]}, \frac{b \left[ n^{1-\alpha} \left( x - \frac{k_x}{n} \right) \right] |x - \frac{k_x}{n}|}{b \left[ n^{1-\alpha} \left( x - \frac{j+1}{n} \right) \right]} \right\},$$

for all  $n > \max\{T + |x|, T^{-1/\alpha}\}$ .

In order to prove the estimate in the theorem we distinguish the following two cases: 1)  $k_x > j$  and 2)  $k_x \leq j$ .

Case 1) Taking into account condition (ii), since  $k_x \in J_{T,n}(x)$  and  $j + 1 \in J_{T,n}(x)$ , by  $x - k_x/n \leq 0$  and  $x - (j + 1)/n \leq 0$ , we immediately get

$$\begin{aligned} E_{n,\alpha}(x) &\leq \frac{b[n^{1-\alpha}(x - \frac{k_x}{n})](\frac{k_x}{n} - x)}{b[n^{1-\alpha}(x - \frac{j+1}{n})]} \leq \frac{M_2}{m_2} \cdot \frac{[T + n^{1-\alpha}(x - \frac{k_x}{n})](\frac{k_x}{n} - x)}{T + n^{1-\alpha}(x - \frac{j+1}{n})} \\ &\leq \frac{M_2}{m_2} \cdot \frac{[T + n^{1-\alpha}(x - \frac{k_x}{n})](\frac{k_x}{n} - x)}{T + n^{1-\alpha}(\frac{-1}{n})} \\ &= \frac{M_2}{m_2} \cdot \frac{n^\alpha[T + n^{1-\alpha}(x - \frac{k_x}{n})](\frac{k_x}{n} - x)}{Tn^\alpha - 1}. \end{aligned}$$

Since  $[T + n^{1-\alpha}(x - \frac{k_x}{n})](\frac{k_x}{n} - x) = -n^{1-\alpha}(\frac{k_x}{n} - x - \frac{T}{2n^{1-\alpha}})^2 + \frac{T^2}{4n^{1-\alpha}} \leq \frac{T^2}{4n^{1-\alpha}}$ , it easily follows that

$$E_{n,\alpha}(x) \leq \frac{M_2}{4m_2} \cdot \frac{T^2 n^{2\alpha-1}}{Tn^\alpha - 1} = \frac{M_2}{4m_2} \cdot \frac{T^2 n^\alpha}{Tn^\alpha - 1} \cdot n^{\alpha-1}.$$

Supposing, in addition, that  $n > (2/T)^{1/\alpha}$  (where clearly  $(2/T)^{1/\alpha} > T^{-1/\alpha}$ ), it follows that

$$\frac{n^\alpha}{Tn^\alpha - 1} = \frac{1}{T} \left( 1 + \frac{1/T}{n^\alpha - 1/T} \right) \leq \frac{1}{T} \left( 1 + \frac{1/T}{2/T - 1/T} \right) = \frac{2}{T},$$

which implies

$$E_{n,\alpha}(x) \leq \frac{TM_2}{2m_2} \cdot n^{\alpha-1},$$

for all  $n > \max\{T + |x|, (2/T)^{1/\alpha}\}$ .

Case 2) Taking into account condition (i), since  $k_x \in J_{T,n}(x)$  and  $j \in J_{T,n}(x)$ , by  $x - k_x/n \geq 0$  and  $x - j/n \geq 0$ , we immediately get

$$\begin{aligned} E_{n,\alpha}(x) &\leq \frac{b[n^{1-\alpha}(x - \frac{k_x}{n})](x - \frac{k_x}{n})}{b[n^{1-\alpha}(x - \frac{j}{n})]} \leq \frac{M_1}{m_1} \cdot \frac{[T - n^{1-\alpha}(x - \frac{k_x}{n})](x - \frac{k_x}{n})}{T - n^{1-\alpha}(x - \frac{j}{n})} \\ &\leq \frac{M_1}{m_1} \cdot \frac{[T - n^{1-\alpha}(x - \frac{k_x}{n})](x - \frac{k_x}{n})}{T - n^{1-\alpha}(\frac{1}{n})} \\ &= \frac{M_1}{m_1} \cdot \frac{n^\alpha[T - n^{1-\alpha}(x - \frac{k_x}{n})](x - \frac{k_x}{n})}{Tn^\alpha - 1}. \end{aligned}$$

Since  $[T - n^{1-\alpha}(x - \frac{k_x}{n})](x - \frac{k_x}{n}) = -n^{1-\alpha}(x - \frac{k_x}{n} - \frac{T}{2n^{1-\alpha}})^2 + \frac{T^2}{4n^{1-\alpha}} \leq \frac{T^2}{4n^{1-\alpha}}$ , reasoning exactly as in the Case 1), we obtain

$$E_{n,\alpha}(x) \leq \frac{TM_1}{2m_1} \cdot n^{\alpha-1},$$

for all  $n > \max\{T + |x|, (2/T)^{1/\alpha}\}$ .

Now, applying Corollary 2.2 for  $\delta = \max\{\frac{TM_2}{2m_2} \cdot n^{1-\alpha}, \frac{TM_1}{2m_1} \cdot n^{1-\alpha}\}$  and from the property  $\omega_1(f, \lambda\delta)_{\mathbb{R}} \leq (\lambda + 1)\omega_1(f, \delta)_{\mathbb{R}}$ , we obtain the desired conclusion.  $\square$

**Corollary 3.3.** *Let  $b(x)$  be a centered bell-shaped function, continuous and with compact support  $[-T, T]$ ,  $T > 0$  and  $0 < \alpha < 1$ . If  $0 < \lim_{x \nearrow T} \frac{b(x)}{T-x} < \infty$  and  $0 < \lim_{x \searrow -T} \frac{b(x)}{T+x} < \infty$  then for all  $f \in CB_+(R)$ ,  $x \in \mathbb{R}$  and for all  $n \in \mathbb{N}$  satisfying  $n > \max\{T + |x|, (2/T)^{1/\alpha}\}$  there exists  $c \in R_+$  independent of  $n$  and  $f$  such that*

$$|f(x) - C_{n,\alpha}^{(M)}(f)(x)| \leq c\omega_1(f; n^{\alpha-1})_{\mathbb{R}}.$$

**Proof.** Let us consider the function  $g : [0, T] \rightarrow \mathbb{R}$ ,  $g(x) = \frac{b(x)}{T-x}$  if  $x \in [0, T)$  and  $g(T) = \lim_{x \nearrow T} \frac{b(x)}{T-x}$ . From our assumptions we get that  $g$  is continuous and strictly positive. By the Weierstrass' theorem it follows that  $g$  attains its minimum and maximum. Hence there exist  $0 < m_1 \leq M_1 < \infty$  such that  $m_1 \leq g(x) \leq M_1$  for all  $x \in [0, T]$ . It follows that  $m_1(T-x) \leq b(x) \leq M_1(T-x)$  for all  $x \in [0, T]$ . Since  $b(T) = 0$  we easily get that  $m_1(T-x) \leq b(x) \leq M_1(T-x)$  for all  $x \in [0, T]$ .

Now, let us consider the function  $h : [-T, 0]$ ,  $h(x) = \frac{b(x)}{T+x}$  if  $x \in (-T, 0]$  and  $h(-T) = \lim_{x \searrow -T} \frac{b(x)}{T+x}$ . Again, it is easy to prove that there exist  $0 < m_2 \leq M_2 < \infty$  such that  $m_2(x+T) \leq b(x) \leq M_2(x+T)$  for all  $x \in [-T, 0]$ .

From the above considerations, applying Theorem 3.2 we easily obtain the desired conclusion.  $\square$

In what follows, we will give some examples of bell-shaped functions for which we can apply Theorem 3.2.

**Example 1.** Let us consider  $b : \mathbb{R} \rightarrow [0, \infty)$ ,  $b(x) = 1 + x$  if  $x \in [-1, 0]$ ,  $b(x) = 1 - x$  if  $x \in [0, 1]$ ,  $b(x) = 0$  elsewhere. Using the same notations as in Theorem 3.2 we have  $T = 1$  and  $m_1 = M_1 = m_2 = M_2 = 1$ . By Theorem 3.2, it follows that for all  $f \in CB_+(R)$ ,  $x \in \mathbb{R}$  and for all  $n \in \mathbb{N}$  satisfying  $n > \max\{T + |x|, (2/T)^{1/\alpha}\}$ , we have the estimate

$$|f(x) - C_{n,\alpha}^{(M)}(f)(x)| \leq 3\omega_1(f; n^{\alpha-1})_{\mathbb{R}}.$$

**Example 2.** Let us consider  $b : \mathbb{R} \rightarrow [0, \infty)$ ,  $b(x) = 1 - x^2$  if  $x \in [-1, 1]$ ,  $b(x) = 0$  elsewhere. We have  $T = 1$ ,  $m_1 = m_2 = 1$ ,  $M_1 = M_2 = 2$ . By Theorem 3.2, it follows that for all  $f \in CB_+(R)$ ,  $x \in \mathbb{R}$  and for all  $n \in \mathbb{N}$  satisfying  $n > \max\{T + |x|, (2/T)^{1/\alpha}\}$ , we have the estimate

$$|f(x) - C_{n,\alpha}^{(M)}(f)(x)| \leq 4\omega_1(f; n^{\alpha-1})_{\mathbb{R}}.$$

**Example 3.** Let us consider  $b : \mathbb{R} \rightarrow [0, \infty)$ ,  $b(x) = \cos x$  if  $x \in [-\pi/2, \pi/2]$ ,  $b(x) = 0$  elsewhere. Since for  $t \in [0, \pi/2]$  we have  $2t/\pi \leq \sin t \leq t$  it follows that  $(2/\pi)(\pi/2 - x) \leq \sin(\pi/2 - x) = \cos x \leq \pi/2 - x$  for all  $x \in [0, \pi/2]$  and  $(2/\pi)(\pi/2 + x) \leq \sin(\pi/2 + x) = \cos x \leq \pi/2 + x$  for all  $x \in [-\pi/2, 0]$ . From the

above inequalities it follows that  $T = \pi/2$ ,  $m_1 = m_2 = 2/\pi$  and  $M_1 = M_2 = 1$ . Applying Theorem 3.2, we obtain

$$|f(x) - C_{n,\alpha}^{(M)}(f)(x)| \leq 7\omega_1(f; n^{\alpha-1})_{\mathbb{R}}.$$

**Remark.** In what follows we will prove that in general, if the bell-shaped function  $b$  satisfies the hypothesis of Theorem 3.2, then the order of approximation of the expression  $E_{n,\alpha}(x)$  in Theorem 3.2. cannot be improved. Firstly, let us notice that from the conclusion of Theorem 3.2 it suffices to prove that we cannot improve the order of approximation of the expression  $E_{n,\alpha}(x)$  for the case when  $b(x) = T + x$  if  $x \in [-T, 0]$ ,  $b(x) = T - x$  if  $x \in [0, T]$ ,  $b(x) = 0$  elsewhere. Without any loss of generality we may assume that  $T = 1$ . For  $n \in \mathbb{N}$ ,  $n > (2/T)^{1/\alpha}$ , take  $x_n = 1/2n$ . It is easy to check that for all  $n \geq 2$ , we have  $n > \max\{T + |x_n|, (2/T)^{1/\alpha}\}$ . Since  $x_n \in (0, 1/n)$ , by Lemma 3.1, it follows that  $\bigvee_{k=-n^2}^{k=n^2} b[n^{1-\alpha}(x_n - \frac{k}{n})] = \max\{b(n^{1-\alpha}x_n), b[n^{1-\alpha}(x_n - \frac{1}{n})]\}$ . Through simple calculus we get

$$\bigvee_{k=-n^2}^{k=n^2} b\left[n^{1-\alpha}\left(x_n - \frac{k}{n}\right)\right] = \frac{2n^\alpha - 1}{2n^\alpha}.$$

This, immediately implies

$$E_{n,\alpha}(x_n) = \frac{\bigvee_{k=-n^2}^{k=n^2} b\left[n^{1-\alpha}\left(x_n - \frac{k}{n}\right)\right] |x_n - \frac{k}{n}|}{(2n^\alpha - 1)/2n^\alpha}.$$

From the above equality it follows that for all  $k \in \mathbb{Z}$ ,  $-n^2 \leq k \leq n^2$ , we have

$$E_{n,\alpha}(x_n) \geq \frac{b\left[n^{1-\alpha}\left(x_n - \frac{k}{n}\right)\right] |x_n - \frac{k}{n}|}{(2n^\alpha - 1)/2n^\alpha}. \quad (2)$$

Let us take  $k_n = [\frac{5n^\alpha+3}{6}] - 1$ . It is easy to check that  $-n^2 \leq k_n \leq n^2$ . Also, for  $n$  sufficiently large we have  $x_n \leq k_n/n$ . Then,

$$\begin{aligned} & b\left[n^{1-\alpha}\left(x_n - \frac{k_n}{n}\right)\right] |x_n - \frac{k_n}{n}| \\ &= [1 + n^{1-\alpha}\left(\frac{1}{2n} - \frac{k_n}{n}\right)]\left(\frac{k_n}{n} - x_n\right) = -n^{1-\alpha}\left(\frac{k_n}{n} - \frac{1}{2n} - \frac{1}{2n^{1-\alpha}}\right)^2 + \frac{1}{4n^{1-\alpha}} \\ &= -n^{1-\alpha}\left(\frac{2k_n - 1}{2n} - \frac{1}{2n^{1-\alpha}}\right)^2 + \frac{1}{4n^{1-\alpha}}. \end{aligned}$$

Since

$$\begin{aligned} & \frac{2k_n - 1}{2n} - \frac{1}{2n^{1-\alpha}} \\ &= \frac{2\left([\frac{5n^\alpha+3}{6}] - 1\right) - 1}{2n} - \frac{1}{2n^{1-\alpha}} \geq \frac{2\left(\frac{5n^\alpha+3}{6} - 2\right) - 1}{2n} - \frac{1}{2n^{1-\alpha}} = \frac{1}{3n^{1-\alpha}} - \frac{2}{n}, \end{aligned}$$



it follows that for  $n \geq 6^{1/\alpha}$  we have  $\frac{2k_n-1}{2n} - \frac{1}{2n^{1-\alpha}} \geq 0$ . Therefore, for  $n \geq 6^{1/\alpha}$  we have

$$\begin{aligned} & -n^{1-\alpha} \left( \frac{2k_n-1}{2n} - \frac{1}{2n^{1-\alpha}} \right)^2 + \frac{1}{4n^{1-\alpha}} \\ &= -n^{1-\alpha} \left( \frac{2 \left( \left[ \frac{5n^\alpha+3}{6} \right] - 1 \right) - 1}{2n} - \frac{1}{2n^{1-\alpha}} \right)^2 + \frac{1}{4n^{1-\alpha}} \\ &\geq -n^{1-\alpha} \left( \frac{2 \cdot \frac{5n^\alpha+3}{6} - 1}{2n} - \frac{1}{2n^{1-\alpha}} \right)^2 + \frac{1}{4n^{1-\alpha}} = \frac{5}{36} \cdot n^{\alpha-1}. \end{aligned}$$

Taking into account relation (2) and the above inequality, we get

$$\begin{aligned} E_{n,\alpha}(x_n) &\geq \frac{b \left[ n^{1-\alpha} \left( x_n - \frac{k_n}{n} \right) \right] |x_n - \frac{k_n}{n}|}{(2n^\alpha - 1)/2n^\alpha} = \frac{-n^{1-\alpha} \left( \frac{2k_n-1}{2n} - \frac{1}{2n^{1-\alpha}} \right)^2 + \frac{1}{4n^{1-\alpha}}}{(2n^\alpha - 1)/2n^\alpha} \\ &\geq \frac{\frac{5}{36} \cdot n^{\alpha-1}}{(2n^\alpha - 1)/2n^\alpha} = \frac{5n^\alpha}{18(2n^\alpha - 1)} \cdot n^{\alpha-1}. \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} \frac{5n^\alpha}{18(2n^\alpha - 1)} = \frac{5}{36}$ , it follows that for  $n$  sufficiently large we get

$$E_{n,\alpha}(x_n) \geq \frac{1}{8} \cdot n^{\alpha-1},$$

which implies the desired conclusion.

## 4 Conclusion

The linear Cardaliaguet-Euvrard operators  $C_{n,\alpha}(f)(x)$  were introduced in Cardaliaguet & Euvrard [6], where it is proved the convergence on compacta to the approximated function. The results were of qualitative type. The first quantitative type estimates in the approximation by  $C_{n,\alpha}(f)(x)$  was obtained in Anastassiou [1]-[3] and then improved in Zhang, Cao, & Xu [8], where at the page 1164 the following type of quantitative estimate is obtained :

$$|C_{n,\alpha}(f)(x) - f(x)| \leq \frac{C_1}{n^\alpha} + C_2 \omega_1(f; n^{\alpha-1})_{\mathbb{R}},$$

for all  $n > \max\{T + |x|, T^{-1/\alpha}\}$ , where  $C_1, C_2 > 0$  are constants independent on  $n$  but depending on  $b$  and  $f$ .

If we suppose now that  $f$  is a Lipschitz function on  $\mathbb{R}$ , that is there exists  $L > 0$  such that  $|f(x) - f(y)| \leq L|x - y|$ , for all  $x, y \in \mathbb{R}$ , from the above estimate we get the following order of approximation by the linear Cardaliaguet-Euvrard operator :

$$|C_{n,\alpha}(f)(x) - f(x)| = \mathcal{O}\left(\frac{1}{n^\alpha}\right) + \mathcal{O}\left(\frac{1}{n^{1-\alpha}}\right), \text{ for all } n > \max\{T + |x|, T^{-1/\alpha}\}.$$

On the other hand, for  $f \in CB_+(\mathbb{R})$  a Lipschitz function, in the case of max-product Cardaliaguet-Euvrard operator, by Theorem 3.2 we get the order of approximation

$$|C_{n,\alpha}^{(M)}(f)(x) - f(x)| = \mathcal{O}\left(\frac{1}{n^{1-\alpha}}\right), \text{ for all } n > \max\{T + |x|, (2/T)^{1/\alpha}\}.$$

It is clear that for  $\frac{1}{2} \leq \alpha < 1$ , we get the same order of approximation  $\mathcal{O}\left(\frac{1}{n^{1-\alpha}}\right)$  for both operators  $C_{n,\alpha}(f)(x)$  and  $C_{n,\alpha}^{(M)}(f)(x)$ , while for  $0 < \alpha < \frac{1}{2}$ , the approximation order obtained by the max-product operator  $C_{n,\alpha}^{(M)}(f)(x)$  is essentially better than that obtained by the linear operator  $C_{n,\alpha}(f)(x)$ .

This shows the advantage we can have by using the max-product Cardaliaguet-Euvrard operator introduced by this paper.

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## On joint distributions of order statistics arising from random vectors

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**Abstract.** In this study, the joint distributions of order statistics arising from *innid* random vectors are expressed. Then, some theorems connecting distributions of order statistics of *innid* random vectors to that of order statistics of *iid* random vectors are given.

**Keywords.** Order statistics, permanent, joint distribution, *iid* random variable, *innid* random variable.

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### 1. Introduction

Arnold et al.[1] considered several identities and recurrence relations in terms of distribution function(*df*) of single and several order statistics of independent and identically distributed(*iid*) random variables.

Balakrishnan[2] considered recent developments on order statistics arising from independent and not necessarily identically distributed(*innid*) random variables based primarily on the theory of permanents.

Balasubramanian and Beg[3] defined linear identities for distribution functions of order statistics from an *iid* sample.

Balasubramanian et al.[4] established identities satisfied by distributions of order statistics from non-independent non-identical variables through operator methods based on difference and differential operators.

Bapat and Beg[5] expressed the joint *df* of order statistics of *innid* random variables in terms of the *df* of the random variables using permanents.

Beg[6] obtained several recurrence relations and identities for product moments of order statistics of *innid* random variables using permanents.

Cao and West[7] obtained recurrence relationships among the distribution functions of order statistics arising from *innid* random variables.

Childs and Balakrishnan[8] obtained, using multinomial arguments, the probability density function(*pdf*) of  $X_{r:n+1}$  ( $1 \leq r \leq n+1$ ) if another independent random variable with *df*  $F_i$  and *pdf*  $f_i$  ( $i=1,2,\dots,n$ ) is added to the original  $n$  variables  $X_1, X_2, \dots, X_n$ .

Corley[9] defined a multivariate generalization of classical order statistics for random samples from a continuous multivariate distribution.

David[10] considered the fundamental distribution theory of order statistics.

Gan and Bain[11] obtained the joint probability function(*pf*) of any  $k$  order statistics and also conditional distributions of discrete order statistics from a general discrete parent by “tie-runs”.

Goldie and Maller[12] derived expressions for generalized joint densities of order statistics of *innid* random variables in terms of Radon-Nikodym derivatives with respect to product measures based on *df*.

Guilbaud[13] expressed probability of the functions of *innid* random vectors as a linear combination of probabilities of the functions of *iid* random vectors and thus also for order statistics of random variables.

Khatri[14] examined the *pf* and *df* of a single order statistics, the joint *pf* and *df* of any two order statistics and joint *df* of any three order statistics of *iid* random variables from a discrete parent.

Reiss[15] considered the joint *pdf*, marginal *pdf* and *df* of any  $k$  order statistics of *iid* random variables under a continuous *df* and discontinuous *df*. He also considered *pdf* of bivariate order statistics by marginal ordering of bivariate *iid* random vectors with a continuous *df* by means of multinomial probabilities of appropriate “cell frequency vectors”, defining multivariate order statistics by marginal ordering of *iid* random vectors with a continuous *df*.

Vaughan and Venables[16] denoted the joint *pdf* and marginal *pdf* of order statistics of *innid* random variables by means of permanents.

From now on, the subscripts and superscripts are defined in the first place in which they are used and these definitions will be valid unless they are redefined.

If  $a_1, a_2, \dots$  are defined as column vectors, then the matrix obtained by taking  $m_1$  copies  $a_1$ ,  $m_2$  copies  $a_2, \dots$  can be denoted as

$$\begin{bmatrix} a_1 & a_2 & \dots \\ m_1 & m_2 & \dots \end{bmatrix}$$

and  $\text{per}A$  denotes the permanent of a square matrix  $A$ , which is defined as similar to determinants except that all terms in the expansion have a positive sign.

Consider  $x = (x^{(1)}, x^{(2)}, \dots, x^{(b)})$  and  $y = (y^{(1)}, y^{(2)}, \dots, y^{(b)})$ , then it can be written as  $x \leq y$  if  $x^{(v)} \leq y^{(v)}$  ( $v=1, 2, \dots, b$ ) and  $x + y = (x^{(1)} + y^{(1)}, x^{(2)} + y^{(2)}, \dots, x^{(b)} + y^{(b)})$ .

Let  $\xi_i = (\xi_i^{(1)}, \xi_i^{(2)}, \dots, \xi_i^{(b)})$  ( $i=1, 2, \dots, n$ ) be *iinid* continuous random vectors which components of  $\xi_i$  are independent. The expression

$$X_{r:n}^{(v)} = Z_{r:n}(\xi_1^{(v)}, \xi_2^{(v)}, \dots, \xi_n^{(v)}) \quad (1.1)$$

is stated as the  $r$ th order statistic of the  $v$ th components of  $\xi_1, \xi_2, \dots, \xi_n$ .

From (1.1), the ordered values of the  $v$ th components of  $\xi_1, \xi_2, \dots, \xi_n$  are expressed as

$$X_{1:n}^{(v)} \leq X_{2:n}^{(v)} \leq \dots \leq X_{n:n}^{(v)}. \quad (1.2)$$

From (1.2), we can write

$$X_{r:n} = (X_{r:n}^{(1)}, X_{r:n}^{(2)}, \dots, X_{r:n}^{(b)}) \quad (1 \leq r \leq n).$$

Also,  $x_w = (x_w^{(1)}, x_w^{(2)}, \dots, x_w^{(b)})$ ,  $x_w^{(v)} \in R$  ( $w=1, 2, \dots, d$ ;  $d=1, 2, \dots, n$ ).

Let  $F_i$  and  $f_i$  be *df* and *pdf* of  $\xi_i^{(v)}$ , respectively. Moreover,  $X_{1:n}^{(v),s}, X_{2:n}^{(v),s}, \dots, X_{n:n}^{(v),s}$  are order statistics of *iid* random variables with *df*  $F^s$  and *pdf*  $f^s$ , respectively, defined by

$$F^s = \frac{1}{n_s} \sum_{i \in s} F_i \quad (1.3)$$

and

$$f^s = \frac{1}{n_s} \sum_{i \in s} f_i. \quad (1.4)$$

Here,  $s$  is a non-empty subset of the integers  $\{1, 2, \dots, n\}$  with  $n_s \geq 1$  elements.  $A[s/.]$  is the matrix obtained from  $A$  by taking rows whose indices are in  $s$ .

In this study, the *df* and *pdf* of  $X_{r_1:n}, X_{r_2:n}, \dots, X_{r_d:n}$  ( $1 \leq r_1 < r_2 < \dots < r_d \leq n$ ) will be given.

Let  $X^{(v)} = (X_{r_1:n}^{(v)}, X_{r_2:n}^{(v)}, \dots, X_{r_d:n}^{(v)})$  and  $x^{(v)} = (x_1^{(v)}, x_2^{(v)}, \dots, x_d^{(v)})$ . For notational convenience we

write  $\sum \sum$ ,  $\sum_{m_d, \dots, m_2, m_1}^{n, \dots, m_3, m_2}$ ,  $\sum_{t_d, \dots, t_2, t_1}^{n, \dots, m_3, m_2}$  and  $\sum_{t_d, \dots, t_2, t_1}^{n, \dots, r_3-1, r_2-1}$  instead of  $\sum_{k=1}^n (-1)^{n-k} \frac{K^n}{n!} \sum_{n_s=K}^n$ ,  $\sum_{m_d=r_d}^n \dots \sum_{m_2=r_2}^{m_3} \sum_{m_1=r_1}^{m_2}$ ,  $\sum_{t_d=m_d}^n \dots \sum_{t_2=m_2}^{m_3} \sum_{t_1=m_1}^{m_2}$  and  $\sum_{t_d=r_d}^n \dots \sum_{t_2=r_2}^{r_3-1} \sum_{t_1=r_1}^{r_2-1}$  in the expressions below, respectively.

## 2. Theorems for distribution function

In this section, the theorems related to  $df$  of  $X_{r_1:n}, X_{r_2:n}, \dots, X_{r_d:n}$  are given. The theorems connect the  $df$  of order statistics of *innid* random vectors to that of order statistics of *iid* random vectors using (1.3).

### Theorem 2.1.

$$F_{r_1, r_2, \dots, r_d:n}(x_1, x_2, \dots, x_d) = \prod_{v=1}^b \left\{ \sum_{m_d, \dots, m_2, m_1}^{n, \dots, m_3, m_2} C \sum_{t_d, \dots, t_2, t_1}^{n, \dots, m_3, m_2} (-1)^{\sum_{w=1}^d (m_{w+1} - t_w)} \prod_{w=1}^d \binom{m_{w+1} - m_w}{t_w - m_w} \right. \\ \left. \cdot \sum_{n_s = n - t_d + m_d} (t_d - m_d)! \text{per} [F(x_1^{(v)}) \dots F(x_d^{(v)})] [s/.] \right\}, \quad x_1 < x_2 < \dots < x_d, \quad (2.1)$$

where  $F(x_w^{(v)}) = (F_1(x_w^{(v)}), F_2(x_w^{(v)}), \dots, F_n(x_w^{(v)}))'$  is column vector,  $C = [\prod_{w=1}^{d+1} (m_w - m_{w-1})!]^{-1}$ ,  $m_0 = 0$ ,  $m_{d+1} = n$  and  $t_0 = m_1$ .

**Proof.** It can be written

$$F_{r_1, r_2, \dots, r_d:n}(x_1, x_2, \dots, x_d) = P\{X_{r_1:n} \leq x_1, X_{r_2:n} \leq x_2, \dots, X_{r_d:n} \leq x_d\}. \quad (2.2)$$

(2.2) can be expressed as

$$F_{r_1, r_2, \dots, r_d:n}(x_1, x_2, \dots, x_d) = P\{X^{(1)} \leq x^{(1)}, X^{(2)} \leq x^{(2)}, \dots, X^{(b)} \leq x^{(b)}\} \\ = \prod_{v=1}^b P\{X^{(v)} \leq x^{(v)}\} \\ = \prod_{v=1}^b P\{X_{r_1:n}^{(v)} \leq x_1^{(v)}, X_{r_2:n}^{(v)} \leq x_2^{(v)}, \dots, X_{r_d:n}^{(v)} \leq x_d^{(v)}\} \\ = \prod_{v=1}^b \sum_{m_d, \dots, m_2, m_1}^{n, \dots, m_3, m_2} C \text{per} A, \quad (2.3)$$

where  $A = [F(x_1^{(v)}) \dots F(x_d^{(v)})]$  is matrix,

$$F(x_w^{(v)}) - F(x_{w-1}^{(v)}) = (F_1(x_w^{(v)}) - F_1(x_{w-1}^{(v)}), F_2(x_w^{(v)}) - F_2(x_{w-1}^{(v)}), \dots, F_n(x_w^{(v)}) - F_n(x_{w-1}^{(v)}))' \quad (w = 1, 2, \dots, d+1),$$

$F_i(x_0^{(v)}) = 0$  and  $F_i(x_{d+1}^{(v)}) = 1$ . Using properties of permanent, we can write

$$\text{per} A = \text{per} [F(x_1^{(v)}) \dots F(x_d^{(v)})] \\ = \sum_{t_d=0}^{n-m_d} (-1)^{n-m_d-t_d} \binom{n-m_d}{t_d} \dots \sum_{t_2=0}^{m_3-m_2} (-1)^{m_3-m_2-t_2} \binom{m_3-m_2}{t_2} \sum_{t_1=0}^{m_2-m_1} (-1)^{m_2-m_1-t_1} \binom{m_2-m_1}{t_1}$$

$$\begin{aligned}
& \cdot \text{per}[F(x_1^{(v)}) \quad F(x_2^{(v)}) \quad \dots \quad 1 \quad F(x_d^{(v)})] \\
& \quad \quad \quad m_2-t_1 \quad m_3-m_2-t_2+t_1 \quad t_d \quad n-m_d-t_d+t_{d-1} \\
& = \sum_{t_d=0}^{n-m_d} \dots \sum_{t_2=0}^{m_3-m_2} \sum_{t_1=0}^{m_2-m_1} (-1)^{n-m_1-\sum_{w=1}^d t_w} \prod_{w=1}^d \binom{m_{w+1}-m_w}{t_w} \sum_{n_s=n-t_d} t_d! \text{per}[F(x_1^{(v)}) \quad F(x_2^{(v)}) \quad \dots \quad F(x_d^{(v)})] [s/.] \\
& \quad \quad \quad m_2-t_1 \quad m_3-m_2-t_2+t_1 \quad n-m_d-t_d+t_{d-1} \\
& = \sum_{t_d=m_d}^n \dots \sum_{t_2=m_2}^{m_3} \sum_{t_1=m_1}^{m_2} (-1)^{\sum_{w=1}^d (m_{w+1}-t_w)} \prod_{w=1}^d \binom{m_{w+1}-m_w}{t_w-m_w} \sum_{n_s=n+m_d-t_d} (t_d-m_d)! \\
& \quad \quad \quad \cdot \text{per}[F(x_1^{(v)}) \quad F(x_2^{(v)}) \quad \dots \quad F(x_d^{(v)})] [s/.], \quad (2.4) \\
& \quad \quad \quad m_2-t_1+m_1 \quad m_3-m_1-t_2+t_1 \quad n-m_{d-1}-t_d+t_{d-1}
\end{aligned}$$

where  $1 = (I, I, \dots, I)'$ .

Using (2.4) in (2.3), (2.1) is obtained.

### Theorem 2.2.

$$\begin{aligned}
F_{r_1, r_2, \dots, r_d; n}(x_1, x_2, \dots, x_d) &= \prod_{v=1}^b \left\{ \sum \sum_{m_d, \dots, m_2, m_1}^{n, \dots, m_3, m_2} n! C[F^s(x_1^{(v)})]^{m_1} \right. \\
& \quad \cdot \prod_{w=2}^{d+1} \sum_{t=m_{w-1}}^{m_w} (-1)^{m_w-t} \binom{m_w-m_{w-1}}{t-m_{w-1}} [F^s(x_w^{(v)})]^{t-m_{w-1}} [F^s(x_{w-1}^{(v)})]^{m_w-t} \}, \quad (2.5)
\end{aligned}$$

where  $F^s(x_{d+1}^{(v)}) = 1$ .

**Proof.** From (2.3), we can write

$$F_{r_1, r_2, \dots, r_d; n}(x_1, x_2, \dots, x_d) = \prod_{v=1}^b \sum \sum P\{X_{r_1; n}^{(v), s} \leq x_1^{(v)}, X_{r_2; n}^{(v), s} \leq x_2^{(v)}, \dots, X_{r_d; n}^{(v), s} \leq x_d^{(v)}\} \quad (2.6)$$

(2.6) is expressed as

$$\begin{aligned}
F_{r_1, r_2, \dots, r_d; n}(x_1, x_2, \dots, x_d) &= \prod_{v=1}^b \left\{ \sum \sum_{m_d, \dots, m_2, m_1}^{n, \dots, m_3, m_2} n! C \prod_{w=1}^{d+1} [F^s(x_w^{(v)}) - F^s(x_{w-1}^{(v)})]^{m_w - m_{w-1}} \right\} \\
&= \prod_{v=1}^b \left\{ \sum \sum_{m_d, \dots, m_2, m_1}^{n, \dots, m_3, m_2} n! C [F^s(x_1^{(v)})]^{m_1} \prod_{w=2}^{d+1} [F^s(x_w^{(v)}) - F^s(x_{w-1}^{(v)})]^{m_w - m_{w-1}} \right\}, \quad (2.7)
\end{aligned}$$

where  $F^s(x_0^{(v)}) = 0$ .

Using binomial expansion in (2.7), (2.5) is obtained.

**Theorem 2.3.**

$$\begin{aligned}
F_{r:n}(x) &= \sum_{m=r}^n \frac{1}{m!(n-m)!} \sum_{t=m}^n (-1)^{n-t} \binom{n-m}{t-m} \sum_{n_s=n-t+m}^{n-t+m} (t-m)! \text{per}[F(x)][s/.] \\
&= \sum \sum \sum_{m=r}^n \binom{n}{m} \sum_{t=m}^n (-1)^{n-t} \binom{n-m}{t-m} [F^s(x)]^{n-t+m}.
\end{aligned} \quad (2.8)$$

**Proof.** In (2.1) and (2.5), if  $b = 1$ ,  $d = 1$ , (2.8) is obtained.

**Theorem 2.4.**

$$\begin{aligned}
F_{1:n}(x) &= 1 - \frac{1}{n!} \sum_{t=0}^n (-1)^{n-t} \binom{n}{t} \sum_{n_s=n-t}^{n-t} t! \text{per}[F(x)][s/.] \\
&= \sum \sum \left\{ 1 - \sum_{t=0}^n (-1)^{n-t} \binom{n}{t} [F^s(x)]^{n-t} \right\}.
\end{aligned} \quad (2.9)$$

**Proof.** In (2.8), if  $r = 1$ , (2.9) is obtained.

**Theorem 2.5.**

$$\begin{aligned}
F_{n:n}(x) &= \frac{1}{n!} \text{per}[F(x)]_n \\
&= \sum \sum [F^s(x)]^n.
\end{aligned} \quad (2.10)$$

**Proof.** In (2.8), if  $r = n$ , (2.10) is obtained.

**3. Theorems for probability density function**

In this section, the theorems related to *pdf* of  $X_{r_1:n}, X_{r_2:n}, \dots, X_{r_d:n}$  are given. The theorems connect the *pdf* of order statistics of *innid* random vectors to that of order statistics of *iid* random vectors using (1.3) and (1.4).

**Theorem 3.1.**

$$\begin{aligned}
f_{r_1, r_2, \dots, r_d:n}(x_1, x_2, \dots, x_d) &= \prod_{v=1}^b \left\{ D \sum_{t_d, \dots, t_2, t_1}^{n, \dots, r_3-1, r_2-1} (-1)^{-d + \sum_{w=1}^d (r_{w+1} - t_w)} \prod_{w=1}^d \binom{r_{w+1} - r_w - 1}{t_w - r_w} \right. \\
&\quad \cdot \sum_{n_s=n+r_d-t_d} (t_d - r_d)! \text{per}[F(x_1^{(v)})_{r_2+r_1-2-t_1} F(x_2^{(v)})_{r_3-r_1-1-t_2+t_1} \dots F(x_d^{(v)})_{n-r_{d-1}-t_d+t_{d-1}} f(x_1^{(v)})_1 f(x_2^{(v)})_1 \dots f(x_d^{(v)})_1] [s/.] \}, \quad (3.1)
\end{aligned}$$



where  $f(x_w^{(v)}) = (f_1(x_w^{(v)}), f_2(x_w^{(v)}), \dots, f_n(x_w^{(v)}))'$ ,  $D = [\prod_{w=1}^{d+1} (r_w - r_{w-1} - 1)]^{-1}$ ,  $r_0 = 0$ ,  $r_{d+1} = n+1$ , and  $t_0 = r_1 - 1$ .

**Proof.** Consider the event

$$\{x_1 < X_{r_1:n} \leq x_1 + \delta x_1, x_2 < X_{r_2:n} \leq x_2 + \delta x_2, \dots, x_d < X_{r_d:n} \leq x_d + \delta x_d\}.$$

It can be written

$$\begin{aligned} & P\{x_1 < X_{r_1:n} \leq x_1 + \delta x_1, x_2 < X_{r_2:n} \leq x_2 + \delta x_2, \dots, x_d < X_{r_d:n} \leq x_d + \delta x_d\} \\ &= P\{x^{(1)} < X^{(1)} \leq x^{(1)} + \delta x^{(1)}, x^{(2)} < X^{(2)} \leq x^{(2)} + \delta x^{(2)}, \dots, x^{(b)} < X^{(b)} \leq x^{(b)} + \delta x^{(b)}\} \\ &= \prod_{v=1}^b P\{x^{(v)} < X^{(v)} \leq x^{(v)} + \delta x^{(v)}\} \\ &= \prod_{v=1}^b P\{x_1^{(v)} < X_{r_1:n}^{(v)} \leq x_1^{(v)} + \delta x_1^{(v)}, x_2^{(v)} < X_{r_2:n}^{(v)} \leq x_2^{(v)} + \delta x_2^{(v)}, \dots, x_d^{(v)} < X_{r_d:n}^{(v)} \leq x_d^{(v)} + \delta x_d^{(v)}\}, \end{aligned} \quad (3.2)$$

where  $\delta x_w = (\delta x_w^{(1)}, \delta x_w^{(2)}, \dots, \delta x_w^{(b)})$  and  $\delta x^{(v)} = (\delta x_1^{(v)}, \delta x_2^{(v)}, \dots, \delta x_d^{(v)})$ .

Dividing (3.2) by  $\prod_{v=1}^b \prod_{w=1}^d \delta x_w^{(v)}$  and then letting  $\delta x_1^{(v)}, \delta x_2^{(v)}, \dots, \delta x_d^{(v)}$  tend to zero, we obtain

$$f_{r_1, r_2, \dots, r_d:n}(x_1, x_2, \dots, x_d) = \prod_{v=1}^b D \text{per} B, \quad (3.3)$$

where  $B = [F(x_1^{(v)})_{r_1-1} \quad f(x_1^{(v)})_1 \quad F(x_2^{(v)}) - F(x_1^{(v)})_{r_2-r_1-1} \quad f(x_2^{(v)})_1 \quad \dots \quad f(x_d^{(v)})_1 \quad 1 - F(x_d^{(v)})_{n-r_d}]$  is matrix.

Using properties of permanent, we can write

$$\begin{aligned} \text{per} B &= \text{per}[F(x_1^{(v)})_{r_1-1} \quad f(x_1^{(v)})_1 \quad F(x_2^{(v)}) - F(x_1^{(v)})_{r_2-r_1-1} \quad f(x_2^{(v)})_1 \quad \dots \quad f(x_d^{(v)})_1 \quad 1 - F(x_d^{(v)})_{n-r_d}] \\ &= \sum_{t_d=0}^{n-r_d} (-1)^{n-r_d-t_d} \binom{n-r_d}{t_d} \dots \sum_{t_2=0}^{r_3-r_2-1} (-1)^{r_3-r_2-1-t_2} \binom{r_3-r_2-1}{t_2} \sum_{t_1=0}^{r_2-r_1-1} (-1)^{r_2-r_1-1-t_1} \binom{r_2-r_1-1}{t_1} \\ &\quad \cdot \text{per}[F(x_1^{(v)})_{r_2-2-t_1} \quad f(x_1^{(v)})_1 \quad F(x_2^{(v)}) - F(x_1^{(v)})_{r_3-r_2-1-t_2+t_1} \quad f(x_2^{(v)})_1 \quad \dots \quad f(x_d^{(v)})_1 \quad 1 - F(x_d^{(v)})_{n-r_d-t_d+t_{d-1}}] \\ &= \sum_{t_d=0}^{n-r_d} \dots \sum_{t_2=0}^{r_3-r_2-1} \sum_{t_1=0}^{r_2-r_1-1} (-1)^{n+1-r_1-d-\sum_{w=1}^d t_w} \prod_{w=1}^d \binom{r_{w+1}-r_w-1}{t_w} \\ &\quad \cdot \sum_{n_s=n-t_d} t_d! \text{per}[F(x_1^{(v)})_{r_2-2-t_1} \quad F(x_2^{(v)})_{r_3-r_2-1-t_2+t_1} \quad \dots \quad F(x_d^{(v)})_{n-r_d-t_d+t_{d-1}} \quad f(x_1^{(v)})_1 \quad f(x_2^{(v)})_1 \quad \dots \quad f(x_d^{(v)})_1][s/.] \end{aligned}$$

$$\begin{aligned}
&= \sum_{t_d=r_d}^n \dots \sum_{t_2=r_2}^{r_3-1} \sum_{t_1=r_1}^{r_2-1} (-1)^{-d+\sum_{w=1}^d (r_{w+1}-t_w)} \prod_{w=1}^d \binom{r_{w+1}-r_w-1}{t_w-r_w} \\
&\quad \cdot \sum_{n_s=n+r_d-t_d} (t_d-r_d)! \text{per} [F(x_1^{(v)}) \quad F(x_2^{(v)}) \quad \dots \quad F(x_d^{(v)}) \quad f(x_1^{(v)}) \quad f(x_2^{(v)}) \quad \dots \quad f(x_d^{(v)})] [s/.] . \quad (3.4)
\end{aligned}$$

Using (3.4) in (3.3), (3.1) is obtained.

**Theorem 3.2.**

$$\begin{aligned}
f_{r_1, r_2, \dots, r_d; n}(x_1, x_2, \dots, x_d) &= \prod_{v=1}^b \left\{ \sum \sum n! D[F^s(x_1^{(v)})]^{r_1-1} \right. \\
&\quad \cdot \prod_{w=2}^{d+1} \sum_{t=r_{w-1}}^{r_w-1} (-1)^{r_w-1-t} \binom{r_w-r_{w-1}-1}{t-r_{w-1}} [F^s(x_{w-1}^{(v)})]^{r_w-1-t} [F^s(x_w^{(v)})]^{t-r_{w-1}} \prod_{w=1}^d f^s(x_w^{(v)}) \left. \right\} . \quad (3.5)
\end{aligned}$$

**Proof.** (3.2) can be expressed as

$$\prod_{v=1}^b \sum \sum P\{x_1^{(v)} < X_{r_1; n}^{(v),s} \leq x_1^{(v)} + \delta x_1^{(v)}, x_2^{(v)} < X_{r_2; n}^{(v),s} \leq x_2^{(v)} + \delta x_2^{(v)}, \dots, x_d^{(v)} < X_{r_d; n}^{(v),s} \leq x_d^{(v)} + \delta x_d^{(v)}\} . \quad (3.6)$$

Dividing (3.6) by  $\prod_{v=1}^b \prod_{w=1}^d \delta x_w^{(v)}$  and then letting  $\delta x_1^{(v)}, \delta x_2^{(v)}, \dots, \delta x_d^{(v)}$  tend to zero, we obtain

$$\begin{aligned}
f_{r_1, r_2, \dots, r_d; n}(x_1, x_2, \dots, x_d) &= \prod_{v=1}^b \left\{ \sum \sum n! D \prod_{w=1}^{d+1} [F^s(x_w^{(v)}) - F^s(x_{w-1}^{(v)})]^{r_w-r_{w-1}-1} \prod_{w=1}^d f^s(x_w^{(v)}) \right\} \\
&= \prod_{v=1}^b \left\{ \sum \sum n! D[F^s(x_1^{(v)})]^{r_1-1} \prod_{w=2}^{d+1} [F^s(x_w^{(v)}) - F^s(x_{w-1}^{(v)})]^{r_w-r_{w-1}-1} \prod_{w=1}^d f^s(x_w^{(v)}) \right\} . \quad (3.7)
\end{aligned}$$

Using binomial expansion in (3.7), (3.5) is obtained.

**Theorem 3.3.**

$$\begin{aligned}
f_{r; n}(x) &= \frac{1}{(r-1)!(n-r)!} \sum_{t=r}^n (-1)^{n-t} \binom{n-r}{t-r} \sum_{n_s=n+r-t} (t-r)! \text{per} [F(x) \quad f(x)] [s/.] \\
&= \sum \sum r \binom{n}{r} \sum_{t=r}^n (-1)^{n-t} \binom{n-r}{t-r} [F^s(x)]^{n+r-1-t} f^s(x) . \quad (3.8)
\end{aligned}$$

**Proof.** In (3.1) and (3.5), if  $b = 1$ ,  $d = 1$ , (3.8) is obtained.

**Theorem 3.4.**

$$\begin{aligned}
 f_{1n}(x) &= \frac{1}{(n-1)!} \sum_{t=1}^n (-1)^{n-t} \binom{n-1}{t-1} \sum_{n_s=n+1-t} (t-1)! \text{per}_{n-t} [F(x) \quad f(x)]_1 [s/.] \\
 &= \sum \sum n \sum_{t=1}^n (-1)^{n-t} \binom{n-1}{t-1} [F^s(x)]^{n-t} f^s(x). \quad (3.9)
 \end{aligned}$$

**Proof.** In (3.8), if  $r = 1$ , (3.9) is obtained.

**Theorem 3.5.**

$$\begin{aligned}
 f_{nn}(x) &= \frac{1}{(n-1)!} \text{per}_{n-1} [F(x) \quad f(x)]_1 \\
 &= \sum \sum n [F^s(x)]^{n-1} f^s(x). \quad (3.10)
 \end{aligned}$$

**Proof.** In (3.8), if  $r = n$ , (3.10) is obtained.

**Theorem 3.6.**

$$\begin{aligned}
 f_{1,nn}(x_1, x_2) &= \frac{1}{(n-2)!} \sum_{t=1}^{n-1} (-1)^{n-1-t} \binom{n-2}{t-1} \text{per}_{n-1-t} [F(x_1) \quad F(x_2) \quad f(x_1) \quad f(x_2)]_{t-1} \\
 &= \sum \sum n(n-1) \sum_{t=1}^{n-1} (-1)^{n-1-t} \binom{n-2}{t-1} [F^s(x_1)]^{n-1-t} [F^s(x_2)]^{t-1} f^s(x_1) f^s(x_2), \quad x_1 < x_2. \quad (3.11)
 \end{aligned}$$

**Proof.** In (3.1) and (3.5), if  $b = 1$ ,  $d = 2$  and  $r_1 = 1$ ,  $r_2 = n$ , (3.11) is obtained.

**Theorem 3.7.**

$$\begin{aligned}
 f_{1,2,\dots,k;n}(x_1, x_2, \dots, x_k) &= \prod_{v=1}^b \left\{ \frac{1}{(n-k)!} \sum_{t=k}^n (-1)^{n-t} \binom{n-k}{t-k} \sum_{n_s=n-t+k} (t-k)! \right. \\
 &\quad \left. \cdot \text{per}_{n-t} [F(x_k^{(v)}) \quad f(x_1^{(v)}) \quad f(x_2^{(v)}) \dots f(x_k^{(v)})]_1 [s/.] \right\} \\
 &= \prod_{v=1}^b \left\{ \sum \sum \frac{n!}{(n-k)!} \sum_{t=k}^n (-1)^{n-t} \binom{n-k}{t-k} [F^s(x_k^{(v)})]^{n-t} \prod_{w=1}^k f^s(x_w^{(v)}) \right\}. \quad (3.12)
 \end{aligned}$$

**Proof.** In (3.1) and (3.5), if  $d = k$  and  $r_1 = 1$ ,  $r_2 = 2, \dots, r_k = k$ , (3.12) is obtained.

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# A new result on absolute summability factors

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## Abstract

In the present paper, a known result of Mazhar [5] on  $|\bar{N}, p_n|_k$  summability has been generalized for  $|\bar{N}, p_n, \theta_n|_k$  summability factors. Some new results have also been obtained.

## 1 Introduction

Let  $\sum a_n$  be a given infinite series with partial sums  $(s_n)$ . Let  $(p_n)$  be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^n p_v \rightarrow \infty \quad \text{as } n \rightarrow \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \quad (1)$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v \quad (2)$$

defines the sequence  $(\sigma_n)$  of the  $(\bar{N}, p_n)$  mean of the sequence  $(s_n)$ , generated by the sequence of coefficients  $(p_n)$ . The series  $\sum a_n$  is said to be summable  $|\bar{N}, p_n|_k, k \geq 1$ , if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |\Delta \sigma_{n-1}|^k < \infty, \quad (3)$$

where

$$\Delta \sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n P_{v-1} a_v, \quad n \geq 1. \quad (4)$$

In the special case  $p_n = 1$  for all values of  $n$ ,  $|\bar{N}, p_n|_k$  summability is the same as  $|C, 1|_k$  (see [4]) summability. Let  $(\theta_n)$  be any sequence of positive constants. The series  $\sum a_n$  is said to be summable  $|\bar{N}, p_n, \theta_n|_k, k \geq 1$ , if (see [6])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |\Delta \sigma_{n-1}|^k < \infty. \quad (5)$$

If we take  $\theta_n = \frac{P_n}{p_n}$ , then  $|\bar{N}, p_n, \theta_n|_k$  summability reduces to  $|\bar{N}, p_n|_k$  summability. Also, if we take  $\theta_n = n$  and  $p_n = 1$  for all values of  $n$ , then we get  $|C, 1|_k$  summability. Furthermore, if we take  $\theta_n = n$ , then  $|\bar{N}, p_n, \theta_n|_k$  summability reduces to  $|R, p_n|_k$  (see [2]) summability.

**2. Known result.** Mazhar [5] has proved the following theorem.

**Theorem A.** If

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n} |s_n \lambda_n|^k < \infty, \quad (6)$$

and

$$\sum_{n=1}^{\infty} |s_n| |\Delta \lambda_n| < \infty, \quad (7)$$

then the series  $\sum a_n \lambda_n$  is summable  $|\bar{N}, p_n|_k, k \geq 1$ .

**3. Main result.** The aim of this paper is to generalize Theorem A for  $|\bar{N}, p_n, \theta_n|_k$  summability. Now we shall prove the following theorem.

**Theorem.** Let  $\left(\frac{\theta_n p_n}{P_n}\right)$  be a non-increasing sequence. If the conditions of Theorem A are satisfied with the condition (6) replaced by:

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k |s_n \lambda_n|^k < \infty, \quad (8)$$

then the series  $\sum a_n \lambda_n$  is summable  $|\bar{N}, p_n, \theta_n|_k, k \geq 1$ . If we take  $\theta_n = \frac{P_n}{p_n}$ , then we get Theorem A. In this case condition (8) reduces to condition (6) and the condition  $\left(\frac{\theta_n p_n}{P_n}\right)$  which is a non-increasing sequence is automatically satisfied.

**4. Proof of the Theorem.** Let  $(T_n)$  be the sequence of  $(\bar{N}, p_n)$  mean of the series  $\sum a_n \lambda_n$ . Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^n p_v \sum_{r=1}^v a_r \lambda_r. \quad (9)$$

Then , using Abel's transformation, we get

$$\begin{aligned} T_n - T_{n-1} &= -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \Delta \lambda_v + \frac{p_n}{P_n} s_n \lambda_n \\ &= T_{n,1} + T_{n,2} + T_{n,3}, \quad \text{say.} \end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3. \quad (10)$$

Firstly , by using Hölder's inequality, we have that

$$\begin{aligned} \sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n,1}|^k &= \sum_{n=2}^{m+1} \theta_n^{k-1} \left| \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v \right|^k \\ &\leq \sum_{n=2}^{m+1} \theta_n^{k-1} \left( \frac{p_n}{P_n} \right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v |s_v \lambda_v|^k \\ &\quad \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\ &= O(1) \sum_{v=1}^m p_v |s_v \lambda_v|^k \left( \frac{\theta_v p_v}{P_v} \right)^{k-1} \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\ &= O(1) \sum_{v=1}^m p_v |s_v \lambda_v|^k \left( \frac{\theta_v p_v}{P_v} \right)^{k-1} \frac{1}{P_v} \\ &= O(1) \sum_{v=1}^m \theta_v^{k-1} \left( \frac{p_v}{P_v} \right)^k |s_v \lambda_v|^k = O(1) \quad \text{as } m \rightarrow \infty, \end{aligned}$$

in view of (8). Also , we get that

$$\begin{aligned} \sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n,2}|^k &= \sum_{n=2}^{m+1} \theta_n^{k-1} \left| \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \Delta \lambda_v \right|^k \\ &\leq \sum_{n=2}^{m+1} \theta_n^{k-1} \left( \frac{p_n}{P_n} \right)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |s_v| |\Delta \lambda_v| \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |s_v| |\Delta \lambda_v| \right\}^{k-1} \\ &= O(1) \sum_{v=1}^m P_v |s_v| |\Delta \lambda_v| \left( \frac{\theta_v p_v}{P_v} \right)^{k-1} \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\ &= O(1) \sum_{v=1}^m |s_v| |\Delta \lambda_v| \left( \frac{\theta_v p_v}{P_v} \right)^{k-1} \\ &= O(1) \left( \frac{\theta_1 p_1}{P_1} \right)^{k-1} \sum_{v=1}^m |s_v| |\Delta \lambda_v| \end{aligned}$$

$$= O(1) \sum_{v=1}^m |s_v| |\Delta \lambda_v| = O(1) \quad \text{as } m \rightarrow \infty,$$

in view of (7). Finally, we have that

$$\sum_{n=1}^m \theta_n^{k-1} |T_{n,3}|^k = \sum_{n=1}^m \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k |s_n \lambda_n|^k = O(1) \quad \text{as } m \rightarrow \infty,$$

in view of (8). Therefore we get that

$$\sum_{n=1}^m \theta_n^{k-1} |T_{n,r}|^k = O(1) \quad \text{as } m \rightarrow \infty, \quad \text{for } r = 1, 2, 3.$$

This completes the proof of the Theorem. If we take  $p_n = 1$  for all values of  $n$  and  $\theta_n = n$ , then we get a result for  $|C, 1|_k$  summability factors. Also, if we take  $p_n = 1$  for all values of  $n$ , then we have a new result for  $|C, 1, \theta_n|_k$  summability factors. Furthermore, if we take  $\theta_n = n$ , then we have another new result for  $|R, p_n|_k$  summability factors.

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# STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES IN TOPOLOGICAL GROUPS

H. ÇAKALLI AND E. SAVAS

**ABSTRACT.** The idea of statistical convergence of double sequences was first introduced by Mursaleen and Edeley [12]) while the idea of statistical convergence of single sequences was first studied by Fast [7] and the rapid developments were started after the papers of Salat [14] and Fridy [8]. Nowadays it has become one of the most active area of research in the field of summability. In this paper, we give an extension of statistical convergence of double sequences to topological groups and give theorems in this general setting.

## 1. INTRODUCTION

Looking through historically to statistical convergence of single sequences, we recall that the concept of statistical convergence of sequences was first introduced by Fast [7] as an extension of the usual concept of sequential limits and also independently by Buck [1]. Schoenberg [15] gave some basic properties of statistical convergence and also studied the concept as a summability method. Over the years statistical convergence has been examined in trigonometric series in [16], studied by J.Connor [3], J.S.Connor [4], Salat [14], and Fridy [8]. Most of the existing works on statistical convergence have been restricted to real or complex sequences except the works of Kolk, Maddox and Çakallı. This notion was used by Kolk in [9] to extend statistical convergence to normed spaces and Maddox [11] extended to locally convex Hausdorff topological linear spaces giving a representation of statistical convergence in terms of strong summability by using a modulus function, introduced in [10]; and also in [5] and [6], Çakallı extended this notation to topological Hausdorff groups.

Recall that for a subset  $M$  of  $\mathbf{N}$  the asymptotic density of  $M$ , denoted by  $\delta(M)$ , is given by

$$\delta(M) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : k \in M\}|,$$

if this limit exists, where  $|\{k \leq n : k \in M\}|$  denotes the cardinality of the set  $\{k \leq n : k \in M\}$ . A sequence  $\mathbf{x} = (x_n)$  is statistically convergent to  $\ell$  if

$$\delta(\{n : |x_n - \ell| \geq \epsilon\}) = 0,$$

for every  $\epsilon > 0$ . In this case  $\ell$  is called the statistical limit of  $\mathbf{x}$ .

A sequence  $(x_k)$  of points in a topological group  $X$ , is called statistically convergent to an element  $\ell$  of  $X$  if  $\delta(M_U) = 0$  where  $M_U = \{k : x_k - \ell \notin U\}$  for every

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neighbourhood  $U$  of 0, i.e.

$$\delta(M_U) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : x_k - \ell \notin U\}| = 0,$$

for every neighbourhood  $U$  of 0 and a sequence  $(x_k)$  of points in a topological group  $X$ , is called statistically Cauchy if for every neighbourhood  $U$  of 0 there exists a number  $N = N(\epsilon)$  such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : x_k - x_N \notin U\}| = 0.$$

The concept of statistical convergence of double sequences was first introduced by Mursaleen and Edeley [12] who have given main definition for double sequences  $\mathbf{x} = (x_{jk})$  and proved some related results supporting by some interesting examples. By the convergence of a double sequence we mean the convergence in Pringsheims sense [13]. A double sequence  $\mathbf{x} = (x_{jk})_{j,k=0}^{\infty}$  is said to be convergent in the Pringsheims sense if for every  $\epsilon > 0$  there exists  $N \in \mathbf{N}$  such that  $|x_{jk} - L| < \epsilon$  whenever  $j, k \geq N$ .  $L$  is called the Pringsheim limit of  $\mathbf{x}$ . A double sequence  $\mathbf{x} = (x_{jk})$  is said to be Cauchy sequence if for every  $\epsilon > 0$  there exists  $N \in \mathbf{N}$  such that  $|x_{pq} - x_{jk}| < \epsilon$  for all  $p \geq j \geq N$  and  $q \geq k \geq N$ .

In a topological group  $X$ , the above definitions become as in the following: a double sequence  $\mathbf{x} = (x_{jk})_{j,k=0}^{\infty}$  of points in  $X$  is said to be convergent to a point  $\ell$  in  $X$  in the Pringsheims sense if for every neighbourhood  $U$  of 0 there exists  $N \in \mathbf{N}$  such that  $x_{jk} - \ell \in U$  whenever  $j, k \geq N$ .  $\ell$  is called the Pringsheim limit of  $\mathbf{x}$ . A double sequence  $\mathbf{x} = (x_{jk})$  of points in  $X$  is said to be a Cauchy sequence if for every neighbourhood  $U$  of 0 there exists  $N \in \mathbf{N}$  such that  $x_{pq} - x_{jk} \in U$  for all  $p \geq j \geq N$  and  $q \geq k \geq N$ .

The purpose of this paper is to study statistical convergence of double sequences in topological groups and to give some important theorems.

## 2. DEFINITIONS AND NOTATION

By  $X$ , we will denote an abelian topological Hausdorff group, written additively, which satisfies the first axiom of countability. For a subset  $A$  of  $X$ ,  $s(A)$  will denote the set of all sequences  $(x_n)$  such that  $x_n$  is in  $A$  for  $n = 1, 2, \dots$ ;  $c(X)$  will denote the set of all convergent sequences.

In [6], a sequence  $(x_n)$  in  $X$  is called to be statistically convergent to an element  $\ell$  of  $X$  if for each neighbourhood  $U$  of 0,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : x_k - \ell \notin U\}| = 0,$$

and is called statistically Cauchy in  $X$  if for each neighbourhood  $U$  of 0 there exists a positive integer  $n_0(U)$ , depending on the neighbourhood  $U$ , such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : x_k - x_{n_0(U)} \notin U\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. The set of all statistically convergent sequences in  $X$  is denoted by  $st(X)$  and the set of all statistically Cauchy sequences in  $X$  is denoted by  $stC(X)$ . It is known that  $stC(X) = st(X)$  when  $X$  is complete. By the convergence of a double sequence in a topological group  $X$  we mean the convergence in Pringsheims sense. A double sequence  $\mathbf{x} = (x_{jk})_{j,k=0}^{\infty}$  is said to be convergent in the Pringsheims sense if for every neighbourhood  $U$  of 0 there exists  $N \in \mathbf{N}$  such that  $x_{jk} - \ell \in U$  whenever

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$j, k \geq N$ .  $\ell$  is called the Pringsheim limit of  $\mathbf{x}$ . A double sequence  $\mathbf{x} = (x_{jk})_{j,k=0}^{\infty}$  is said to be a Cauchy sequence if for every neighbourhood  $U$  of 0 there exists  $N \in \mathbf{N}$  such that  $x_{pq} - x_{jk} \in U$  whenever  $p \geq j \geq N$  and  $q \geq k \geq N$ .

## 3. STATISTICAL CONVERGENCE

Let  $K \subset \mathbf{N} \times \mathbf{N}$  be a two-dimensional set of positive integers and let  $K(n, m)$  be the numbers of  $(i, j)$  in  $K$  such that  $i \leq n$  and  $j \leq m$ . Then the two-dimensional analogue of natural density can be defined as follows. The lower asymptotic density of a set  $K \subset \mathbf{N} \times \mathbf{N}$  is defined as

$$\underline{\delta}_2(K) = \liminf_{n,m} \frac{K(n, m)}{nm}$$

In case the sequence  $(\frac{K(n, m)}{nm})$  has a limit in Pringsheims sense then we say that  $K$  has a double natural density and is defined as

$$\delta_2(K) = \lim_{n,m} \frac{K(n, m)}{nm}$$

For example, let  $K = \{(i^2, j^2) : i, j \in \mathbf{N}\}$ . Then

$$\delta_2(K) = \lim_{n,m} \frac{K(n, m)}{nm} \leq \lim_{n,m} \frac{\sqrt{n}\sqrt{m}}{nm} = 0$$

i.e., the set  $K$  has double natural density zero, while the set  $\{(i, 2j) : i, j \in \mathbf{N}\}$  has double natural density  $1/2$ . Note that, if we set  $n = m$ , we have a two-dimensional natural density considered by Christopher [2]. Mursaleen and Edely called a real double sequence  $\mathbf{x} = (x_{jk})$  statistically convergent to the number  $\ell$  if for each  $\epsilon > 0$ , the set  $\{(j, k), j \leq n \text{ and } k \leq m : |x_{jk} - \ell| \geq \epsilon\}$  has double natural density zero. In this case we write  $st_2 - \lim_{j,k} x_{jk} = \ell$  and we denote the set of all statistically convergent double sequences by  $st_2$ . Now we give definition of statistical convergence of double sequences  $\mathbf{x} = (x_{jk})_{j,k=0}^{\infty}$  of points in a topological group in the following.

In a topological group, double sequence  $\mathbf{x} = (x_{jk})$  is called statistically convergent to a point  $\ell$  of  $X$  if for each neighbourhood  $U$  of 0 the set

$$\{(j, k), j \leq n; \text{ and } k \leq m : x_{jk} - \ell \notin U\}$$

has double natural density zero. In this case we write  $st_2(X) - \lim_{j,k} x_{jk} = \ell$  and we denote the set of all statistically convergent double sequences by  $st_2(X)$ .

If  $\mathbf{x}$  is a convergent double sequence of points in  $X$ , then it is also statistically convergent to the same point. Since there are only a finite number of bounded (unbounded) rows and/or columns,

$$K(n, m) \leq s_1 n + s_2 m$$

where  $s_1$  and  $s_2$  are finite numbers, which we can conclude that  $\mathbf{x}$  is statistically convergent.

A statistically convergent double sequence has a unique limit, i.e. if  $\mathbf{x}$  is statistically convergent to elements  $\ell_1$  and  $\ell_2$  of  $X$ , then  $\ell_1 = \ell_2$ .

If  $\mathbf{x}$  is statistically convergent, then  $\mathbf{x}$  need not be convergent. For example, let  $\mathbf{x} = (x_{jk})$  be defined as  $x_{jk} = jkz$ , if  $j$  and  $k$  are squares;  $z$ , otherwise where  $z$

is a fixed non-zero element of  $X$ . It is easy to see that  $st_2 - \lim x_{jk} = z$ , since the cardinality of the set  $\{(j, k) : x_{jk} - z \notin U\}$  satisfies the inequality

$$|\{(j, k) : x_{jk} - z \notin U\}| \leq \sqrt{j}\sqrt{k}$$

for every neighborhood  $U$  of 0. But  $\mathbf{x}$  is not convergent.

For a subset  $A$  of  $X$ ,  $s_2(A)$  will denote the set of all double sequences  $(x_{nm})$  of points in  $A$ ;  $c_2(X)$  will denote the set of all convergent double sequence of points in  $X$ .

*Theorem 1.* A double sequence  $\mathbf{x} = (x_{jk})$  of points in  $X$  is statistically convergent to  $\ell$  if and only if there exists a subset  $K \subset \mathbf{N} \times \mathbf{N}$  such that  $\delta_2(K) = 1$  and  $\lim_{n,k \rightarrow \infty} x_{jk} = \ell$  where limit is being taken over the set  $K$ , i.e.,  $(j, k) \in K$ .

*Proof.* Let  $\mathbf{x}$  be statistically convergent to  $\ell$ , and  $(U_r)$  be a base of nested closed neighbourhoods of 0. Write

$$K_r = \{(j, k) \in \mathbf{N} \times \mathbf{N} : x_{jk} - \ell \notin U_r\}$$

$$M_r = \{(j, k) \in \mathbf{N} \times \mathbf{N} : x_{jk} - \ell \in U_r\} \quad (r = 1, 2, \dots).$$

Then  $\delta_2(K_r) = 0$  and

$$(1) \quad M_1 \supset M_2 \supset \dots \supset M_i \supset M_{i+1} \supset \dots$$

and

$$(2) \quad \delta_2(M_r) = 1, \quad r = 1, 2, \dots$$

Now we will show that for  $(j, k) \in M_r$ ,  $(x_{jk})$  is convergent to  $\ell$ . Suppose that  $(x_{jk})$  is not convergent to  $\ell$  so that there is a neighbourhood  $U$  of 0 such that

$$x_{jk} - \ell \notin U$$

for infinitely many terms. Let  $U_r \subset U$  ( $r = 1, 2, \dots$ ) and  $M_U = \{(j, k) : x_{jk} - \ell \in U\}$ . Then

$$\delta_2(M_U) = 0,$$

and by (1),  $M_r \subset M_U$ . Hence  $\delta_2(M_r) = 0$  which is a contradiction to (2). Thus  $(x_{jk})$  is convergent to  $\ell$ .

Conversely, suppose that there exists a subset  $K = \{(j, k)\} \subseteq \mathbf{N} \times \mathbf{N}$  such that  $\delta_2(K) = 1$  and  $\lim_{j,k} x_{jk} = \ell$ , i.e. there exists an  $n_o \in \mathbf{N}$  such that for each neighbourhood  $U$  of 0,

$$x_{jk} - \ell \in U \text{ for every } j, k \geq n_o$$

Now

$$K_U = \{(j, k) : x_{jk} - \ell \notin U\} \subseteq \mathbf{N} \times \mathbf{N} \setminus \{(j_{n_o+1}, k_{n_o+1}), (j_{n_o+2}, k_{n_o+2}), \dots\}$$

Hence

$$\delta_2(K_U) \leq 1 - 1 = 0$$

It follows that  $\mathbf{x}$  is statistically convergent to  $\ell$ . □

*Corollary 2.* If a double sequence  $(x_{jk})$  is statistically convergent to a point  $\ell$ , then there exists a sequence  $(y_{jk})$  such that  $\lim_{j,k} y_{jk} = \ell$  and  $\delta_2\{(j, k) : x_{jk} = y_{jk}\} = 1$ , i.e.,  $x_{jk} = y_{jk}$  for almost all  $j, k$ .

## 4. STATISTICALLY CAUCHY DOUBLE SEQUENCES IN TOPOLOGICAL GROUPS

In a topological group, double sequence  $\mathbf{x} = (x_{jk})$  is called statistically Cauchy if for each neighbourhood  $U$  of 0 there exists  $N = N(U)$  and  $M = M(U)$  such that for all  $j, p \geq N, k, q \geq M$  the set  $\{(j, k) : j \leq n, k \leq m : x_{jk} - x_{pq} \notin U\}$  has double natural density zero. In this case we denote the set of all statistically Cauchy double sequences by  $st_2C(X)$ .

*Theorem 3.* Let  $X$  be complete. A double sequence  $\mathbf{x} = (x_{jk})$  of points in a  $X$  is statistically convergent if and only if  $\mathbf{x}$  is statistically Cauchy.

*Proof.* Take any statistically convergent double sequence  $\mathbf{x} = (x_{jk})$  with statistical limit  $\ell$ . Let  $U$  be any neighbourhood of 0. Then we may choose a symmetric neighbourhood  $W$  of 0 such that

$$W + W \subset U.$$

Then for this neighbourhood  $W$  of 0, the set

$$\{(j, k) : j \leq n, k \leq m \text{ and } x_{jk} - \ell \in W\}$$

has double natural density 0. For each neighbourhood  $U$  of 0, the set  $\{(j, k) : j \leq n, k \leq m : x_{jk} - \ell \notin U\}$  has double natural density zero. Then we may choose numbers  $M$  and  $N$  such that  $x_{NM} - \ell \notin U$ . Now write

$$A_U = \{(j, k) : j \leq n, k \leq m : x_{jk} - x_{NM} \notin U\}$$

$$B_W = \{(j, k) : j \leq n, k \leq m : x_{jk} - \ell \notin W\}$$

$$C_W = \{(j, k) : j = N \leq n, k = M \leq m : x_{NM} - \ell \notin W\}.$$

Then  $A_U \subset B_W \cup C_W$  and hence  $\delta_2(A_U) \leq \delta_2(B_W) + \delta_2(C_W) = 0$ . Therefore we get that  $\mathbf{x}$  is statistically Cauchy.

To prove the converse suppose that there is a statistically Cauchy sequence  $\mathbf{x}$  which is not statistically convergent. Then we may find natural numbers  $N$  and  $M$  such that the set  $A_U$  has double natural density zero. It follows from this that the set

$$E_U = \{(j, k) : j \leq n, k \leq m : x_{jk} - x_{NM} \in U\}$$

has double natural density 1. We may choose a neighborhood  $W$  of 0 such that  $W + W \subset U$ . Now take any fixed non-zero element  $\ell$  of  $X$ . Write  $x_{jk} - x_{NM} = x_{jk} - \ell + \ell - x_{NM}$ . It follows from this equality that  $x_{jk} - x_{NM} \in U$  if  $x_{jk} - \ell \in W$ . Since  $x$  is not statistically convergent to  $\ell$ , the set  $B_W$  has double natural density 1, i.e., the set  $\{(j, k) : j \leq n, k \leq m : x_{jk} - \ell \notin W\}$  has double natural density 0. Hence the set  $\{(j, k) : j \leq n, k \leq m : x_{jk} - x_{NM} \in U\}$  has double natural density 0, i.e. the set  $A_U$  has double natural density 1 which is a contradiction. Hence this completes the proof.  $\square$

Finally from theorems 1 and 3 we can state the following theorem and the proof is easy and omitted.

*Theorem 4.* If  $X$  is complete, then the following conditions are equivalent:

- (a)  $\mathbf{x}$  is statistically convergent to  $\ell$ ;
- (b)  $\mathbf{x}$  is statistically Cauchy;
- (c) there exists a subsequence  $\mathbf{y}$  of  $\mathbf{x}$  such that  $\lim_{j,k} y_{jk} = \ell$ .

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# Approximately left derivations: An alternative fixed point approach

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**Abstract.** Using fixed point methods, we investigate the generalized Hyers–Ulam–Rassias stability of left derivations and left Jordan derivations on Banach algebras associated with the following generalized Cauchy–Jensen type functional equation

$$\mu f\left(\frac{x+y}{2}+z\right)+\mu f\left(\frac{x-y}{2}+z\right)=f(\mu x)+2f(\mu z) \quad (\mu \in \mathbb{T}).$$

## 1. INTRODUCTION

The stability of functional equations was first introduced by S. M. Ulam [21] in 1940. More precisely, he proposed the following problem: Given a group  $G_1$ , a metric group  $(G_2, d)$  and a positive number  $\epsilon$ , does there exist a  $\delta > 0$  such that if a function  $f : G_1 \rightarrow G_2$  satisfies the inequality  $d(f(xy), f(x)f(y)) < \delta$  for all  $x, y \in G_1$ , then there exists a homomorphism  $T : G_1 \rightarrow G_2$  such that  $d(f(x), T(x)) < \epsilon$  for all  $x \in G_1$ ? As mentioned above, when this problem has a solution, we say that the homomorphisms from  $G_1$  to  $G_2$  are stable. In 1941, D. H. Hyers [10] gave a partial solution of *Ulam's* problem for the case of approximate additive mappings under the assumption that  $G_1$  and  $G_2$  are Banach spaces. In 1978, Th. M. Rassias [18] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences.

This phenomenon of stability that was introduced by Th. M. Rassias [18] is called the Hyers–Ulam–Rassias stability. According to Th. M. Rassias theorem:

**Theorem 1.1.** *Let  $f : E \rightarrow E'$  be a mapping from a norm vector space  $E$  into a Banach space  $E'$  subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

*for all  $x, y \in E$ , where  $\epsilon$  and  $p$  are constants with  $\epsilon > 0$  and  $p < 1$ . Then there exists a unique additive mapping  $T : E \rightarrow E'$  such that*

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2-2^p} \|x\|^p$$

*for all  $x \in E$ . If  $p < 0$  then inequality (1.3) holds for all  $x, y \neq 0$ , and (1.4) for  $x \neq 0$ . Also, if the function  $t \mapsto f(tx)$  from  $\mathbb{R}$  into  $E'$  is continuous for each fixed  $x \in E$ , then  $T$  is linear.*

During the last decades several stability problems of functional equations have been investigated by many mathematicians. A large list of references concerning the stability of functional equations can be found in [7, 11, 14, 19].

Approximate derivations was first investigated by K.-W. Jun and D.-W. Park [13]. Recently,

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the stability of derivations have been investigated by some authors; see [1, 12, 13, 15] and references therein.

On the other hand Cădariu and Radu applied the fixed point method to the investigation of the functional equations. (see also [4, 5, 6, 16, 17, 20]).

Let  $A$  be an algebra and let  $X$  be a Banach left  $A$ -module. Note that a linear map  $D : A \rightarrow X$  is called a left derivation (left Jordan derivation) if

$$D(ab) = aD(b) + bD(a) \quad (D(a^2) = 2aD(a))$$

for all  $a, b \in A$ . In this paper, we will adopt the fixed point alternative of Cădariu and Radu to prove the generalized Hyers–Ulam–Rassias stability of left derivations and Jordan left derivations on Banach algebras associated with the following generalized Cauchy–Jensen type functional equation

$$\mu f\left(\frac{x+y}{2} + z\right) + \mu f\left(\frac{x-y}{2} + z\right) = f(\mu x) + 2f(\mu z) \quad (\mu \in \mathbb{T}).$$

Throughout this paper assume that  $A$  is a Banach algebra and  $X$  is a left Banach  $A$ -module.

## 2. LEFT DERIVATIONS

Before proceeding to the main results, we will state the following theorem.

**Theorem 2.1.** *(The alternative of fixed point [3]). Suppose that we are given a complete generalized metric space  $(\Omega, d)$  and a strictly contractive mapping  $T : \Omega \rightarrow \Omega$  with Lipschitz constant  $L$ . Then for each given  $x \in \Omega$ , either*

$$d(T^m x, T^{m+1} x) = \infty \quad \text{for all } m \geq 0,$$

*or other exists a natural number  $m_0$  such that*

$$d(T^m x, T^{m+1} x) < \infty \quad \text{for all } m \geq m_0;$$

*the sequence  $\{T^m x\}$  is convergent to a fixed point  $y^*$  of  $T$ ;*

*$y^*$  is the unique fixed point of  $T$  in the set  $\Lambda = \{y \in \Omega : d(T^{m_0} x, y) < \infty\}$ ;*

$$d(y, y^*) \leq \frac{1}{1-L} d(y, Ty) \quad \text{for all } y \in \Lambda.$$

By a following similar way as in [16], we obtain the next theorem.

**Theorem 2.2.** *Let  $f : A \rightarrow X$  be a mapping for which there exists a function  $\phi : A^5 \rightarrow [0, \infty)$  such that*

$$\left\| \mu f\left(\frac{x+y}{2} + z\right) + \mu f\left(\frac{x-y}{2} + z\right) - f(\mu x) - 2f(\mu z) + f(ab) - af(b) - bf(a) \right\| \leq \phi(x, y, z, a, b), \quad (2.1)$$

*for all  $\mu \in \mathbb{T}$  and all  $x, y, z, a, b \in A$ . If there exists an  $L < 1$  such that  $\phi(x, y, z, a, b) \leq 2L\phi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{a}{2}, \frac{b}{2})$  for all  $x, y, z, a, b \in A$ , then there exists a unique left derivation  $D : A \rightarrow X$  such that*

$$\|f(x) - D(x)\| \leq \frac{L}{1-L} \phi(x, 0, 0, 0, 0) \quad (2.2)$$

*for all  $x \in A$ .*



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*Proof.* It follows from  $\phi(x, y, z, a, b) \leq 2L\phi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{a}{2}, \frac{b}{2})$  that

$$\lim_j 2^{-j} \phi(2^j x, 2^j y, 2^j z, 2^j a, 2^j b) = 0 \quad (2.3)$$

for all  $x, y, z, a, b \in A$ .

Put  $\mu = 1, y = z = a = b = 0$  in (2.1) to obtain

$$\|2f(\frac{x}{2}) - f(x)\| \leq \phi(x, 0, 0, 0, 0) \quad (2.4)$$

for all  $x \in A$ . Hence,

$$\|\frac{1}{2}f(2x) - f(x)\| \leq \frac{1}{2}\phi(2x, 0, 0, 0, 0) \leq L\phi(x, 0, 0, 0, 0) \quad (2.5)$$

for all  $x \in A$ .

Consider the set  $X := \{g \mid g : A \rightarrow B\}$  and introduce the generalized metric on  $X$ :

$$d(h, g) := \inf\{C \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq C\phi(x, 0, 0, 0, 0) \forall x \in A\}.$$

It is easy to show that  $(X, d)$  is complete. Now we define the linear mapping  $J : X \rightarrow X$  by

$$J(h)(x) = \frac{1}{2}h(2x)$$

for all  $x \in A$ . By Theorem 3.1 of [3],

$$d(J(g), J(h)) \leq Ld(g, h)$$

for all  $g, h \in X$ .

It follows from (2.5) that

$$d(f, J(f)) \leq L.$$

By Theorem 2.1,  $J$  has a unique fixed point in the set  $X_1 := \{h \in X : d(f, h) < \infty\}$ . Let  $D$  be the fixed point of  $J$ .  $D$  is the unique mapping with

$$D(2x) = 2D(x)$$

for all  $x \in A$  satisfying there exists  $C \in (0, \infty)$  such that

$$\|D(x) - f(x)\| \leq C\phi(x, 0, 0, 0, 0)$$

for all  $x \in A$ . On the other hand we have  $\lim_n d(J^n(f), D) = 0$ . It follows that

$$\lim_n \frac{1}{2^n} f(2^n x) = D(x) \quad (2.6)$$

for all  $x \in A$ . It follows from  $d(f, D) \leq \frac{1}{1-L}d(f, J(f))$ , that

$$d(f, D) \leq \frac{L}{1-L}.$$

This implies the inequality (2.2). It follows from (2.1), (2.3) and (2.6) that

$$\begin{aligned} & \|D(\frac{x+y}{2} + z) + D(\frac{x-y}{2} + z) - D(x) - 2D(z)\| \\ &= \lim_n \frac{1}{2^n} \|f(2^{n-1}(x+y) + 2^n z) + f(2^{n-1}(x-y) + 2^n z) - f(2^n x) - 2f(2^n z)\| \\ &\leq \lim_n \frac{1}{2^n} \phi(2^n x, 2^n y, 2^n z, 0, 0) \\ &= 0 \end{aligned}$$

for all  $x, y \in A$ . So

$$D(\frac{x+y}{2} + z) + D(\frac{x-y}{2} + z) = D(x) + 2D(z)$$

for all  $x, y, z \in A$ . By Lemma 2.1 of [2], the mapping  $D : A \rightarrow A$  is Cauchy additive. By putting  $y = x, z = a = 0$  in (2.1), we have

$$\|\mu f(\frac{2x}{2}) - f(\mu x)\| \leq \phi(x, x, 0, 0, 0)$$

for all  $x \in A$ . It follows that

$$\|D(2\mu x) - 2\mu D(x)\| = \lim_m \frac{1}{2^m} \|f(2\mu 2^m x) - 2\mu f(2^m x)\| \leq \lim_m \frac{1}{2^m} \phi(2^m x, 2^m x, 0, 0, 0) = 0$$

for all  $\mu \in \mathbb{T}$ , and all  $x \in A$ . One can show that the mapping  $D : A \rightarrow X$  is  $\mathbb{C}$ -linear. By putting  $x = y = z = 0$  in (2.1) it follows that

$$\begin{aligned} & \|D(ab) - aD(b) - bD(a)\| \\ &= \lim_n \left\| \frac{1}{4^n} f((4^n ab) - \frac{1}{2^n} af(2^n b) - \frac{1}{2^n} bf(2^n a) \right\| \\ &\leq \lim_n \frac{1}{4^n} \phi(0, 0, 0, 2^n a, 2^n b) \leq \lim_n \frac{1}{2^n} \phi(0, 0, 0, 2^n a, 2^n b) \\ &= 0 \end{aligned}$$

for all  $a \in A$ . This means that  $D : A \rightarrow X$  is a left derivation satisfying (2.2), as desired.  $\square$

We prove the following Hyers–Ulam–Rassias stability problem for left derivations on Banach algebras.

**Corollary 2.3.** *Let  $p \in (0, 1), \theta \in [0, \infty)$  be real numbers. Suppose  $f : A \rightarrow X$  satisfies*

$$\begin{aligned} & \|\mu f(\frac{x+y}{2} + z) + \mu f(\frac{x-y}{2} + z) - f(\mu x) - 2f(\mu z) + f(ab) - af(b) - bf(a)\| \\ & \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|a\|^p + \|b\|^p), \end{aligned}$$

for all  $\mu \in \mathbb{T}$  and all  $x, y, a, b \in A$ . Then there exists a unique left derivation  $D : A \rightarrow X$  such that

$$\|f(x) - D(x)\| \leq \frac{2^p \theta}{2 - 2^p} \|x\|^p$$

for all  $x \in A$ .

*Proof.* It follows from Theorem 2.2, by putting  $\phi(x, y, a, b) := \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|a\|^p + \|b\|^p)$  all  $x, y, z, a, b \in A$  and  $L = 2^{p-1}$ .  $\square$

**Theorem 2.4.** *Let  $f : A \rightarrow X$  be an odd mapping for which there exists a function  $\phi : A^5 \rightarrow [0, \infty)$  such that*

$$\|\mu f(\frac{x+y}{2} + z) + \mu f(\frac{x-y}{2} + z) - f(\mu x) - 2f(\mu z) + f(ab) - af(b) - bf(a)\| \leq \phi(x, y, z, a, b), \quad (2.7)$$

for all  $\mu \in \mathbb{T}$  and all  $x, y, z, a, b \in A$ . If there exists an  $L < 1$  such that  $\phi(x, 3x, z, a, b) \leq 2L\phi(\frac{x}{2}, \frac{3x}{2}, \frac{z}{2}, \frac{a}{2}, \frac{b}{2})$  for all  $x, y, a, b \in A$ , then there exists a unique left derivation  $D : A \rightarrow X$  such that

$$\|f(x) - D(x)\| \leq \frac{1}{2 - 2L} \phi(x, 3x, 0, 0, 0) \quad (2.8)$$

for all  $x \in A$ .

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*Proof.* Putting  $\mu = 1, y = 3x, z = a = b = 0$  in (2.7), it follows by oddness of  $f$  that

$$\|f(2x) - 2f(x)\| \leq \phi(x, 3x, 0, 0, 0)$$

for all  $x \in A$ . Hence,

$$\|\frac{1}{2}f(2x) - f(x)\| \leq \frac{1}{2}\phi(x, 3x, 0, 0, 0) \leq L\phi(x, 3x, 0, 0, 0) \quad (2.9)$$

for all  $x \in A$ .

Consider the set  $X := \{g \mid g : A \rightarrow B\}$  and introduce the generalized metric on  $X$ :

$$d(h, g) := \inf\{C \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq C\phi(x, 0, 0, 0, 0) \forall x \in A\}.$$

It is easy to show that  $(X, d)$  is complete. Now we define the linear mapping  $J : X \rightarrow X$  by

$$J(h)(x) = \frac{1}{2}h(2x)$$

for all  $x \in A$ . By Theorem 3.1 of [3],

$$d(J(g), J(h)) \leq Ld(g, h)$$

for all  $g, h \in X$ .

It follows from (2.12) that

$$d(f, J(f)) \leq L.$$

By Theorem 2.1,  $J$  has a unique fixed point in the set  $X_1 := \{h \in X : d(f, h) < \infty\}$ . Let  $D$  be the fixed point of  $J$ .  $D$  is the unique mapping with

$$D(2x) = 2D(x)$$

for all  $x \in A$  satisfying there exists  $C \in (0, \infty)$  such that

$$\|D(x) - f(x)\| \leq C\phi(x, 3x, 0, 0, 0)$$

for all  $x \in A$ . On the other hand we have  $\lim_n d(J^n(f), D) = 0$ . It follows that

$$\lim_n \frac{1}{2^n} f(2^n x) = D(x)$$

for all  $x \in A$ . It follows from  $d(f, D) \leq \frac{1}{1-L}d(f, J(f))$ , which implies that

$$d(f, D) \leq \frac{1}{2-2L}.$$

This implies the inequality (2.8). The rest of proof is similar to the proof of Theorem 2.2.  $\square$

**Corollary 2.5.** Let  $0 < r < \frac{1}{2}, \theta \in [0, \infty)$  be real numbers. Suppose  $f : A \rightarrow X$  satisfies

$$\begin{aligned} & \|\mu f(\frac{x+y}{2} + z) + \mu f(\frac{x-y}{2} + z) - f(\mu x) - 2f(\mu z) + f(ab) - af(b) - bf(a)\| \\ & \leq \theta(\|x\|^r \|y\|^r + \|z\|^r + \|a\|^{2r} + \|b\|^{2r}), \end{aligned}$$

for all  $\mu \in \mathbb{T}$  and all  $x, y, z, a, b \in A$ . Then there exists a unique  $n$ -Jordan derivation  $D : A \rightarrow X$  such that

$$\|f(x) - D(x)\| \leq \frac{3^r \theta}{2-2^r} \|x\|^{2r}$$

for all  $x \in A$ .

*Proof.* It follows from Theorem 2.3, by putting  $\phi(x, y, z, a, b) := \theta(\|x\|^r \|y\|^r + \|z\|^r + \|a\|^{2r} + \|b\|^{2r})$  all  $x, y, z, a, b \in A$  and  $L = 2^{2r-1}$ .  $\square$

## 3. JORDAN LEFT DERIVATIONS

In this section we establish the stability of Jordan left derivations.

**Theorem 3.1.** *Let  $f : A \rightarrow X$  be a mapping for which there exists a function  $\phi : A^4 \rightarrow [0, \infty)$  such that*

$$\|\mu f(\frac{x+y}{2} + z) + \mu f(\frac{x-y}{2} + z) - f(\mu x) - 2f(\mu z) + f(a^2) - 2af(a)\| \leq \phi(x, y, z, a), \quad (3.1)$$

for all  $\mu \in \mathbb{T}$  and all  $x, y, z, a \in A$ . If there exists an  $L < 1$  such that  $\phi(x, y, z, a) \leq 2L\phi(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{a}{2})$  for all  $x, y, z, a \in A$ , then there exists a unique left derivation  $D : A \rightarrow X$  such that

$$\|f(x) - D(x)\| \leq \frac{L}{1-L} \phi(x, 0, 0) \quad (3.2)$$

for all  $x \in A$ .

*Proof.* By hypothesis, we can show that

$$\lim_j 2^{-j} \phi(2^j x, 2^j y, 2^j z, 2^j a) = 0 \quad (3.3)$$

for all  $x, y, z, a \in A$ .

Put  $\mu = 1, z = y = a = 0$  in (3.1) to obtain

$$\|2f(\frac{x}{2}) - f(x)\| \leq \phi(x, 0, 0, 0) \quad (3.4)$$

for all  $x \in A$ . Hence,

$$\|\frac{1}{2}f(2x) - f(x)\| \leq \frac{1}{2}\phi(2x, 0, 0, 0) \leq L\phi(x, 0, 0, 0) \quad (3.5)$$

for all  $x \in A$ .

Consider the set  $X := \{g \mid g : A \rightarrow B\}$  and introduce the generalized metric on  $X$ :

$$d(h, g) := \inf\{C \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq C\phi(x, 0, 0, 0) \forall x \in A\}.$$

It is easy to show that  $(X, d)$  is complete. Now we define the linear mapping  $J : X \rightarrow X$  by

$$J(h)(x) = \frac{1}{2}h(2x)$$

for all  $x \in A$ . By Theorem 3.1 of [3],

$$d(J(g), J(h)) \leq Ld(g, h)$$

for all  $g, h \in X$ .

It follows from (3.5) that

$$d(f, J(f)) \leq L.$$

By Theorem 2.1,  $J$  has a unique fixed point in the set  $X_1 := \{h \in X : d(f, h) < \infty\}$ . Let  $D$  be the fixed point of  $J$ .

By a same reasoning as proof of Theorem 2.2, one can show that the mapping  $D : A \rightarrow X$  is  $\mathbb{C}$ -linear which satisfies (3.2). By putting  $x = y = z = 0$  in (3.1) it follows that

$$\begin{aligned} & \|D(a^2) - 2aD(a)\| \\ &= \lim_n \left\| \frac{1}{4^n} f((4^n a^2) - \frac{1}{2^{n-1}} a f(2^n a) \right\| \leq \\ & \lim_n \frac{1}{4^n} \phi(0, 0, 2^n a) \leq \lim_n \frac{1}{2^n} \phi(0, 0, 2^n a) \\ &= 0 \end{aligned}$$

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for all  $a \in A$ . This means that  $D : A \rightarrow X$  is a Jordan left derivation satisfying (3.2), as desired.  $\square$

We prove the following Hyers–Ulam–Rassias stability problem for left derivations on Banach algebras.

**Corollary 3.2.** *Let  $p \in (0, 1)$ ,  $\theta \in [0, \infty)$  be real numbers. Suppose  $f : A \rightarrow X$  satisfies*

$$\|\mu f(\frac{x+y}{2}+z) + \mu f(\frac{x-y}{2}+z) - f(\mu x) - 2f(\mu z) + f(a^2) - 2af(a)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|a\|^p),$$

*for all  $\mu \in \mathbb{T}$  and all  $x, y, z, a \in A$ . Then there exists a unique left derivation  $D : A \rightarrow X$  such that*

$$\|f(x) - D(x)\| \leq \frac{2^p \theta}{2 - 2^p} \|x\|^p$$

*for all  $x \in A$ .*

*Proof.* It follows from Theorem 3.1, by putting  $\phi(x, y, z, a) := \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|a\|^p)$  all  $x, y, a \in A$  and  $L = 2^{p-1}$ .  $\square$

**Theorem 3.3.** *Let  $f : A \rightarrow X$  be an odd mapping for which there exists a function  $\phi : A^3 \rightarrow [0, \infty)$  such that*

$$\|\mu f(\frac{x+y}{2}+z) + \mu f(\frac{x-y}{2}+z) - f(\mu x) - 2f(\mu z) + f(a^2) - 2af(a)\| \leq \phi(x, y, z, a), \quad (3.6)$$

*for all  $\mu \in \mathbb{T}$  and all  $x, y, z, a \in A$ . If there exists an  $L < 1$  such that  $\phi(x, 3x, z, a) \leq 2L\phi(\frac{x}{2}, \frac{3x}{2}, \frac{z}{2}, \frac{a}{2})$  for all  $x, y, z, a \in A$ , then there exists a unique left derivation  $D : A \rightarrow X$  such that*

$$\|f(x) - D(x)\| \leq \frac{1}{2 - 2L} \phi(x, 3x, 0, 0) \quad (3.7)$$

*for all  $x \in A$ .*

*Proof.* Putting  $\mu = 1, y = 3x, z = a = 0$  in (3.6), it follows by oddness of  $f$  that

$$\|f(2x) - 2f(x)\| \leq \phi(x, 3x, 0, 0)$$

for all  $x \in A$ . Hence,

$$\|\frac{1}{2}f(2x) - f(x)\| \leq \frac{1}{2}\phi(x, 3x, 0, 0) \leq L\phi(x, 3x, 0, 0) \quad (3.8)$$

for all  $x \in A$ .

Consider the set  $X := \{g \mid g : A \rightarrow B\}$  and introduce the generalized metric on  $X$ :

$$d(h, g) := \inf\{C \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq C\phi(x, 0, 0, 0) \forall x \in A\}.$$

It is easy to show that  $(X, d)$  is complete. Now we define the linear mapping  $J : X \rightarrow X$  by

$$J(h)(x) = \frac{1}{2}h(2x)$$

for all  $x \in A$ . By Theorem 3.1 of [3],

$$d(J(g), J(h)) \leq Ld(g, h)$$

for all  $g, h \in X$ .

It follows from (3.8) that

$$d(f, J(f)) \leq L.$$

By Theorem 2.1,  $J$  has a unique fixed point in the set  $X_1 := \{h \in X : d(f, h) < \infty\}$ . Let  $D$  be the fixed point of  $J$ .  $D$  is the unique mapping with

$$D(2x) = 2D(x)$$

for all  $x \in A$  satisfying there exists  $C \in (0, \infty)$  such that

$$\|D(x) - f(x)\| \leq C\phi(x, 3x, 0, 0)$$

for all  $x \in A$ . On the other hand we have  $\lim_n d(J^n(f), D) = 0$ . It follows that

$$\lim_n \frac{1}{2^n} f(2^n x) = D(x)$$

for all  $x \in A$ . It follows from  $d(f, D) \leq \frac{1}{1-L} d(f, J(f))$ , which implies that

$$d(f, D) \leq \frac{1}{2-2L}.$$

This implies the inequality (3.7). The rest of proof is similar to the proof of Theorem 3.1.  $\square$

**Corollary 3.4.** Let  $0 < r < \frac{1}{2}, \theta \in [0, \infty)$  be real numbers. Suppose  $f : A \rightarrow X$  satisfies

$$\|\mu f(\frac{x+y}{2}+z) + \mu f(\frac{x-y}{2}+z) - f(\mu x) - 2f(\mu z) + f(a^2) - 2af(a)\| \leq \theta(\|x\|^r \|y\|^r + \|z\|^r + \|a\|^{2r}),$$

for all  $\mu \in \mathbb{T}$  and all  $x, y, z, a \in A$ . Then there exists a unique Jordan left derivation  $D : A \rightarrow X$  such that

$$\|f(x) - D(x)\| \leq \frac{3^r \theta}{2-2^r} \|x\|^{2r}$$

for all  $x \in A$ .

*Proof.* It follows from Theorem 2.3, by putting  $\phi(x, y, z, a) := \theta(\|x\|^r \|y\|^r + \|z\|^r + \|a\|^{2r})$  all  $x, y, a \in A$  and  $L = 2^{2r-1}$ .  $\square$

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## ON THE CATEGORY OF INTUITIONISTIC FUZZY METRIC SPACES

HAKAN EFE, SERKAN GÜMÜŞ, AND CEMIL YILDIZ

**ABSTRACT.** The aim of this paper to give the category of intuitionistic fuzzy metric spaces with the objects are complete intuitionistic fuzzy metric spaces defined in the sense of Alaca et.al. Furthermore, the existence of solution for domain equation in these intuitionistic fuzzy settings by defining a categorical contraction mapping in the sense of Alaca et.al. is investigated.

### 1. INTRODUCTION

After the introduction of the concept of fuzzy sets by Zadeh [17] in 1965, many authors have introduced the fuzzy metric spaces in different ways [4,5,7,8]. Grabiec [5] extends two fixed point theorems of Banach and Edelstein to contractive mappings of complete and compact fuzzy metric spaces, respectively.

In [2,3], Alessi et.al, have studied on solution of metric domain equation in the categories of complete metric spaces. In their papers, a new method for solving domain equations in categories of metric spaces is studied. Rafi [12] introduced the category of fuzzy metric spaces with the objects are complete fuzzy metric spaces defined in the sense of Kramosil and Michalek [8]. As an application, he investigate the existence of solution for domain equation in these fuzzy settings by defining a categorical contraction mapping in the sense of Grabiec [5]. Recently, Rafi [13] studied on probabilistic nonexpansive mappings between probabilistic metric spaces and proved a fixed point theorem in category of probabilistic metric spaces.

Park [11] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy (briefly, *IF*) metric spaces with the help of continuous *t*-norm and continuous *t*-conorm as a generalization of fuzzy metric space due to George and Veeramani [4]. Alaca et. al. [1] using the idea of *IF*-sets, defined the notion of *IF*-metric space as Park [11] with the help of continuous *t*-norms and continuous *t*-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [8]. Further, they introduced the notion of Cauchy sequences in an *IF*-metric spaces and proved the well-known fixed point theorems of Banach and Edelstein extended to *IF*-metric spaces with the help of Grabiec [5]. Many authors studied on fixed point theorems in *IF*-metric spaces [6,10,15].

In this paper, we give the category of *IF*-metric spaces with the objects are complete *IF*-metric spaces and morphisms are  $(\varepsilon - \delta)$ -*IF*-adjoint pairs defined in the sense of Alaca et.al [1]. Furthermore, we investigate the existence of solution for domain equation in these *IF* settings by defining a categorical contraction mapping in the sense of Alaca et.al [1].

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## 2. PRELIMINARIES AND SOME RESULTS

**Definition 1** ([16]). A category  $C$  consist of

- (i) A class of objects
- (ii) For every ordered pair of objects  $X$  and  $Y$ , a set  $\text{hom}(X, Y)$  of morphisms with domain  $X$  and range  $Y$ ; if  $f \in \text{hom}(X, Y)$ , we write  $f : X \longrightarrow Y$  or  $X \xrightarrow{f} Y$ .
- (iii) For every ordered triple of objects  $X, Y$  and  $Z$ , a function associating to a pair of morphisms  $f : X \longrightarrow Y$  and  $g : Y \longrightarrow Z$  their composite,

$$gf = g \circ f : X \longrightarrow Z$$

these satisfy the following two axioms:

*Associativity:* If  $f : X \longrightarrow Y$ ,  $g : Y \longrightarrow Z$  and  $h : Z \longrightarrow W$ , then

$$h \circ (g \circ f) : (h \circ g) \circ f : X \longrightarrow W$$

*Identity:* For every object  $Y$  there is a morphism  $\text{id}_Y : Y \longrightarrow Y$  such that if  $f : X \longrightarrow Y$ , then  $\text{id}_Y \circ f = f$ , and if  $h : Y \longrightarrow Z$  and  $h \circ \text{id}_Y = h$ .

For example; the category of topological spaces and continuous maps or category of groups and homomorphisms.

**Definition 2** ([16]). Let  $C$  and  $D$  be categories. A covariant functor (or contravariant functor)  $T$  from  $C$  to  $D$  consist of an object function which assigns to every object  $X$  of  $C$  an object  $T(X)$  of  $D$  and a morphism function which assigns to every morphism  $f : X \longrightarrow Y$  of  $C$  a morphism  $T(f) : T(X) \longrightarrow T(Y)$  [or  $T(f) : T(Y) \longrightarrow T(X)$ ] of  $D$  such that,

- (i)  $T(1_X) = 1_{T(X)}$ ,
- (ii)  $T(g \circ f) = T(g) \circ T(f)$  [or  $T(g \circ f) = T(f) \circ T(g)$ ]

For example; there is a covariant functor from the category of topological spaces and continuous maps to the category of sets and functions which assigns to every topological space its underlying set.

**Definition 3** ([14]). A binary operation  $*$  :  $[0, 1] \times [0, 1] \longrightarrow [0, 1]$  is a continuous  $t$ -norm if it satisfies the following conditions: (i)  $*$  is associative and commutative; (ii)  $*$  is continuous; (iii)  $a * 1 = a$  for all  $a \in [0, 1]$ ; (iv)  $a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$ ; for each  $a, b, c, d \in [0, 1]$ .

**Definition 4** ([14]). A binary operation  $\diamond$  :  $[0, 1] \times [0, 1] \longrightarrow [0, 1]$  is a continuous  $t$ -conorm if it satisfies the following conditions: (i)  $\diamond$  is associative and commutative; (ii)  $\diamond$  is continuous; (iii)  $a \diamond 0 = a$  for all  $a \in [0, 1]$ ; (iv)  $a \diamond b \leq c \diamond d$  whenever  $a \leq c$  and  $b \leq d$ ; for each  $a, b, c, d \in [0, 1]$ .

**Definition 5** ([1]). A 5-tuple  $(X, M, N, *, \diamond)$  is said to be an IF-metric space if  $X$  is an arbitrary set,  $*$  is a continuous  $t$ -norm,  $\diamond$  is a continuous  $t$ -conorm and  $M, N$  are fuzzy sets on  $X^2 \times [0, \infty)$  satisfying the following conditions: for all  $x, y, z \in X$ ,  $s, t > 0$

- (i)  $M(x, y, t) + N(x, y, t) \leq 1$ ,
- (ii)  $M(x, y, 0) = 0$ ,
- (iii)  $M(x, y, t) = 1$  for all  $t > 0$  if and only if  $x = y$ ,
- (iv)  $M(x, y, t) = M(y, x, t)$ ,
- (v)  $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$  for all  $x, y, z \in X$ ,  $s, t > 0$
- (vi)  $M(x, y, \cdot) : [0, \infty) \longrightarrow [0, 1]$  is left continuous,

- (vii)  $\lim_{t \rightarrow \infty} M(x, y, t) = 1$  for all  $x, y$  in  $X$
- (viii)  $N(x, y, 0) = 1$ ,
- (ix)  $N(x, y, t) = 0$  for all  $t > 0$  if and only if  $x = y$ ,
- (x)  $N(x, y, t) = N(y, x, t)$ ,
- (xi)  $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$  for all  $x, y, z \in X$ ,  $s, t > 0$ ,
- (xii)  $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$  is right continuous,
- (xiii)  $\lim_{t \rightarrow \infty} N(x, y, t) = 0$  for all  $x, y$  in  $X$ .

Then  $(M, N)$  is called an *IF-metric* on  $X$ . The functions  $M(x, y, t)$  and  $N(x, y, t)$  denote the degree of nearness and the degree of non-nearness between  $x$  and  $y$  with respect to  $t$ , respectively.

**Remark 1.** Every fuzzy metric space  $(X, M, *)$  is an *IF-metric* space of the form  $(X, M, 1 - M, *, \diamond)$  such that  $t$ -norm  $*$  and  $t$ -conorm  $\diamond$  are associated [9], i.e.  $x \diamond y = 1 - ((1 - x) * (1 - y))$  for any  $x, y \in X$ .

**Remark 2.** In *IF-metric* space  $X$ ,  $M(x, y, \cdot)$  is non-decreasing and  $N(x, y, \cdot)$  is non-increasing for all  $x, y \in X$ .

**Definition 6** ([1]). Let  $(X, M, N, *, \diamond)$  be a *IF-metric* space. Then

(a) a sequence  $\{x_n\}$  in  $X$  is said to be *Cauchy sequence* if for each  $t > 0$  and  $p > 0$ ,

$$\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, t) = 1 \text{ and } \lim_{n \rightarrow \infty} N(x_{n+p}, x_n, t) = 0$$

(b) a sequence  $\{x_n\}$  in  $X$  is *converging to  $x$  in  $X$*  if for each  $t > 0$ ,

$$\lim_{n \rightarrow \infty} M(x_n, x, t) = 1 \text{ and } \lim_{n \rightarrow \infty} N(x_n, x, t) = 0$$

(Since  $*$  and  $\diamond$  are continuous, the limit is uniquely determined from (v) and (xi).)

A *IF-metric* space is said to be *complete* if and only if every Cauchy sequence is convergent.

Now, we define the basic notion of  $(\varepsilon - \delta)$ -*IF*-adjoint and  $(\varepsilon - \delta)$ -*IF*-isometry.

**Definition 7.** Let  $(X, M_1, N_1, *, \diamond)$  and  $(Y, M_2, N_2, *, \diamond)$  be *IF-metric* spaces under the same  $t$ -norm  $*$  and  $t$ -conorm  $\diamond$ . A mapping  $f : X \rightarrow Y$  is called *IF-non-expansive* if for all  $x, x' \in X$  and  $t > 0$  the following conditions hold:

$$M_2(fx, fx', t) \geq M_1(x, x', t) \text{ and } N_2(fx, fx', t) \leq N_1(x, x', t).$$

**Lemma 1.** Let  $F^{XY} = \{f : X \rightarrow Y \mid f \text{ is IF-non-expansive}\}$

$$\mathcal{M}_{XY}(f, g, t) = \inf_{x \in X} M_2(fx, gx, t) \text{ and } \mathcal{N}_{XY}(f, g, t) = \sup_{x \in X} N_2(fx, gx, t)$$

for every  $f, g \in F^{XY}$ . Then,  $(F^{XY}, \mathcal{M}_{XY}, \mathcal{N}_{XY}, *, \diamond)$  is an *IF-metric* space if and only if  $(Y, M_2, N_2, *, \diamond)$  is an *IF-metric* space.

**Definition 8.** Let  $(X, M_1, N_1, *, \diamond)$  and  $(Y, M_2, N_2, *, \diamond)$  be *IF-metric* spaces.

(i) A mapping  $f \in F^{XY}$  is called *IF-mapping* if there exists a  $k \in (0, 1)$  such that for all  $x, x' \in X$  and for all  $t > 0$ ,

$$M_2(fx, fx', kt) \geq M_1(x, x', t) \text{ and } N_2(fx, fx', kt) \leq N_1(x, x', t).$$

$k$  is called the *contractive constant* of  $f$ .

(ii) A mapping  $f \in F^{XY}$  is called *IF-isometric embedding* if for all  $x, x' \in X$  and for all  $t > 0$ ,

$$M_2(fx, fx', t) = M_1(x, x', t) \text{ and } N_2(fx, fx', t) = N_1(x, x', t).$$

If  $f$  is bijection then it is an *IF-isometry*.

**Definition 9.** For  $r_1, r_2, r'_1, r'_2, \varepsilon, \delta \in [0, 1]$  where  $\varepsilon + \delta \leq 1$ , let

$$\begin{aligned} r_1 &\approx_{\varepsilon} r_2 \iff r_1 * r_2 \geq \varepsilon \text{ and} \\ r'_1 &\approx_{\delta} r'_2 \iff r'_1 \diamond r'_2 \leq \delta. \end{aligned}$$

**Definition 10.** Let  $(X, M_1, N_1, *, \diamond)$  and  $(Y, M_2, N_2, *, \diamond)$  be *IF-metric spaces*. Two *IF-non-expansive mappings*  $f : X \longrightarrow Y$  and  $g : Y \longrightarrow X$  are said to be  $(\varepsilon - \delta)$ -*IF-adjoint* if for all  $x \in X, y \in Y, t > 0$ ,

$$M_2(fx, y, t) \approx_{\varepsilon} M_1(x, gy, t) \text{ and } N_2(fx, y, t) \approx_{\delta} N_1(x, gy, t).$$

If  $f$  and  $g$  are  $(1, 0)$ -*IF-adjoint*, then  $\langle f, g \rangle$  is called a *proper IF-adjoint pair*.

**Definition 11.** Let  $(X, M_1, N_1, *, \diamond)$  and  $(Y, M_2, N_2, *, \diamond)$  be *IF-metric spaces*. Two *IF-non-expansive mappings*  $f : X \longrightarrow Y$  and  $g : Y \longrightarrow X$ ,

$$\begin{aligned} \Im(f, g, t) &= \min\{\mathcal{M}_{XX}(id_x, g \circ f, t), \mathcal{M}_{YY}(f \circ g, id_Y, t)\} \text{ and} \\ \aleph(f, g, t) &= \max\{\mathcal{N}_{XX}(id_x, g \circ f, t), \mathcal{N}_{YY}(f \circ g, id_Y, t)\}. \end{aligned}$$

**Definition 12.** Let  $(X, M_1, N_1, *, \diamond)$  and  $(Y, M_2, N_2, *, \diamond)$  be *IF-metric spaces*. A pair of non-expansive mappings  $f : X \longrightarrow Y$  and  $g : Y \longrightarrow X$  with  $\Im(f, g, t) \approx_{\varepsilon} 1$  and  $\aleph(f, g, t) \approx_{\delta} 0$  is called an  $(\varepsilon - \delta)$ -*IF-isometry* where  $\varepsilon, \delta \in [0, 1]$  and  $\varepsilon + \delta \leq 1$ .

Note that by definition, any pair  $\langle f, g \rangle$  of *IF-non-expansive mappings* is an  $(\varepsilon - \delta)$ -*IF-isometry*, for  $\varepsilon = \Im(f, g, t)$  and  $\delta = \aleph(f, g, t)$ .

The above definition can be justified by observation that  $(1, 0)$ -*IF-isometries* satisfy  $id_X = g \circ f$  and  $f \circ g = id_Y$  for all  $t > 0$  (from definition 5) and consequently  $f$  (and also  $g$ ) is an isometry.

Under some strict condition on the  $t$ -norm and  $t$ -conorm, we have the following equivalence of mappings.

**Theorem 1.** Let  $(X, M_1, N_1, *, \diamond)$  and  $(Y, M_2, N_2, *, \diamond)$  be *IF-metric spaces* under the same  $t$ -norm  $*$  and  $t$ -conorm  $\diamond$  such that  $a * b = \min\{a, b\}$  and  $a \diamond b = \max\{a, b\}$  and let  $\varepsilon, \delta \in [0, 1]$  where  $\varepsilon + \delta \leq 1$ . For all non-expansive mappings  $f : X \longrightarrow Y$  and  $g : Y \longrightarrow X$ .  $\langle f, g \rangle$  is an  $(\varepsilon - \delta)$ -*IF-adjoint* if and only if  $\langle f, g \rangle$  is an  $(\varepsilon - \delta)$ -*IF-isometry*.

*Proof.* Let  $f$  and  $g$  are  $(\varepsilon - \delta)$ -*IF-adjoint*. Then for any  $x \in X$ ,

$$\begin{aligned} M_1(x, g \circ f(x), t) &\approx_{\varepsilon} M_2(f(x), f(x), t) = 1 \text{ and} \\ N_1(x, g \circ f(x), t) &\approx_{\delta} N_2(f(x), f(x), t) = 0. \end{aligned}$$

Thus,  $\mathcal{M}_{XX}(id_x, g \circ f, t) \approx_{\varepsilon} 1$  and  $\mathcal{N}_{XX}(id_x, g \circ f, t) \approx_{\delta} 0$ . Similarly, for any  $y \in Y$ ,

$$\begin{aligned} M_2(f \circ g(y), y, t) &\approx_{\varepsilon} M_1(g(y), g(y), t) = 1 \text{ and} \\ N_2(f \circ g(y), y, t) &\approx_{\delta} N_1(g(y), g(y), t) = 0 \end{aligned}$$

which mean  $\mathcal{M}_{YY}(f \circ g, id_Y, t) \approx_{\varepsilon} 1$  and  $\mathcal{N}_{YY}(f \circ g, id_Y, t) \approx_{\delta} 0$ . Hence,  $\Im(f, g, t) \approx_{\varepsilon} 1$  and  $\aleph(f, g, t) \approx_{\delta} 0$ . We conclude that  $\langle f, g \rangle$  is an  $(\varepsilon - \delta)$ -*IF-isometry*.

Conversely, suppose that  $\Im(f, g, t) \approx_\varepsilon 1$  and  $\aleph(f, g, t) \approx_\delta 0$ . For all  $x \in X$ ,  $y \in Y$ , and  $k > 1$ , we have

$$\begin{aligned} M_1(x, g(y), t) &\geq M_1(x, g \circ f(x), (1-k)t) * M_1(g \circ f(x), g(y), kt) \\ &\geq \varepsilon * M_1(g \circ f(x), g(y), kt) \\ &\geq \min\{\varepsilon, M_1(g \circ f(x), g(y), kt)\} \\ &\geq \min\{\varepsilon, M_2(f(x), y, kt)\}. \end{aligned}$$

Similarly,

$$\begin{aligned} M_2(f(x), y, t) &\geq M_2(f(x), f \circ g(y), kt) * M_2(f \circ g(y), y, (1-k)t) \\ &\geq M_2(f(x), f \circ g(y), kt) * \varepsilon \\ &\geq \min\{M_1(x, g(y), kt), \varepsilon\}. \end{aligned}$$

Hence,

$$\min \left\{ \begin{array}{l} M_1(x, g(y), t), \\ M_2(f(x), y, t) \end{array} \right\} \geq \min\{\varepsilon, M_1(x, g(y), kt), M_2(f(x), y, kt)\} \geq \varepsilon$$

i.e.,  $M_1(x, g(y), t) * M_2(f(x), y, t) \geq \varepsilon$ , which implies

$$(2.1) \quad M_2(f \circ g(y), y, t) \approx_\varepsilon M_1(g(y), g(y), t).$$

On the other hand for all  $x \in X$ ,  $y \in Y$ , and  $k > 1$ , we have

$$\begin{aligned} N_1(x, g(y), t) &\leq N_1(x, g \circ f(x), (1-k)t) \Diamond N_1(g \circ f(x), g(y), kt) \\ &\leq \delta \Diamond N_1(g \circ f(x), g(y), kt) \\ &\leq \max\{\delta, N_1(g \circ f(x), g(y), kt)\} \\ &\leq \max\{\delta, N_2(f(x), y, kt)\}. \end{aligned}$$

Similarly,

$$\begin{aligned} N_2(f(x), y, t) &\leq N_2(f(x), f \circ g(y), kt) \Diamond N_2(f \circ g(y), y, (1-k)t) \\ &\leq N_2(f(x), f \circ g(y), kt) \Diamond \delta \\ &\leq \max\{N_1(x, g(y), kt), \delta\}. \end{aligned}$$

Hence,

$$\max \left\{ \begin{array}{l} N_1(x, g(y), t), \\ N_2(f(x), y, t) \end{array} \right\} \leq \max\{\delta, N_1(x, g(y), kt), N_2(f(x), y, kt)\} \leq \delta$$

i.e.,  $N_1(x, g(y), t) \Diamond N_2(f(x), y, t) \leq \delta$ , which implies

$$(2.2) \quad N_2(f \circ g(y), y, t) \approx_\delta N_1(g(y), g(y), t).$$

Therefore from 2.1 and 2.2  $\langle f, g \rangle$  is an  $(\varepsilon - \delta)$ -IF-adjoint.  $\square$

### 3. THE CATEGORY OF IF-METRIC SPACES

**Definition 13.** Let  $IFMS^\approx$  denote the category of IF-metric spaces that has nonempty complete IF-metric spaces as objects and  $(\varepsilon - \delta)$ -IF-adjoint pairs as morphisms. The composition of a pair of morphisms

$$\begin{aligned} h_1 &= \langle f_1, g_1 \rangle : (X, M_1, N_1, *, \Diamond) \longrightarrow (Y, M_2, N_2, *, \Diamond) \text{ and} \\ h_2 &= \langle f_2, g_2 \rangle : (Y, M_2, N_2, *, \Diamond) \longrightarrow (Z, M_3, N_3, *, \Diamond) \end{aligned}$$

is defined as

$$h_2 \circ h_1 = \langle f_2 \circ f_1, g_1 \circ g_2 \rangle : (X, M_1, N_1, *, \Diamond) \longrightarrow (Z, M_3, N_3, *, \Diamond).$$

**Definition 14.** Let  $IFMS^\approx$  be a category.

(i) An IF-tower in  $IFMS^\approx$  is a sequence  $\{(X_n, M_n, N_n, *, \diamond), h_n\}_n$  of objects and morphisms such that for all  $n \in \mathbb{N}$ ,

$$h_n : (X_n, M_n, N_n, *, \diamond) \longrightarrow (X_{n+1}, M_{n+1}, N_{n+1}, *, \diamond).$$

(ii) A tower  $\{(X_n, M_n, N_n, *, \diamond), h_n\}_n$  in  $IFMS^\approx$ , with  $h_n = \langle f_n, g_n \rangle$ , is called a Cauchy if for

$$\lim_{n \rightarrow \infty} \Im(h_{n,n+p}, t) = 1 \text{ and } \lim_{n \rightarrow \infty} \Re(h_{n,n+p}, t) = 0$$

for each  $n \in \mathbb{N}$  and  $p, t > 0$ , where  $h_{n,n+p} = h_{n+p-1} \circ h_{n+p-2} \circ \cdots \circ h_n$ .

**Definition 15.** Let  $\{(X_n, M_n, N_n, *, \diamond), h_n\}_n$  be a Cauchy tower in  $IFMS^\approx$ , where  $h_n = \langle f_n, g_n \rangle$ . The direct limit of  $\{(X_n, M_n, N_n, *, \diamond), h_n\}_n$  is a fuzzy cone  $((X, M, N, *, \diamond), \{\gamma_n\}_n)$ , where  $\gamma_n = (\alpha_n, \beta_n)$ , which is defined as follows:

(i) The IF-metric space  $(X, M, N, *, \diamond)$  is given by

$$X = \{\{x_n\}_n : \forall n \in \mathbb{N}, x_n \in X_n \text{ and } x_n = g_n(x_{n+1})\}$$

and

$$\begin{aligned} M(\{x_n\}_n, \{x'_n\}_n, t) &= \inf_{n \in \mathbb{N}} M_n(x_n, x'_n, t) \text{ and} \\ N(\{x_n\}_n, \{x'_n\}_n, t) &= \sup_{n \in \mathbb{N}} N_n(x_n, x'_n, t). \end{aligned}$$

(ii) Morphisms  $\gamma_n = (\alpha_n, \beta_n) : X_n \longrightarrow X$  are defined as follows:

- $\alpha_n : X_n \longrightarrow X$  where  $\alpha_n(x_n) = \{x_k\}_k$  with  $x_k = \lim_{r \rightarrow \infty} g_{kr} \circ f_{nr}(x)$
- $\beta_n : X \longrightarrow X_n$  where  $\beta_n(\{x_k\}_k) = x_n$ .

The notion of initial object of a category can be defined as follows:

**Definition 16.** An initial object of a category  $IFMS^\approx$  is an object  $(\vartheta, M_\vartheta, N_\vartheta, *, \diamond)$  in  $IFMS^\approx$  such that for every object  $(X, M, N, *, \diamond)$  in  $IFMS^\approx$ , there exists a unique morphism  $\tau : (\vartheta, M_\vartheta, N_\vartheta, *, \diamond) \longrightarrow (X, M, N, *, \diamond)$ .

**Lemma 2.** If  $\lim_{n \rightarrow \infty} \Im(\gamma_n, t) = 1$  and  $\lim_{n \rightarrow \infty} \Re(\gamma_n, t) = 0$ , then  $((X, M, N, *, \diamond), \{\gamma_n\}_n)$  with  $\gamma_n = (\alpha_n, \beta_n)$  will be the initial cone of the Cauchy tower  $\{(X_n, M_n, N_n, *, \diamond), h_n\}_n$ .

*Proof.* Let  $((X', M', N', *, \diamond), \{\gamma'_n\}_n)$  with  $\gamma'_n = (\alpha'_n, \beta'_n)$  be another cone for  $\{(X_n, M_n, N_n, *, \diamond), h_n\}_n$ . We show that there exists a unique morphism  $\tau : (X, M, N, *, \diamond) \longrightarrow (X', M', N', *, \diamond)$  such that for all  $n \in \mathbb{N}$ ,  $\gamma'_n = \tau \circ \delta_n$ . Note that  $\{\alpha'_n \circ \beta_n\}$  and  $\{\alpha_n \circ \beta'_n\}$  are Cauchy sequence, since  $\{(X_n, M_n, N_n, *, \diamond), h_n\}_n$  is a Cauchy sequence. Furthermore, the objects of  $IFMS^\approx$  are complete, so we can define  $\lim_{n \rightarrow \infty} (\alpha'_n \circ \beta_n) = i$  and  $\lim_{n \rightarrow \infty} (\alpha_n \circ \beta'_n) = j$ . Obviously, this defines a morphism  $\tau = (i, j) : (X, M, N, *, \diamond) \longrightarrow (X', M', N', *, \diamond)$ . It follows from the facts that  $\lim_{n \rightarrow \infty} \Im(\delta_n, t) = 1$  and  $\lim_{n \rightarrow \infty} \Re(\gamma_n, t) = 0$  that  $\gamma'_n = \tau \circ \delta_n$ ,  $\tau$  is the unique morphism with this property. This proves that  $((X, M, N, *, \diamond), \{\gamma_n\}_n)$  is the initial cone of the tower  $\{(X_n, M_n, N_n, *, \diamond), h_n\}_n$ .  $\square$

As a consequence of the above lemma, we have

**Corollary 1.** The direct limit of a Cauchy tower is an initial cone for that tower.

**Remark 3.** We note that if  $((X', M', N', *, \diamond), \{\gamma'_n\}_n)$  with  $\gamma'_n = (\alpha'_n, \beta'_n)$  be another initial cone for the Cauchy tower  $\{(X_n, M_n, N_n, *, \diamond), h_n\}_n$ , then by the above corollary, we have  $(X, M, N, *, \diamond) \simeq (X', M', N', *, \diamond)$ . Thus, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathfrak{I}(\gamma_n, t) &= \lim_{n \rightarrow \infty} \mathfrak{I}(\gamma'_n, t) = 1 \text{ and} \\ \lim_{n \rightarrow \infty} \mathfrak{N}(\gamma_n, t) &= \lim_{n \rightarrow \infty} \mathfrak{N}(\gamma'_n, t) = 0. \end{aligned}$$

**Lemma 3.** (Initiality Lemma) Let  $\{(X_n, M_n, N_n, *, \diamond), h_n\}_n$  be an *IF*-Cauchy tower in  $IFMS^\approx$  and let  $((X, M, N, *, \diamond), \{\gamma_n\}_n)$ , with  $\gamma_n = (\alpha_n, \beta_n)$  be a cone. Then  $((X, M, N, *, \diamond), \{\gamma_n\}_n)$  is an initial cone if and only if  $\lim_{n \rightarrow \infty} \mathfrak{I}(\gamma_n, t) = 1$  and  $\lim_{n \rightarrow \infty} \mathfrak{N}(\gamma_n, t) = 0$ .

*Proof.* It follows by Lemma 2 and the Remark 3.  $\square$

#### 4. AN APPLICATION TO FIXED POINT THEORY

In this section, we give a fixed point theorem in category of complete *IF*-metric spaces. Before we prove the fixed point theorem in the category  $IFMS^\approx$ , we state the following *IF*-Banach contraction theorem due to Alaca, [1].

**Theorem 2** ([1]). (*IF*-Banach contraction theorem). Let  $(X, M, N, *, \diamond)$  be a complete *IF*-metric space and  $T : X \rightarrow X$  be a mapping satisfying

$$M(Tx, Ty, kt) \geq M(x, y, t) \text{ and } N(Tx, Ty, kt) \leq N(x, y, t)$$

for all  $x, y$  in  $X$ ,  $0 < k < 1$ . Then  $T$  has a unique fixed point.

In category  $IFMS^\approx$ , we have the following definition.

**Definition 17.** A functor  $T : IFMS^\approx \rightarrow IFMS^\approx$  is called  $CAT^\approx$ -contraction, if there exists a  $k \in (0, 1)$  such that for each morphism  $\tau : (X_1, M_1, N_1, *, \diamond) \rightarrow (X_2, M_2, N_2, *, \diamond)$ ,

$$\mathfrak{I}(T\tau, kt) \geq \mathfrak{I}(\tau, t) \text{ and } \mathfrak{N}(T\tau, kt) \leq \mathfrak{N}(\tau, t)$$

where  $T\tau = (Tf, Tg)$  for  $\tau = (f, g)$ .

By the Initiality Lemma, a  $CAT^\approx$ -contraction functor preserves Cauchy tower and the initial cones, in a similar way as *IF*-contracting functions preserves Cauchy sequence and their limits.

We prove the following theorem which shows the existence of fixed points for contracting functors on the category  $IFMS^\approx$ .

**Theorem 3.** Let  $T : IFMS^\approx \rightarrow IFMS^\approx$  be a  $CAT^\approx$ -contraction functor. Then,  $T$  has a fixed point, i.e., there exists a complete *IF*-metric space  $(X, M, N, *, \diamond)$  such that  $(X, M, N, *, \diamond) \simeq (TX, M, N, *, \diamond)$ .

*Proof.* Let  $(X_0, M_0, N_0, *, \diamond)$  be a complete *IF*-metric space in  $IFMS^\approx$  and let  $f_0 : X_0 \rightarrow \xi X_0$  be any morphism. Consider the *IF*-tower  $\{(T^n X_0, M_0, N_0, *, \diamond), T^n f_0\}_n$ . Since  $T$  is a  $CAT^\approx$ -contraction functor, it is a Cauchy tower in  $IFMS^\approx$ . Thus, it has a direct limit,  $((X, M, N, *, \diamond), \{\gamma_n\}_n)$ , which is an initial cone for the tower. Hence, we conclude that  $(X, M, N, *, \diamond) \simeq (TX, M, N, *, \diamond)$ .  $\square$

As given in the Remark 3.9 of [2], the contractiveness is not a necessary condition in order that a functor has fixed points. As an example, the identity functor is not contracting.

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# A note on the oscillation of second order differential equations with damping

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## **Abstract**

We present some new oscillation criteria for the second order nonlinear differential equation with damping of the form

$$(r(t)\psi(x(t))K(x'(t)))' + p(t)K(x'(t)) + q(t)x = 0.$$

Our results generalize and extend some known oscillation criteria in the literature.

**Key words :** Oscillatory, second order differential equations.

**AMS (MOS) Subject Classification:** 34A30, 34C10.

## **1 Introduction**

We shall be concerned here with the oscillatory properties for the second order nonlinear differential equation with damping term

$$(r(t)\psi(x(t))K(x'(t)))' + p(t)K(x'(t)) + q(t)x = 0, \quad t \geq t_0 > 0, \quad (1.1)$$

where  $r : [t_0, \infty) \rightarrow (0, \infty)$ ,  $p, q : [t_0, \infty) \rightarrow \mathbb{R} = (-\infty, \infty)$ ,  $K : \mathbb{R} \rightarrow \mathbb{R}$  and  $\psi : \mathbb{R} \rightarrow \mathbb{R}^+ = (0, \infty)$  are continuous functions. Throughout this paper we shall also assume that



- (i)  $0 < c \leq \psi(x) \leq c_1$  for all  $x \in R$ ;
- (ii)  $\gamma > 0$  and  $K^2(y) \leq \gamma y K(y)$  for all  $y \neq 0$ , where  $c, c_1$  and  $\gamma$  are constants.

The oscillatory character is considered in the usual sense, i.e., a solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros, otherwise it is said to be nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

The oscillation problem of solutions for various classes of second order differential equations have been widely discussed in the literature (see, for example, [1-15] and the references quoted therein ). There is a great number of papers dealing with particular cases of equation (1.1) such as the linear equations

$$x'' + p(t)x' + q(t)x = 0, \quad t \geq t_0 > 0, \quad (1.2)$$

$$x'' + q(t)x = 0, \quad t \geq t_0 > 0 \quad (1.3)$$

and

$$(r(t)x')' + q(t)x = 0, \quad t \geq t_0 > 0, \quad (1.4)$$

Actually, equation (1.2) can be reduced via suitable Sturm-Liouville transformation to the undamped equation. Here an additional assumption is added on  $p(t)$ , namely  $p(t)$  assumed to be continuously differentiable. By Sturm-Liouville transformation

$$y(t) = x(t) \exp \left( \frac{1}{2} \int_{t_0}^t p(s) ds \right) \quad (1.5)$$

reduces equation (1.2) to

$$y'' + \left( q(t) - \frac{p^2(t)}{4} - \frac{p'(t)}{2} \right) y = 0, \quad t \geq t_0 > 0 \quad (1.6)$$

which is in the form (1.3). Although (1.2) can be put in the form (1.3), there are advantages in obtaining direct oscillation theorems for (1.2). Oscillation criteria of (1.2)-(1.4) have been extensively studied by many authors. Many of these criteria involve the integral of the coefficients. Downstairs is a list of some well known oscillation criteria for equation (1.3) that exist in the literature

$$\lim_{t \rightarrow \infty} Q(t) = \lim_{t \rightarrow \infty} \int_{t_0}^t q(s) ds \quad (1.7)$$

(Fite-Winter-Leighton [4,5,7]);

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_{t_0}^t Q(s) ds = \infty \quad (1.8)$$

(Wintner [5]);

$$\limsup_{t \rightarrow \infty} \frac{1}{t^a} \int_{t_0}^t (t-s)^{a-1} Q(s) ds = \infty, \quad \text{for some } a > 1 \quad (1.9)$$

(Kamenev [6]).

Even though (1.5) is an oscillation-preserving transformation under the additional that  $p(t)$  is differentiable or at least  $p(t)$  is absolutely continuous so that  $p'(t)$  is defined. But this superfluous condition was not assumed in Sobol's paper (see [12]). By using polar coordinates transformation, he proved that

$$\lim_{t \rightarrow \infty} G(t) = \lim_{t \rightarrow \infty} \left[ Q(t) - \frac{p(t)}{2} - \frac{1}{4} \int_{t_0}^t p^2(s) ds \right] = \infty \quad (1.10)$$

was sufficient for equation (1.2) to be oscillatory. Wong [3] noticed this point and gave several oscillation criteria for equation (1.2), which generalized the results due to Wintner and Kamenev.

More recently, Zheng and Liu [1] established oscillation criteria for equation (1.2) and the main results are as follows.

They assume that  $g(t) \in C^2([t_0, \infty); (0, \infty))$  is a given functions,  $f(t) = -\frac{g'(t)}{2g(t)}$  and

$$\Phi(t) = -\frac{1}{2}g(t)p(t) + \int_{t_0}^t g(s) \left[ q(s) - f(s)p(s) + f^2(s) - f'(s) - \frac{p^2(s)}{4} \right] ds.$$

**Theorem A.** Suppose that

$$\int_{t_0}^{\infty} g^{-1}(s) ds = \infty \quad (1.11)$$

holds. Then equation (1.2) is oscillatory provided

$$\lim_{t \rightarrow \infty} \Phi(t) = \infty. \quad (1.12)$$

**Theorem B.** Suppose that

$$\int_{t_0}^{\infty} \left( \int_{t_0}^t g(\tau) d\tau \right)^{-1} ds = \infty \quad (1.13)$$

holds. Then equation (1.2) is oscillatory provided

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_{t_0}^t \Phi(s) ds = \infty. \quad (1.14)$$

**Theorem C.** Suppose that there exists a constant  $a > 1$  such that

$$\limsup_{t \rightarrow \infty} \frac{1}{t^a} \int_{t_0}^t \left[ (t-s)^{a-1} \Phi(s) - \frac{a}{4} g(s) (t-s)^{a-2} \right] ds = \infty. \quad (1.15)$$

Then equation (1.2) is oscillatory.

In this article, by using generalized Riccati transformation, we establish some oscillation criteria for equation (1.1) without any restriction on the sign  $p(t)$  and  $q(t)$ . Our results extend the oscillation criteria in [1].

## 2 Main Results

In order to prove our theorems, we assume that  $g(t) \in C^1([t_0, \infty); (0, \infty))$  is a given function and

$$\Gamma(t) = \int_{t_0}^t \left[ q(s)g(s) - \frac{(\gamma g(s)p(s) - \gamma g'(s)r(s)\psi(x(s)))^2}{4\gamma g(s)r(s)\psi(x(s))} \right] ds - \frac{\gamma g(t)p(t) - \gamma g'(t)r(t)\psi(x(t))}{2}$$

Our results are as follows:

**Theorem 2.1.** In addition to the basic assumptions imposed on the functions  $r$ ,  $p$ ,  $q$ ,  $g$ ,  $\psi$  and  $K$ , suppose the following assumptions are valid:

(i)

$$\int_{t_0}^{\infty} \frac{ds}{g(s)r(s)} = \infty, \quad (2.1)$$

(ii)

$$\lim_{t \rightarrow \infty} \Gamma(t) = \infty. \quad (2.2)$$

Then equation (1.1) is oscillatory

Proof. Let  $x(t)$  be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that  $x(t) > 0$  for  $t \geq t_0$ . Furthermore, we put

$$w(t) = g(t) \frac{r(t)\psi(x(t))K(x'(t))}{x(t)}, \quad t \geq t_0,$$

and using (ii), then it follows (1.1) that

$$w'(t) + \frac{1}{\gamma g(t)r(t)\psi(x(t))} [w^2(t) + (\gamma g(t)p(t) - \gamma g'(t)r(t)\psi(x(t))w] + q(t)g(t) \leq 0.$$

By completing the square we have

$$w'(t) + \frac{1}{\gamma g(t)r(t)\psi(x(t))} \left[ \left( w(t) + \frac{\gamma g(t)p(t) - \gamma g'(t)r(t)\psi(x(t))}{2} \right)^2 - \frac{(\gamma g(t)p(t) - \gamma g'(t)r(t)\psi(x(t)))^2}{4} \right] + q(t)g(t) \leq 0 \quad (2.3)$$

Define  $u(t) = w(t) + \frac{\gamma g(t)p(t) - \gamma g'(t)r(t)\psi(x(t))}{2}$ , rewrite (2.3), and integrate from  $t_0$  to  $t \geq t_0$ , we obtain

$$u(t) + \int_{t_0}^t \frac{u^2(s)}{\gamma g(s)r(s)\psi(x(s))} ds + \Gamma(t) \leq w(t_0). \quad (2.4)$$

From (2.2), we can choose  $t_1$  sufficiently large so that

$$u(t) + \int_{t_0}^t \frac{u^2(s)}{\gamma g(s)r(s)\psi(x(s))} ds \leq 0, \quad t \geq t_1. \quad (2.5)$$

Set  $V(t) = \int_{t_0}^t \frac{u^2(s)}{\gamma g(s)r(s)\psi(x(s))} ds$ . By (2.5), We get

$$\begin{aligned} V^2(t) &= \left( \int_{t_0}^t \frac{u^2(s)}{\gamma g(s)r(s)\psi(x(s))} ds \right)^2 \leq u^2(t) = \gamma g(t)r(t)\psi(x(t))V'(t) \\ &\leq \gamma c_1 g(t)r(t)V'(t), \text{ for } t \geq t_1. \end{aligned} \quad (2.6)$$

Dividing (2.6) through by  $V^2(t)$  and integrating from  $t_1$  to  $t$ , we find

$$\int_{t_1}^t \frac{ds}{g(s)r(s)} \leq \frac{\gamma c_1}{V(t_1)} - \frac{\gamma c_1}{V(t)}, \quad (2.7)$$

due to  $V(t) > 0$ , we have

$$\int_{t_1}^t \frac{ds}{g(s)r(s)} \leq \frac{\gamma c_1}{V(t_1)}, \quad (2.8)$$

which gives a desired contradiction with (2.1) as  $t \rightarrow \infty$ . This completes the proof.

**Theorem 2.2.** In addition to the basic assumptions imposed on the functions  $r, p, q, g, \psi$  and  $K$ , suppose the following assumptions are valid:

(i)

$$\int_{t_0}^{\infty} \left( \int_{t_0}^s g(\tau)r(\tau)d\tau \right)^{-1} ds = \infty, \quad (2.9)$$

(ii)

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_{t_0}^t \Gamma(s)ds = \infty. \quad (2.10)$$

Then equation (1.1) is oscillatory.

Proof. Let  $x(t)$  be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that  $x(t) > 0$  for  $t \geq t_0$ . Follows the proof of theorem 2.1, we obtain (2.4). Afterwards, Integrate (2.4) from  $t_0$  to  $t$  and divide through by  $t$  to obtain

$$\frac{1}{t} \int_{t_0}^t u(s)ds + \frac{1}{t} \int_{t_0}^t V(s)ds + \frac{1}{t} \int_{t_0}^t \Gamma(s)ds \leq w(t_0). \quad (2.11)$$

From (2.9), we can choose  $t_1$  sufficiently large so that  $t \geq t_1$

$$\int_{t_0}^t u(s)ds + \int_{t_0}^t V(s)ds \leq 0 \quad (2.12)$$

Let  $A(t) = \int_{t_0}^t V(s)ds$ . Using Hölder inequality, we find

$$\begin{aligned}
 A^2(t) &\leq \left( \int_{t_0}^t u(s)ds \right)^2 = \left( \int_{t_0}^t \sqrt{\gamma g(s)r(s)\psi(x(s))} \frac{u(s)}{\sqrt{\gamma g(s)r(s)\psi(x(s))}} ds \right)^2 \\
 &\leq \left( \int_{t_0}^t \gamma g(s)r(s)\psi(x(s))ds \right) \left( \int_{t_0}^t \frac{u^2(s)}{\gamma g(s)r(s)\psi(x(s))} ds \right) \\
 &= V(t) \left( \int_{t_0}^t \gamma g(s)r(s)\psi(x(s))ds \right) = A'(t) \left( \int_{t_0}^t \gamma g(s)r(s)\psi(x(s))ds \right) \\
 &\leq \gamma c_1 A'(t) \left( \int_{t_0}^t g(s)r(s)ds \right)
 \end{aligned} \tag{2.13}$$

Dividing (2.13) through by  $A^2(t)$  and integrating from  $t_1$  to  $t$ , we obtain

$$\int_{t_1}^t \left( \int_{t_0}^s g(\tau)r(\tau)d\tau \right)^{-1} ds \leq \frac{\gamma c_1}{A(t_1)} - \frac{\gamma c_1}{A(t)} \leq \frac{\gamma c_1}{A(t_1)}. \tag{2.14}$$

But (2.14) incompatible with (2.9) as  $t \rightarrow \infty$ . Hence the Theorem.

**Theorem 2.3.** Assume that there exists a constant  $a > 1$  such that

$$\limsup_{t \rightarrow \infty} \frac{1}{t^a} \int_{t_0}^t \left[ (t-s)^{a-1} \Gamma(s) - \frac{a}{4} \gamma g(s)r(s)\psi(x(s)) (t-s)^{a-2} \right] ds = \infty. \tag{2.15}$$

Then equation (1.1) is oscillatory.

**Proof.** As in the proof of Theorem 2.1, (2.4) holds. Multiplying (2.4) through by  $(t-s)^{a-1}$ , integrating  $t_0$  to  $t$  and dividing through by  $t^a$ , we obtain

$$\begin{aligned}
 &\frac{1}{t^a} \left\{ \int_{t_0}^t (t-s)^{a-1} u(s) + \int_{t_0}^t (t-s)^{a-1} \int_{t_0}^s \frac{u^2(\tau)}{\gamma g(\tau)r(\tau)\psi(x(\tau))} d\tau ds + \int_{t_0}^t (t-s)^{a-1} \Gamma(s) ds \right\} \\
 &\leq a^{-1} w(t_0)
 \end{aligned} \tag{2.16}$$

Denote  $R(s) = \int_{t_0}^s \frac{u^2(\tau)}{\gamma g(\tau)r(\tau)\psi(x(\tau))} d\tau$ . Integrating the second integral in (2.16) by parts, we obtain

$$\int_{t_0}^t (t-s)^{a-1} R(s) ds = \frac{1}{a} \int_{t_0}^t (t-s)^a \frac{u^2(s)}{\gamma g(s)r(s)\psi(x(s))} ds \tag{2.17}$$

Putting (2.17) into (2.16) and completing squares in  $u(s)$ , we acquire

$$\begin{aligned} & \frac{1}{at^a} \int_{t_0}^t \left[ \frac{(t-s)^{a/2} u(s)}{\sqrt{\gamma g(s) r(s) \psi(x(s))}} + \frac{a \sqrt{\gamma g(s) r(s) \psi(x(s))} (t-s)^{(a-2)/2}}{2} \right]^2 ds \\ & + \frac{1}{t^a} \int_{t_0}^t \left[ (t-s)^{a-1} \Gamma(s) - \frac{a}{4} \gamma g(s) r(s) \psi(x(s)) (t-s)^{a-2} \right] ds \\ & \leq a^{-1} w(t_0). \end{aligned}$$

Therefore, we have

$$\frac{1}{t^a} \int_{t_0}^t \left[ (t-s)^{a-1} \Gamma(s) - \frac{a}{4} \gamma g(s) r(s) \psi(x(s)) (t-s)^{a-2} \right] ds \leq a^{-1} w(t_0).$$

Taking the upper limit as  $t \rightarrow \infty$ , we obtain a contradiction with (2.15). This completes the proof of Theorem 2.3.

**Corollary 2.1.** If there exists a constant  $a > 1$  such that

$$\limsup_{t \rightarrow \infty} \frac{1}{t^a} \int_{t_0}^t \gamma g(s) r(s) \psi(x(s)) (t-s)^{a-2} ds < \infty$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{t^a} \int_{t_0}^t (t-s)^{a-1} \Gamma(s) ds = \infty,$$

then equation (1.1) is oscillatory.

**Remarks.**

**1.** Let  $r(t) = 1, \psi(x(t)) = 1$  and  $K(x'(t)) = x'(t)$  in equation (1.1). Then our equation (1.1) reduces to equation (1.2) which is same as in [1]. In spite of the condition  $g(t) \in C^2([t_0, \infty); (0, \infty))$  is given in Theorems A, B and C, our results depend on  $g(t) \in C^1([t_0, \infty); (0, \infty))$  in Theorems 2.1, 2.2 and 2.3.

**2.** Theorems 2.1, 2.2 and 2.3 generalize and extend Theorems A, B and C to the nonlinear differential equation (1.1).

**3.** Let  $\psi(x(t)) = 1, K(x'(t)) = x'(t)$  and  $q(t)x = q(t)f(x)$ . Then the equation (1.1) reduces to

$$(r(t)x')' + p(t)x' + q(t)f(x) = 0,$$

which considered in [9]. In this situation, the conditions of Theorem 2.1 and Theorem 2.2 are compatible with Theorem 2.1 and Theorem 2.5 of [9], respectively. But, our Theorem 2.3 is not mentioned in [9].

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# Stability of a mixed type additive and quadratic functional equation in non-Archimedean spaces

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**Abstract.** In this paper, we establish generalized Hyres-Ulam-Rassias stability of the mixed type additive and quadratic functional equation

$$f(3x + y) + f(3x - y) = f(x + y) + f(x - y) + 2f(3x) - 2f(x)$$

in non-Archimedean spaces.

## 1. INTRODUCTION

In 1897, Hensel [14] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications [19, 28].

A non-Archimedean field is a field  $K$  equipped with a function (valuation)  $|\cdot|$  from  $K$  into  $[0, \infty)$  such that  $|r| = 0$  if and only if  $r = 0$ ,  $|rs| = |r||s|$ , and  $|r + s| \leq \max\{|r|, |s|\}$  for all  $r, s \in K$ . Clearly  $|1| = |-1| = 1$  and  $|n| \leq 1$  for all  $n \in \mathbb{N}$ . An example of a non-Archimedean valuation is the mapping  $|\cdot|$  taking everything but 0 into 1 and  $|0| = 0$ . This valuation is called trivial.

**Definition 1.1.** Let  $X$  be a vector space over a scalar field  $K$  with a non-Archimedean non-trivial valuation  $|\cdot|$ . A function  $\|\cdot\| : X \rightarrow \mathbb{R}$  is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(NA<sub>1</sub>)  $\|x\| = 0$  if and only if  $x = 0$ ;

(NA<sub>2</sub>)  $\|rx\| = |r|\|x\|$  for all  $r \in K$  and  $x \in X$ ;

(NA<sub>3</sub>)  $\|x + y\| \leq \max\{\|x\|, \|y\|\}$  for all  $x, y \in X$  (the strong triangle inequality).

Then  $(X, \|\cdot\|)$  is called a non-Archimedean space.

**Remark 1.2.** Thanks to the inequality

$$\|x_m - x_l\| \leq \max\{\|x_{j+1} - x_j\| : l \leq j \leq m - 1\} \quad (m > l)$$

a sequence  $\{x_m\}$  is Cauchy if and only if  $\{x_{m+1} - x_m\}$  converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

The most important examples of non-Archimedean spaces are  $p$ -adic numbers. A key property of  $p$ -adic numbers is that they do not satisfy the Archimedean axiom: "for  $x, y > 0$ , there exists  $n \in \mathbb{N}$  such that  $x < ny$ ."

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**Example 1.3.** Let  $p$  be a prime number. For any nonzero rational number  $x = \frac{a}{b}p^{n_x}$  such that  $a$  and  $b$  are integers not divisible by  $p$ , define the  $p$ -adic absolute value  $|x|_p := p^{-n_x}$ . Then  $|\cdot|$  is a non-Archimedean norm on  $\mathbb{Q}$ . The completion of  $\mathbb{Q}$  with respect to  $|\cdot|$  is denoted by  $\mathbb{Q}_p$  which is called the  $p$ -adic number field.

Note that if  $p > 3$ , then  $|2^n| = 1$  in for each integer  $n$ .

The stability problem of functional equations originated from a question of Ulam[27] in 1940, concerning the stability of group homomorphisms. Let  $(G_1, \cdot)$  be a group and let  $(G_2, *, d)$  be a metric group with the metric  $d(\cdot, \cdot)$ . Given  $\epsilon > 0$ , does there exist a  $\delta > 0$ , such that if a mapping  $h : G_1 \rightarrow G_2$  satisfies the inequality  $d(h(xy), h(x) * h(y)) < \delta$  for all  $x, y \in G_1$ , then there exists a homomorphism  $H : G_1 \rightarrow G_2$  with  $d(h(x), H(x)) < \epsilon$  for all  $x \in G_1$ ? In the other words, Under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers[16] gave a first affirmative answer to the question of Ulam for Banach spaces. Let  $f : E \rightarrow E'$  be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all  $x, y \in E$ , and for some  $\delta > 0$ . Then there exists a unique additive mapping  $T : E \rightarrow E'$  such that

$$\|f(x) - T(x)\| \leq \delta$$

for all  $x \in E$ . Moreover if  $f(tx)$  is continuous in  $t \in \mathbb{R}$  for each fixed  $x \in E$ , then  $T$  is linear. In 1978, Th. M. Rassias[23] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded. In 1991, Z. Gajda[11] answered the question for the case  $p > 1$ , which was raised by Rassias. This new concept is known as Hyers-Ulam-Rassias stability of functional equations (see [1, 2, 5], [12, 15, 17] and [24]). The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \quad (1.1)$$

is related to symmetric bi-additive function. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function  $f$  between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function  $B$  such that  $f(x) = B(x, x)$  for all  $x$  (see [1, 18]). The bi-additive function  $B$  is given by

$$B(x, y) = \frac{1}{4}(f(x+y) - f(x-y)) \quad (1.2)$$

Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.1) was proved by Skof for functions  $f : A \rightarrow B$ , where  $A$  is normed space and  $B$  Banach space (see [26, 6, 7, 13]). Borelli and Forti [4] generalized the stability result of quadratic functional equations as follows (cf. [21, 22]): Let  $G$  be an Abelian group, and  $X$  a Banach space. Assume that a mapping  $f : G \rightarrow X$  satisfies the functional inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varphi(x, y)$$

for all  $x, y \in G$ , and  $\varphi : G \times G \rightarrow [0, \infty)$  is a function such that

$$\Phi(x, y) := \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i y) < \infty$$

for all  $x, y \in G$ . Then there exists a unique quadratic mapping  $Q : G \rightarrow X$  with the property

$$\|f(x) - Q(x)\| \leq \Phi(x, x)$$

for all  $x \in G$ .

Arriola and Beyer [3] investigated stability of approximate additive mappings  $f : \mathbb{Q}_p \rightarrow \mathbb{R}$ . They showed that if  $f : \mathbb{Q}_p \rightarrow \mathbb{R}$  is a continuous mapping for which there exists a fixed  $\epsilon$  :

$$|f(x+y) - f(x) - f(y)| \leq \epsilon$$

for all  $x, y \in \mathbb{Q}_p$ , then there exists a unique additive mapping  $T : \mathbb{Q}_p \rightarrow \mathbb{R}$  such that

$$|f(x) - T(x)| \leq \epsilon$$

for all  $x \in \mathbb{Q}_p$ . Additionally in 2007, Moslehian and Rassias [20] proved the generalized Hyers–Ulam stability of the Cauchy functional equation and the quadratic functional equation in non-Archimedean normed spaces (see also [8]).

In this paper, we establish the stability of the additive–quadratic functional equation

$$f(3x+y) + f(3x-y) = f(x+y) + f(x-y) + 2f(3x) - 2f(x) \quad (1.3)$$

in non-Archimedean space. The function  $f(x) = x+x^2$  satisfies the functional equation (1.3), which explains why it is called additive–quadratic functional equation. For more detailed definitions of mixed type functional equations, we can refer to [9] and [10].

## 2. MAIN RESULTS

Throughout this section, we assume that  $G$  is an additive group and  $X$  is a complete non-Archimedean space. Now before taking up the main subject, for a given function  $f : G \rightarrow X$ , we define the difference operator

$$Df(x, y) = f(3x+y) + f(3x-y) - f(x+y) - f(x-y) - 2f(3x) + 2f(x)$$

for all  $x, y \in G$ . we consider the following function inequality:

$$\|Df(x, y)\| \leq \varphi(x, y)$$

for an upper bound  $\varphi : G \times G \rightarrow [0, \infty)$ .

**Theorem 2.1.** *Let  $\varphi : G \times G \rightarrow [0, \infty)$  be a function such that*

$$\lim_{n \rightarrow \infty} \frac{\varphi(2^n x, 2^n y)}{|2|^n} = 0, \quad (2.1)$$

$$\lim_{n \rightarrow \infty} \frac{1}{|2^n|} \max\{\max\{\varphi(\frac{2^n x}{4}, \frac{2^n x}{4}), \varphi(\frac{2^n x}{4}, \frac{3 \cdot 2^n x}{4})\}, \max\{\varphi(\frac{2^n x}{4}, \frac{2^n x}{4}), \varphi(\frac{2^n x}{4}, \frac{5 \cdot 2^n x}{4})\}\} = 0 \quad (2.2)$$

for all  $x, y \in G$ , and let for each  $x \in G$  the limit

$$\begin{aligned} \lim_{n \rightarrow \infty} \max\{\frac{1}{|2^j|} \max\{\max\{\varphi(\frac{2^j x}{4}, \frac{2^j x}{4}), \varphi(\frac{2^j x}{4}, \frac{3 \cdot 2^j x}{4})\}, \\ \max\{\varphi(\frac{2^j x}{4}, \frac{2^j x}{4}), \varphi(\frac{2^j x}{4}, \frac{5 \cdot 2^j x}{4})\}\} : 0 \leq j < n\}, \end{aligned} \quad (2.3)$$

denoted by  $\tilde{\varphi}_A(x)$ , exists. Suppose that  $f : G \rightarrow X$  is an odd function satisfying

$$\|Df(x, y)\| \leq \varphi(x, y) \quad (2.4)$$

for all  $x, y \in G$ . Then there exists an additive function  $A : G \rightarrow X$  such that

$$\|A(x) - f(x)\| \leq \frac{1}{|2|} \tilde{\varphi}_A(x) \quad (2.5)$$

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for all  $x \in G$ , if moreover

$$\lim_{i \rightarrow \infty} \lim_{n \rightarrow \infty} \max\left\{\frac{1}{|2^j|} \max\left\{\max\left\{\varphi\left(\frac{2^j x}{4}, \frac{2^j x}{4}\right), \varphi\left(\frac{2^j x}{4}, \frac{3 \cdot 2^j x}{4}\right)\right\}, \max\left\{\varphi\left(\frac{2^j x}{4}, \frac{2^j x}{4}\right), \varphi\left(\frac{2^j x}{4}, \frac{5 \cdot 2^j x}{4}\right)\right\}\right\} : i \leq j < n + i\right\} = 0,$$

then  $A$  is the unique additive function satisfying (2.5).

*Proof.* Setting  $y = x$  in (2.4), we get

$$\|f(4x) - 2f(3x) + 2f(x)\| \leq \varphi(x, x) \quad (2.6)$$

for all  $x \in G$ . If we let  $y = 3x$  in (2.4), we get by the oddness of  $f$ ,

$$\|f(6x) - 2f(3x) - f(4x) + 2f(x) + f(2x)\| \leq \varphi(x, 3x) \quad (2.7)$$

for all  $x \in G$ . It follows from (2.6) and (2.7) that

$$\|f(6x) - 2f(4x) + f(2x)\| \leq \max\{\varphi(x, x), \varphi(x, 3x)\} \quad (2.8)$$

for all  $x \in G$ . Once again, by letting  $y = 5x$  in (2.4), we get by the oddness of  $f$ ,

$$\|f(8x) - f(2x) - f(6x) + f(4x) - 2f(3x) + 2f(x)\| \leq \varphi(x, 5x) \quad (2.9)$$

for all  $x \in G$ . By (2.6) and (2.9), we get

$$\|f(8x) - f(6x) - f(2x)\| \leq \max\{\varphi(x, x), \varphi(x, 5x)\} \quad (2.10)$$

for all  $x \in G$ . By (2.8) and (2.10), we obtain

$$\|f(8x) - 2f(4x)\| \leq \max\{\max\{\varphi(x, x), \varphi(x, 3x)\}, \max\{\varphi(x, x), \varphi(x, 5x)\}\} \quad (2.11)$$

for all  $x \in G$ . If we replace  $x$  by  $\frac{x}{4}$  in (2.11), and divide both sides of (2.11) by 2, we get that

$$\left\|\frac{1}{2}f(2x) - f(x)\right\| \leq \frac{1}{|2|} \max\left\{\max\left\{\varphi\left(\frac{x}{4}, \frac{x}{4}\right), \varphi\left(\frac{x}{4}, \frac{3x}{4}\right)\right\}, \max\left\{\varphi\left(\frac{x}{4}, \frac{x}{4}\right), \varphi\left(\frac{x}{4}, \frac{5x}{4}\right)\right\}\right\} \quad (2.12)$$

for all  $x \in G$ . Replacing  $x$  by  $2^{n-1}x$  in (2.12), we obtain

$$\begin{aligned} \left\|\frac{1}{2^n}f(2^n x) - \frac{1}{2^{(n-1)}}f(2^{n-1}x)\right\| &\leq \frac{1}{|2^n|} \max\left\{\max\left\{\varphi\left(\frac{2^{n-1}x}{4}, \frac{2^{n-1}x}{4}\right), \varphi\left(\frac{2^{n-1}x}{4}, \frac{3 \cdot 2^{n-1}x}{4}\right)\right\}, \right. \\ &\quad \left. \max\left\{\varphi\left(\frac{2^{n-1}x}{4}, \frac{2^{n-1}x}{4}\right), \varphi\left(\frac{2^{n-1}x}{4}, \frac{5 \cdot 2^{n-1}x}{4}\right)\right\}\right\} \end{aligned} \quad (2.13)$$

for all  $x \in G$ . It follows from (2.2) and (2.13) that the sequence  $\{\frac{f(2^n x)}{2^n}\}$  is Cauchy. Since  $X$  is complete, we conclude that  $\{\frac{f(2^n x)}{2^n}\}$  is convergent. Set  $A(x) := \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$ . By using induction one can show that

$$\begin{aligned} \left\|\frac{f(2^n x)}{2^n} - f(x)\right\| &\leq \frac{1}{|2|} \max\left\{\frac{1}{|2^i|} \max\left\{\max\left\{\varphi\left(\frac{2^i x}{4}, \frac{2^i x}{4}\right), \varphi\left(\frac{2^i x}{4}, \frac{3 \cdot 2^i x}{4}\right)\right\}, \right. \right. \\ &\quad \left. \left. \max\left\{\varphi\left(\frac{2^i x}{4}, \frac{2^i x}{4}\right), \varphi\left(\frac{2^i x}{4}, \frac{5 \cdot 2^i x}{4}\right)\right\}\right\} : 0 \leq i < n\right\} \end{aligned} \quad (2.14)$$

for all  $n \in \mathbb{N}$  and all  $x \in G$ . By taking  $n$  to approach infinity in (2.14) and using (2.3) one obtains (2.5). By (2.1) and (2.4), we get

$$\|DA(x, y)\| = \lim_{n \rightarrow \infty} \frac{1}{|2^n|} \|f(2^n x, 2^n y)\| \leq \lim_{n \rightarrow \infty} \frac{\varphi(2^n x, 2^n y)}{|2|^n} = 0$$

for all  $x, y \in G$ . Therefore the function  $A : G \rightarrow X$  satisfies (1.3). If  $A'$  is another additive function satisfying (2.5), then

$$\begin{aligned} \|A(x) - A'(x)\| &= \lim_{i \rightarrow \infty} |2|^{-i} \|A(2^i x) - A'(2^i x)\| \\ &\leq \lim_{i \rightarrow \infty} |2|^{-i} \max\{ \|A(2^i x) - f(2^i x)\|, \|f(2^i x) - A'(2^i x)\| \} \\ &\leq \frac{1}{|2|} \lim_{i \rightarrow \infty} \lim_{n \rightarrow \infty} \max\{ \frac{1}{|2^j|} \max\{ \varphi(\frac{2^j x}{4}, \frac{2^j x}{4}), \varphi(\frac{2^j x}{4}, \frac{3 \cdot 2^j x}{4}) \} \\ &\quad, \max\{ \varphi(\frac{2^j x}{4}, \frac{2^j x}{4}), \varphi(\frac{2^j x}{4}, \frac{5 \cdot 2^j x}{4}) \} \} : i \leq j < n + i \} \\ &= 0. \end{aligned}$$

for all  $x \in G$ . Therefore  $A = A'$ . This completes the proof of the uniqueness of  $A$ .  $\square$

**Theorem 2.2.** Let  $\varphi : G \times G \rightarrow [0, \infty)$  be a function such that

$$\lim_{n \rightarrow \infty} \frac{\varphi(2^n x, 2^n y)}{|2|^{2n}} = 0 \quad (2.15)$$

$$\lim_{n \rightarrow \infty} \frac{1}{|2^{2n}|} \max\{ \varphi(\frac{2^n x}{4}, \frac{2^n x}{4}), \max\{ \varphi(\frac{2^n x}{4}, \frac{5 \cdot 2^n x}{4}), \varphi(\frac{2^n x}{4}, \frac{-3 \cdot 2^n x}{4}) \} \} = 0 \quad (2.16)$$

for all  $x, y \in G$  and let for each  $x \in G$  the limit

$$\lim_{n \rightarrow \infty} \max\{ \frac{1}{|2^{2j}|} \max\{ \varphi(\frac{2^j x}{4}, \frac{2^j x}{4}), \max\{ \varphi(\frac{2^j x}{4}, \frac{5 \cdot 2^j x}{4}), \varphi(\frac{2^j x}{4}, \frac{-3 \cdot 2^j x}{4}) \} \} : 0 \leq j < n \}, \quad (2.17)$$

denoted by  $\tilde{\varphi}_Q(x)$ , exists. Suppose that  $f : G \rightarrow X$  is an even function satisfying

$$\|Df(x, y)\| \leq \varphi(x, y) \quad (2.18)$$

for all  $x, y \in G$ . Then there exists a quadratic function  $Q : G \rightarrow X$  such that

$$\|Q(x) - f(x)\| \leq \frac{1}{|4|} \tilde{\varphi}_Q(x) \quad (2.19)$$

for all  $x \in G$ , if moreover

$$\begin{aligned} \lim_{i \rightarrow \infty} \lim_{n \rightarrow \infty} \max\{ \frac{1}{|2^{2j}|} \max\{ \varphi(\frac{2^j x}{4}, \frac{2^j x}{4}) \\ , \max\{ \varphi(\frac{2^j x}{4}, \frac{5 \cdot 2^j x}{4}), \varphi(\frac{2^j x}{4}, \frac{-3 \cdot 2^j x}{4}) \} \} : i \leq j < n + i \} = 0, \end{aligned}$$

then  $Q$  is the unique quadratic function satisfying (2.19).

*Proof.* Replacing  $y$  by  $x + y$  in (2.18) to get

$$\|f(4x + y) + f(2x - y) - f(2x + y) - f(y) - 2f(3x) + 2f(x)\| \leq \varphi(x, x + y) \quad (2.20)$$

for all  $x, y \in G$ . If we Replace  $y$  by  $-y$  in (2.20), we obtain

$$\|f(4x - y) + f(2x + y) - f(2x - y) - f(y) - 2f(3x) + 2f(x)\| \leq \varphi(x, x - y) \quad (2.21)$$

for all  $x, y \in G$ . If we add (2.20) to (2.21), we have

$$\|f(4x + y) + f(4x - y) - 2f(y) - 4f(3x) + 4f(x)\| \leq \max\{ \varphi(x, x + y), \varphi(x, x - y) \}. \quad (2.22)$$

Letting  $y = 0$  in (2.22), we get the inequality

$$\|2f(4x) - 4f(3x) + 4f(x)\| \leq \varphi(x, x) \quad (2.23)$$

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for all  $x \in G$ . Once again By letting  $y = 4x$  in (2.22), we get the inequality

$$\|f(8x) - 2f(4x) - 4f(3x) + 4f(x)\| \leq \max\{\varphi(x, 5x), \varphi(x, -3x)\} \quad (2.24)$$

for all  $x \in G$ . By (2.23) and (2.24), we get

$$\|f(8x) - 4f(4x)\| \leq \max\{\varphi(x, x), \max\{\varphi(x, 5x), \varphi(x, -3x)\}\} \quad (2.25)$$

for all  $x \in G$ . If we replace  $x$  in (2.25) by  $\frac{x}{4}$  and divide both sides of (2.25) by 4, we lead to

$$\|\frac{1}{4}f(2x) - f(x)\| \leq \frac{1}{|4|} \max\{\varphi(\frac{x}{4}, \frac{x}{4}), \max\{\varphi(\frac{x}{4}, \frac{5x}{4}), \varphi(\frac{x}{4}, \frac{-3x}{4})\}\} \quad (2.26)$$

for all  $x \in G$ . Replacing  $x$  by  $2^{n-1}x$  in (2.26), we get

$$\begin{aligned} \|\frac{1}{2^{2n}}f(2^n x) - \frac{1}{2^{2(n-1)}}f(2^{n-1}x)\| &\leq \frac{1}{|2^{2n}|} \max\{\varphi(\frac{2^{n-1}x}{4}, \frac{2^{n-1}x}{4}) \\ &\quad , \max\{\varphi(\frac{2^{n-1}x}{4}, \frac{5 \cdot 2^{n-1}x}{4}), \varphi(\frac{2^{n-1}x}{4}, \frac{-3 \cdot 2^{n-1}x}{4})\}\} \end{aligned} \quad (2.27)$$

for all  $x \in G$ . By (2.16) and (2.27), it follows that the sequence  $\{\frac{f(2^n x)}{2^{2n}}\}$  is Cauchy. Since  $X$  is complete, we conclude that  $\{\frac{f(2^n x)}{2^{2n}}\}$  is convergent. Set  $Q(x) := \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^{2n}}$  for all  $x \in X$ .

Using induction one can show that

$$\begin{aligned} \|\frac{f(2^n x)}{2^{2n}} - f(x)\| &\leq \frac{1}{|2^{2n}|} \max\{\frac{1}{|2^{2i}|} \max\{\varphi(\frac{2^i x}{4}, \frac{2^i x}{4}), \max\{\varphi(\frac{2^i x}{4}, \frac{5 \cdot 2^i x}{4}) \\ &\quad , \varphi(\frac{2^i x}{4}, \frac{-3 \cdot 2^i x}{4})\}\} : 0 \leq i < n\} \end{aligned} \quad (2.28)$$

for all  $n \in \mathbb{N}$  and all  $x \in G$ . Letting  $n \rightarrow \infty$  in (2.28) and using (2.17) one can obtain (2.19). By (2.15) and (2.18), we get

$$\|DQ(x, y)\| = \lim_{n \rightarrow \infty} \frac{1}{|2^{2n}|} \|f(2^n x, 2^n y)\| \leq \lim_{n \rightarrow \infty} \frac{\varphi(2^n x, 2^n y)}{|2|^{2n}} = 0$$

for all  $x, y \in G$ . Therefore the function  $Q : G \rightarrow X$  satisfies (1.3). To prove the uniqueness property of  $Q$ , let  $Q'$  be another quadratic function satisfying (2.19), then

$$\begin{aligned} \|Q(x) - Q'(x)\| &= \lim_{i \rightarrow \infty} |2|^{-2i} \|Q(2^i x) - Q'(2^i x)\| \\ &\leq \lim_{i \rightarrow \infty} |2|^{-2i} \max\{\|Q(2^i x) - f(2^i x)\|, \|f(2^i x) - Q'(2^i x)\|\} \\ &\leq \frac{1}{|2^{2i}|} \lim_{i \rightarrow \infty} \lim_{n \rightarrow \infty} \max\{\frac{1}{|2^{2j}|} \max\{\varphi(\frac{2^j x}{4}, \frac{2^j x}{4}) \\ &\quad , \max\{\varphi(\frac{2^j x}{4}, \frac{5 \cdot 2^j x}{4}), \varphi(\frac{2^j x}{4}, \frac{-3 \cdot 2^j x}{4})\}\} : i \leq j < n + i\}. \end{aligned}$$

for all  $x \in G$ . It follows from hypothesis that  $Q = Q'$ . □

**Theorem 2.3.** Let  $\varphi : G \times G \rightarrow [0, \infty)$  be a function such that

$$\lim_{n \rightarrow \infty} \frac{\varphi(2^n x, 2^n y)}{|2|^n} = \lim_{n \rightarrow \infty} \frac{\varphi(2^n x, 2^n y)}{|2|^{2n}} = 0 \quad (2.29)$$

for all  $x, y \in G$  and let for each  $x \in G$  the limit

$$\lim_{n \rightarrow \infty} \max\{\frac{1}{|2^j|} \max\{\max\{\varphi(\frac{2^j x}{4}, \frac{2^j x}{4}), \varphi(\frac{2^j x}{4}, \frac{3 \cdot 2^j x}{4})\}\}$$

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$$, \max\{\varphi(\frac{2^j x}{4}, \frac{2^j x}{4}), \varphi(\frac{2^j x}{4}, \frac{5 \cdot 2^j x}{4})\} : 0 \leq j < n\},$$

denoted by  $\tilde{\varphi}_A(x)$ , and

$$\lim_{n \rightarrow \infty} \max\{\frac{1}{|2^{2j}|} \max\{\varphi(\frac{2^j x}{4}, \frac{2^j x}{4}), \max\{\varphi(\frac{2^j x}{4}, \frac{5 \cdot 2^j x}{4}), \varphi(\frac{2^j x}{4}, \frac{-3 \cdot 2^j x}{4})\}\} : 0 \leq j < n\},$$

denoted by  $\tilde{\varphi}_Q(x)$ , exist. Suppose that  $f : G \rightarrow X$  is a function satisfying

$$\|Df(x, y)\| \leq \varphi(x, y) \quad (2.30)$$

for all  $x, y \in G$ . Then there exist an additive function  $A : X \rightarrow Y$  and a quadratic function  $Q : G \rightarrow X$  such that

$$\|f(x) - A(x) - Q(x)\| \leq \frac{1}{|2^2|} \max\{\max\{\tilde{\varphi}_A(x), \tilde{\varphi}_A(-x)\}, \frac{1}{|2|} \max\{\tilde{\varphi}_Q(x), \tilde{\varphi}_Q(-x)\}\} \quad (2.31)$$

for all  $x \in G$ , if

$$\begin{aligned} \lim_{i \rightarrow \infty} \lim_{n \rightarrow \infty} \max\{\frac{1}{|2^j|} \max\{\max\{\varphi(\frac{2^j x}{4}, \frac{2^j x}{4}), \varphi(\frac{2^j x}{4}, \frac{3 \cdot 2^j x}{4})\} \\ , \max\{\varphi(\frac{2^j x}{4}, \frac{2^j x}{4}), \varphi(\frac{2^j x}{4}, \frac{5 \cdot 2^j x}{4})\}\} : i \leq j < n + i\} = 0, \end{aligned}$$

$$\begin{aligned} \lim_{i \rightarrow \infty} \lim_{n \rightarrow \infty} \max\{\frac{1}{|2^{2j}|} \max\{\varphi(\frac{2^j x}{4}, \frac{2^j x}{4}) \\ , \max\{\varphi(\frac{2^j x}{4}, \frac{5 \cdot 2^j x}{4}), \varphi(\frac{2^j x}{4}, \frac{-3 \cdot 2^j x}{4})\}\} : i \leq j < n + i\} = 0 \end{aligned}$$

then  $A$  is the unique additive function and  $Q$  is the unique quadratic function satisfying (2.31).

*Proof.* Let  $f_o(x) = \frac{1}{2}[f(x) - f(-x)]$  for all  $x \in G$ . Then  $f_o(0) = 0$ ,  $f_o(-x) = -f_o(x)$ , and

$$\|Df_o(x, y)\| \leq \frac{1}{|2|} \max\{\varphi(x, y), \varphi(-x, -y)\}$$

for all  $x, y \in G$ . From Theorem 2.1, it follows that there exists a unique additive function  $A : G \rightarrow X$  satisfying

$$\|f_o(x) - A(x)\| \leq \frac{1}{|2^2|} \max\{\tilde{\varphi}_A(x), \tilde{\varphi}_A(-x)\} \quad (2.32)$$

for all  $x \in G$ .

Let  $f_e(x) = \frac{1}{2}[f(x) + f(-x)]$  for all  $x \in G$ . Then  $f_e(0) = 0$ ,  $f_e(-x) = f_e(x)$ , and

$$\|Df_e(x, y)\| \leq \frac{1}{|2|} \max\{\varphi(x, y), \varphi(-x, -y)\}$$

for all  $x, y \in G$ . From Theorem 2.2, it follows that there exists a unique quadratic function  $Q : G \rightarrow X$  satisfying

$$\|f_e(x) - Q(x)\| \leq \frac{1}{|2^3|} \max\{\tilde{\varphi}_Q(x), \tilde{\varphi}_Q(-x)\} \quad (2.33)$$

for all  $x \in G$ .

Hence, (3.31) follows from (3.32) and (3.33).  $\square$



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# Ternary Jordan $*$ -derivations in $C^*$ -ternary algebras

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**Abstract.** We say a functional equation  $(\xi)$  is stable if any function  $g$  satisfying the equation  $(\xi)$  *approximately* is near to true solution of  $(\xi)$ . In this paper, we prove the Generalized Hyers–Ulam stability of ternary Jordan  $*$ -derivations in  $C^*$ -ternary algebras for the following generalized Cauchy–Jensen additive mapping:

$$rf\left(\frac{s\sum_{j=1}^p x_j + t\sum_{j=1}^d x_j}{r}\right) = s\sum_{j=1}^p f(x_j) + t\sum_{j=1}^d f(x_j).$$

## 1. INTRODUCTION

A  $C^*$ -ternary algebra is a complex Banach space  $A$ , equipped with a ternary product  $(x, y, z) \mapsto [x, y, z]$  of  $A^3$  into  $A$ , which is  $\mathbb{C}$ -linear in the outer variables, conjugate  $\mathbb{C}$ -linear in the middle variable, and associative in the sense that  $[x, y, [z, u, v]] = [x, [u, z, y], v] = [[x, y, z], u, v]$ , and satisfies  $\|[x, y, z]\| \leq \|x\| \cdot \|y\| \cdot \|z\|$  and  $\|[x, x, x]\| = \|x\|^3$  (see [1, 3, 5, 11, 22, 24, 25]). If a  $C^*$ -ternary algebra  $(A, [., ., .])$  has an identity, i.e., an element  $e \in A$  such that  $x = [x, e, e] = [e, e, x]$  for all  $x \in A$ , then it is routine to verify that  $A$ , endowed with  $xy := [x, e, y]$  and  $x^* := [e, x, e]$ , is a unital  $C^*$ -algebra. Conversely, if  $(A, o)$  is a unital  $C^*$ -algebra, then  $[x, y, z] := xoy^*oz$  makes  $A$  into a  $C^*$ -ternary algebra.

Let  $(A, [., ., .])$  be a  $*$ -ternary algebra over a scalar field  $\mathbb{R}$  or  $\mathbb{C}$ . A linear mapping  $D : (A, [., ., .]) \rightarrow (A, [., ., .])$  is called a ternary Jordan  $*$ -derivation, if

$$D([xxx]) = [D(x)xx] + [xD(x)x] + [xxD(x)], \quad D(x^*) = (D(x))^*$$

for all  $x \in A$ .

We say a functional equation  $(\xi)$  is stable if any function  $g$  satisfying the equation  $(\xi)$  *approximately* is near to true solution of  $(\xi)$ . We say that a functional equation is *superstable* if every approximately solution is an exact solution of it.

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The stability of functional equations was first introduced by S. M. Ulam [23] in 1940. More precisely, he proposed the following problem: Given a group  $G_1$ , a metric group  $(G_2, d)$  and a positive number  $\epsilon$ , does there exist a  $\delta > 0$  such that if a function  $f : G_1 \rightarrow G_2$  satisfies the inequality  $d(f(xy), f(x)f(y)) < \delta$  for all  $x, y \in G_1$ , then there exists a homomorphism  $T : G_1 \rightarrow G_2$  such that  $d(f(x), T(x)) < \epsilon$  for all  $x \in G_1$ . As mentioned above, when this problem has a solution, we say that the homomorphisms from  $G_1$  to  $G_2$  are stable. In 1941, D. H. Hyers [8] gave a partial solution of *Ulam's* problem for the case of approximate additive mappings under the assumption that  $G_1$  and  $G_2$  are Banach spaces. In 1950, T. Aoki [2] was the second author to treat this problem for additive mappings. In 1978, Th. M. Rassias [21] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences.

On the other hand J. M. Rassias proved the Hyers stability result by presenting another condition controlled by a product of different powers of norms. According to J. M. Rassias Theorem (see [18,19,20]):

**Theorem 1.1.** *If it is assumed that there exist constants  $\Theta \geq 0$  and  $p_1, p_2 \in \mathbb{R}$  such that  $p = p_1 + p_2 \neq 1$ , and  $f : E \rightarrow E'$  is a map from a norm space  $E$  into a Banach space  $E'$  such that the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \|x\|^{p_1} \|y\|^{p_2}$$

*for all  $x, y \in E$ , then there exists a unique additive mapping  $T : E \rightarrow E'$  such that*

$$\|f(x) - T(x)\| \leq \frac{\Theta}{2-2^p} \|x\|^p,$$

*for all  $x \in E$ . If in addition for every  $x \in E$ ,  $f(tx)$  is continuous in real  $t$  for each fixed  $x$ , then  $T$  is linear.*

During the last decades several stability problems for various functional equations have been investigated by many mathematicians; we refer the reader to the monographs [4, 6, 7, 9, 10, 13, 14, 16, 17].

## 2. STABILITY

Let  $A$  be a  $C^*$ -ternary algebra. For a given mapping  $f : A \rightarrow A$ , we define

$$\Delta_\mu f(x_1, \dots, x_p, y_1, \dots, y_d) := r f\left(\frac{s \sum_{j=1}^p \mu x_j + t \sum_{j=1}^d \mu y_j}{r}\right) - s \sum_{j=1}^p \mu f(x_j) - t \sum_{j=1}^d \mu f(y_j)$$

for all  $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$  and all  $x_1, \dots, x_p, y_1, \dots, y_d \in A$ , and let

$$\Delta f(x, y) = f([x, x, x]) - [f(x), x, x] - [x, f(x), x] - [x, x, f(x)] + f(y^*) - (f(y))^*$$

for all  $x, y \in A$ .

One can easily show that a mapping  $f : A \rightarrow A$  satisfies

$$\Delta_\mu f(x_1, \dots, x_p, y_1, \dots, y_d) = 0$$

for all  $\mu \in \mathbb{T}^1$  and all  $x_1, \dots, x_p, y_1, \dots, y_d \in A$  if and only if

$$f(\mu x + \lambda y) = \mu f(x) + \lambda f(y)$$

for all  $\mu, \lambda \in \mathbb{T}^1$  and all  $x, y \in A$ .

We will use the following lemma in the proof of our main theorem:

**Lemma 2.1.** [15] *Let  $f : A \rightarrow A$  be an additive mapping such that  $f(\mu x) = \mu f(x)$  for all  $x \in A$  and all  $\mu \in \mathbb{T}^1$ . Then the mapping  $f$  is  $\mathbb{C}$ -linear.*

**Theorem 2.2.** Let  $r, \theta$  be non-negative real numbers such that  $r \in (-\infty, 1) \cup (3, +\infty)$ , and let  $f : A \longrightarrow A$  be a mapping such that

$$\|\Delta_\mu f(x_1, \dots, x_p, y_1, \dots, y_d)\| \leq \theta \left( \sum_{j=1}^p \|x_j\|^r + \sum_{j=1}^d \|y_j\|^r \right) \quad (2.1)$$

and that

$$\|\Delta f(x, y)\| \leq 3\theta(\|x\|^r + \|y\|^r) \quad (2.2)$$

for all  $x \in A$ . Then there exists a unique ternary Jordan  $\ast$ -derivation  $\delta : A \longrightarrow A$  such that

$$\|f(x) - \delta(x)\| \leq \frac{2^r(p+d)}{|2(p+2d)^r - (p+2d)2^r|} \theta \|x\|^r \quad (2.3)$$

for all  $x \in A$ .

*Proof.* Letting  $\mu = 1$  and  $x_1 = \dots = x_p = y_1, \dots, y_d = x$  and  $s = 1, t = 2$  in (2.1), we get

$$\|f((p+2d)x) - (p+2d)f(x)\| \leq (p+d)\theta \|x\|^r$$

for all  $x \in A$ . So

$$\|f(x) - (p+2d)f\left(\frac{x}{p+2d}\right)\| \leq \frac{(p+d)\theta}{2^r(p+2d)^r} \|x\|^r$$

for all  $x \in A$ . Hence, one can show that

$$\begin{aligned} \|(p+2d)^l f\left(\frac{x}{(p+2d)^l}\right) - (p+2d)^m f\left(\frac{x}{(p+2d)^m}\right)\| &\leq \sum_{j=l}^{m-1} \|(p+2d)^j f\left(\frac{x}{(p+2d)^j}\right) \\ &\quad - (p+2d)^{j+1} f\left(\frac{x}{(p+2d)^{j+1}}\right)\| \leq \frac{\theta}{2^r} \sum_{j=l}^{m-1} \frac{(p+2d)^j}{(p+2d)^{rj}} \|x\|^r \end{aligned} \quad (2.4)$$

for all non-negative integers  $m$  and  $l$  with  $m > l$  and all  $x \in A$ . It follows from (2.4) that the sequence  $\{(p+2d)^n f(\frac{x}{(p+2d)^n})\}$  is a Cauchy sequence for all  $x \in A$ . Since  $A$  is complete, the sequence  $\{(p+2d)^n f(\frac{x}{(p+2d)^n})\}$  converges. So one can define the mapping  $\delta : A \longrightarrow A$  by

$$\delta(x) := \lim_{n \rightarrow \infty} (p+2d)^n f\left(\frac{x}{(p+2d)^n}\right)$$

for all  $x \in A$ .

Moreover letting  $l = 0$  and passing the limit  $m \rightarrow \infty$  in (2.4), we get (2.3). It follows from (2.1) that

$$\begin{aligned} \|r\delta\left(\frac{(p+2d)x}{r}\right) - (p+2d)\delta(x)\| &\leq \lim_{n \rightarrow \infty} (p+2d)^n \|rf\left(\frac{x}{(p+2d)^{n-1}}\right) \\ &\quad - (p+2d)f\left(\frac{x}{(p+2d)^n}\right)\| \leq \lim_{n \rightarrow \infty} \frac{(p+2d)^n}{(p+2d)^{nr}} (3\theta \|x\|^r) \\ &= 0 \end{aligned}$$

for all  $x \in A$ . So

$$r\delta\left(\frac{s \sum_{j=1}^p x_j + t \sum_{j=1}^d y_j}{r}\right) = s \sum_{j=1}^p \delta(x_j) + t \sum_{j=1}^d \delta(y_j)$$

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for all  $x \in A$ . By Lemma 2.1, the mapping  $\delta : A \longrightarrow A$  is Cauchy additive. By the same reasoning as in the proof of Theorem 2.1 of [15], the mapping  $\delta : A \longrightarrow A$  is  $\mathbb{C}$ -linear.

Setting  $y = 0$  in (2.2), then we have

$$\begin{aligned} & \|\delta([x, x, x]) - [\delta(x), x, x] - [x, \delta(x), x] - [x, x, \delta(x)]\| \\ &= \lim_{n \rightarrow \infty} (p+2d)^{(p+d)n} \|f(\frac{[x, x, x]}{(p+2d)^{(p+d)n}}) - [f(\frac{x}{(p+2d)^{(p+d)n}}), \frac{x}{(p+2d)^n}] \\ & \quad - [\frac{x}{(p+2d)^n}, f(\frac{x}{(p+2d)^n}), \frac{x}{(p+2d)^n}] - [\frac{x}{(p+2d)^n}, \frac{x}{(p+2d)^n}, f(\frac{x}{(p+2d)^n})]\| \\ &\leq \lim_{n \rightarrow \infty} \frac{(p+2d)^{(p+d)n}}{(p+2d)^{nr}} (3\theta \|x\|^r) = 0 \end{aligned}$$

for all  $x \in A$ . So

$$\delta([x, x, x]) = [\delta(x), x, x] + [x, \delta(x), x] + [x, x, \delta(x)]$$

for all  $x \in A$ .

Put  $x = 0$  in (2.2), we lead to

$$\begin{aligned} & \|\delta(x^*) - (\delta(y))^*\| \\ &= \lim_{n \rightarrow \infty} (p+2d)^{(p+d)n} \|f(\frac{y^*}{(p+2d)^{(p+d)n}}) - (p+2d)^{(p+d)n} (f(\frac{y}{(p+2d)^{(p+d)n}}))^*\| \\ &\leq \lim_{n \rightarrow \infty} \frac{(p+2d)^{(p+d)n}}{(p+2d)^{nr}} (3\theta \|y\|^r) = 0 \end{aligned}$$

for all  $y \in A$ . This means that  $\delta$  is  $*$ -preserving. Now, let  $T : A \longrightarrow A$  be another Cauchy-Jensen additive mapping satisfying (2.2). Then we have

$$\begin{aligned} \|\delta(x) - T(x)\| &= (p+2d)^n \|\delta(\frac{x}{(p+2d)^n}) - T(\frac{x}{(p+2d)^n})\| \\ &\leq (p+2d)^n (\|\delta(\frac{x}{(p+2d)^n}) - f(\frac{x}{(p+2d)^n})\| + \|T(\frac{x}{(p+2d)^n}) - f(x(p+2d)^n)\|) \\ &\leq \frac{3(p+2d)^n \theta}{(2^r - 2)(p+2d)^{nr}} \|x\|^r \end{aligned}$$

which tends to zero as  $n \rightarrow \infty$  for all  $x \in A$ . This proves the uniqueness property of  $\delta$ . Thus the mapping  $\delta : A \longrightarrow A$  is unique ternary Jordan  $*$ -derivation satisfying (2.3).  $\square$

**Theorem 2.3.** Let  $r, \theta$  be non-negative real numbers such that  $r \in (-\infty, \frac{1}{p+d}) \cup (1, +\infty)$ , and let  $f : A \longrightarrow A$  be a mapping such that

$$\|\Delta_\mu f(x_1, \dots, x_p, y_1, \dots, y_d)\| \leq \theta \prod_{j=1}^p \|x_j\|^r \cdot \prod_{j=1}^d \|y_j\|^r \quad (2.5)$$

for all  $\mu \in \mathbb{T}^1$  and all  $x, x_1, \dots, x_p, y_1, \dots, y_d \in A$ , and

$$\|\Delta f(x, y)\| \leq \theta \|x\|^{3r} \quad (2.6)$$

for all  $x \in A$ . Then there exists a unique ternary Jordan  $*$ -derivation  $\delta : A \longrightarrow A$  such that

$$\|f(x) - \delta(x)\| \leq \frac{2^{(p+d)r}}{|2(p+2d)^{(p+d)r} - (p+2d)2^{(p+d)r}|} \theta \|x\|^{(p+d)r} \quad (2.7)$$

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for all  $x \in A$ .

*Proof.* Letting  $\mu = 1$  and  $x_1 = \dots = x_p = y_1, \dots, y_d = x$  and  $s = 1, t = 2$  in (2.5), we get

$$\|f((p+2d)x) - (p+2d)f(x)\| \leq (p+d)\theta\|x\|^{3r} \quad (2.8)$$

for all  $x \in A$ . So

$$\|f(x) - (p+2d)f(\frac{x}{p+2d})\| \leq \frac{\theta}{(p+2d)^{(p+d)r}} \|x\|^{(p+d)r}$$

for all  $x \in A$ . Hence,

$$\begin{aligned} & \|(p+2d)^l f(\frac{x}{(p+2d)^l}) - (p+2d)^m f(\frac{x}{(p+2d)^m})\| \\ & \leq \sum_{j=l}^{m-1} \|(p+2d)^j f(\frac{x}{(p+2d)^j}) - (p+2d)^{j+1} f(\frac{x}{(p+2d)^{j+1}})\| \\ & \leq \frac{\theta}{(p+2d)^{(p+d)r}} \sum_{j=l}^{m-1} \frac{(p+2d)^j}{(p+2d)^{(p+d)rj}} \|x\|^{(p+d)r} \end{aligned} \quad (2.9)$$

for all non-negative integers  $m$  and  $l$  with  $m > l$  and all  $x \in A$ . It follows from (2.9) that the sequence  $\{(p+2d)^n f(\frac{x}{(p+2d)^n})\}$  is a Cauchy sequence for all  $x \in A$ . Since  $A$  is complete, the sequence  $\{(p+2d)^n f(\frac{x}{(p+2d)^n})\}$  converges. So one can define the mapping  $\delta : A \rightarrow A$  by

$$\delta(x) := \lim_{n \rightarrow \infty} (p+2d)^n f(\frac{x}{(p+2d)^n})$$

for all  $x \in A$ .

Moreover letting  $l = 0$  and passing the limit  $m \rightarrow \infty$  in (2.9), it follows from (2.6) that  $\delta$  is a ternary Jordan derivation. The rest of the proof is similar to the proof of Theorem 2.2.  $\square$

Now, we investigate superstability of ternary Jordan homomorphisms in  $C^*$ -ternary algebras associated with the functional equation  $\Delta_\mu f(x_1, \dots, x_p, y_1, \dots, y_d) = 0$ .

**Theorem 2.4.** Let  $t \in \{-1, 1\}$ , and let  $\varphi : A^{(p+d)} \rightarrow [0, \infty)$  and  $\psi : A^3 \rightarrow [0, \infty)$  be mappings such that

$$\tilde{\varphi}(x) := \sum_{n=0}^{\infty} \gamma^{-tn} \varphi(\gamma^{tn} x, \dots, \gamma^{tn} x) < \infty, \quad (2.10)$$

$$\lim_{n \rightarrow \infty} \gamma^{-tn} \varphi(\gamma^{tn} x_1, \dots, \gamma^{tn} x_p, \gamma^{tn} y_1, \dots, \gamma^{tn} y_d) = 0, \quad (2.11)$$

$$\lim_{n \rightarrow \infty} \gamma^{-3tn} \psi(\gamma^{tn} x, \gamma^{tn} x, \gamma^{tn} x) = 0, \quad \lim_{n \rightarrow \infty} \gamma^{-2tn} \psi(\gamma^{tn} x, \gamma^{tn} x, x) = 0 \quad (2.12)$$

for all  $x, x_1, \dots, x_p, y_1, \dots, y_d \in A$  where  $\gamma = \frac{p+2d}{2}$ . Suppose that  $f : A \rightarrow A$  is a mapping satisfying

$$\|\Delta_\mu f(x_1, \dots, x_p, y_1, \dots, y_d)\|_A \leq \varphi(x_1, \dots, x_p, y_1, \dots, y_d), \quad (2.13)$$

$$\|\Delta f(x, y)\| \leq \psi(x, x, x) \quad (2.14)$$

for all  $\mu \in \mathbb{T}^1$  and all  $x, x_1, \dots, x_p, y_1, \dots, y_d \in A$ . Then the mapping  $f : A \rightarrow A$  is a ternary Jordan  $*$ -derivation.

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*Proof.* By the same reasoning as the proof of Theorem 3.3 of [12], the sequence  $\{\frac{1}{\gamma^{tn}}f(\gamma^{tn}x)\}$  is converges. Thus one can define the mapping  $\delta : A \longrightarrow A$  by

$$\delta(x) := \lim_{n \rightarrow \infty} \frac{1}{\gamma^{tn}} f(\gamma^{tn}x)$$

for all  $x \in A$  moreover,  $\delta(\lambda x + \mu y) = \lambda\delta(x) + \mu\delta(y)$  for all  $\lambda, \mu \in \mathbb{T}^1$  and all  $x, y \in A$ . Therefore, by Lemma 2.1 the mapping  $\delta : A \longrightarrow A$  is  $\mathbb{C}$ -linear. It follows from (2.11) and (2.14) that

$$\|\Delta\delta(x, x, x)\| = \lim_{n \rightarrow \infty} \frac{1}{\gamma^{3tn}} \|\Delta f(\gamma^{tn}x, \gamma^{tn}x, \gamma^{tn}x)\| \leq \lim_{n \rightarrow \infty} \frac{1}{\gamma^{3tn}} \psi(\gamma^{tn}x, \gamma^{tn}x, \gamma^{tn}x) = 0$$

for all  $x \in A$ . Hence

$$\delta([x, x, x]) = [\delta(x), x, x] + [x, \delta(x), x] + [x, x, \delta(x)] \quad (2.15)$$

for all  $x \in A$ . So the mapping  $\delta : A \longrightarrow A$  is a  $C^*$ -ternary Jordan derivation. It follows from (2.12) and (2.14)

$$\begin{aligned} & \|\delta([x, x, x]) - [\delta(x), x, x] - [x, \delta(x), x] - [x, x, \delta(x)]\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{\gamma^{2tn}} \|f[\gamma^{tn}x, \gamma^{tn}x, x] - [f(\gamma^{tn}x), \gamma^{tn}x, x] \\ & \quad - [\gamma^{tn}x, f(\gamma^{tn}x), x] - [\gamma^{tn}x, \gamma^{tn}x, f(x)]\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{\gamma^{2tn}} \psi(\gamma^{tn}x, \gamma^{tn}x, x) = 0 \end{aligned}$$

for all  $x \in A$ . This means that

$$\delta[x, x, x] = [\delta(x), x, x] + [x, \delta(x), x] + [x, x, \delta(x)] \quad (2.16)$$

for all  $x \in A$ . Hence we get from (2.15) and (2.16) that

$$[x, x, \delta(x)] = [x, x, f(x)] \quad (2.17)$$

for all  $x \in A$ . Letting  $x = f(x) - \delta(x)$  in (2.17), we get

$$\|f(x) - \delta(x)\|^3 = \|[f(x) - \delta(x), f(x) - \delta(x), f(x) - \delta(x)]\| = 0$$

for all  $x \in A$ . Hence  $f(x) = \delta(x)$  for all  $x \in A$ . So the mapping  $f : A \longrightarrow A$  is a  $C^*$ -ternary Jordan derivation, as desired.  $\square$

**Corollary 2.5.** Let  $r < 1, s < 2$  and  $\theta$  be non-negative real numbers, and let  $f : A \longrightarrow A$  be a mapping satisfying (2.1) and

$$\|\Delta f(x, y)\| \leq 3\theta\|x\|^s \quad (2.18)$$

for all  $x \in A$ . Then the mapping  $f : A \longrightarrow A$  is a  $C^*$ -ternary Jordan derivation.

*Proof.* It follows from Theorem 2.4 by putting  $t = 1$ ,

$$\varphi(x_1, \dots, x_p, y_1, \dots, y_d) = \theta \left( \sum_{j=1}^p \|x_j\|_A^r + \sum_{j=1}^d \|y_j\|^r \right)$$

and

$$\psi(x, x, x) = 3\theta\|x\|^s$$

for all  $x, x_1, \dots, x_p, y_1, \dots, y_d \in A$ .  $\square$



**Corollary 2.6.** *Let  $r, s$  and  $\theta$  be non-negative real numbers such that  $s, r(p + d) < 1$ , and let  $f : A \longrightarrow A$  be a mapping satisfying (2.5) and*

$$\|\Delta f(x, y)\| \leq \|x\|^{3s} \quad (2.19)$$

*for all  $x \in A$ . Then the mapping  $f : A \longrightarrow A$  is a  $C^\ast$ -ternary Jordan derivation.*

*Proof.* It follows from Theorem 2.4 by putting  $t = 1$ ,

$$\varphi(x_1, \dots, x_p, y_1, \dots, y_d) = \theta \prod_{j=1}^p \|x_j\|_A^r \prod_{j=1}^d \|y_j\|^r$$

and

$$\psi(x, x, x) = \theta \|x\|^{3s}$$

for all  $x, x_1, \dots, x_p, y_1, \dots, y_d \in A$ . □

Similarly by putting  $t = -1$  in Theorem 2.4, we can prove the following corollary.

**Corollary 2.7.** *Let  $r, s$  and  $\theta$  be non-negative real numbers such that  $s, r(p + d) > 1$ , and let  $f : A \longrightarrow A$  be a mapping satisfying (2.5) and (2.19). Then the mapping  $f : A \longrightarrow A$  is a  $C^\ast$ -ternary Jordan derivation.*

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# On the Second Moment of Rational Bernstein Functions

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## Abstract

We study the dependance of the second moment of the rational Bernstein functions on the weight numbers. For the second degree rational Bernstein functions we prove that the minimal value of the second moment uniform on  $[0, 1]$  is attained when all weight numbers are equal. At the end we present a quantitative variant of Voronovskaja's Theorem.

**Key words and phrases:** Bernstein operator; Rational Bernstein functions; Moduli of continuity; Voronovskaja's Theorem.

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## 1 Introduction and main results

The NURBS ("non-uniform" rational B-splines) play an important role in Computer Aided Geometric Design (CAGD). The standart source on this method is is the book of Piegl and Tiller [4]. Special cases of NURBS curves are the famous Schoenberg spline curve and rational Bernstein-Bézier curves. Adapted to the context of approximation of functions the latter was generalized by H.Gonska in [1] as follows:

$$R_n(f; x) := \frac{\sum_{k=0}^n \omega_k \cdot f\left(\frac{k}{n}\right) \cdot p_{n,k}(x)}{\sum_{k=0}^n \omega_k \cdot p_{n,k}(x)}, \quad (1.1)$$

where  $\omega_k > 0$ ,  $0 \leq k \leq n$  are the so-called weight numbers and

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

are the basic Bernstein polynomials of degree  $n$ . Suppose that  $\omega_k = \omega$ ,  $0 \leq k \leq n$ . Then

$$R_n(f; x) := B_n(f; x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \cdot p_{n,k}(x),$$

i.e. the rational Bernstein function (1.1) reduces to the Bernstein polynomial of degree  $n$ . We consider  $R_n(f; x)$  as a linear positive operator (rational Bernstein operator), acting on the space of continuous functions  $f \in C[0, 1]$ . Some properties of  $R_n(f; x)$  are:

- (i)  $R_n(f; x)$  is a positive linear operator reproducing constant functions.
- (ii) One has  $R_n(f; 0) = f(0)$  and  $R_n(f; 1) = f(1)$ .
- (iii) The following representation of the first moment of  $R_n(f; x)$  was established in [1]-(see Proposition 3):

$$R_n(f; x) - x = x(1-x) \cdot \frac{1}{N} \cdot \sum_{k=0}^{n-1} (\omega_{k+1} - \omega_k) p_{n-1,k}(x), \quad (1.2)$$

where

$$N = \sum_{k=0}^n \omega_k p_{n,k}(x).$$

As far as we know the exact presentation of the second moment of  $R_n(f; x)$ , namely  $R_n((t-x)^2; x)$  in a form, similar to (1.2) is still missing. From (1.2) it follows also that the operator  $R_n(f; x)$  reproduces linear functions if and only if all weight numbers  $\omega_k$  are equal. With several examples it was shown in [1] that even when not all weights  $\omega_k$  are equal, by an appropriate choice of the numbers  $\omega_k$  pointwise convergence

$$R_n(f; x) \rightarrow f(x)$$

and also uniform convergence

$$\|R_n(f) - f\|_{C[0,1]} \rightarrow 0$$

when  $n \rightarrow \infty$  is possible.

From the viewpoint of quantitative approximation theory an essential role play the estimates of the first and of the second moments of the operator  $R_n(f; x)$ :

$$R_n((t-x)^2; x) := \frac{\sum_{k=0}^n \omega_k \left(\frac{k}{n} - x\right)^2 p_{n,k}(x)}{\sum_{k=0}^n \omega_k \cdot p_{n,k}(x)}. \quad (1.3)$$

It is known that Bernstein operator reproduces linear functions and that

$$B_n((t-x)^2; x) = \frac{x(1-x)}{n}, \quad (1.4)$$

for all  $n \in \mathbb{N}$ -the set of natural numbers and  $x \in [0, 1]$ . Our first result concerns the second moment of  $R_2(f; x)$  when  $n = 2$  is:

**Theorem 1.** *Let  $n = 2$  and  $\omega_k > 0$ ,  $k = 0, 1, 2$ . Then for all  $x \in [0, 1]$*

$$\min_{\omega_k, 0 \leq k \leq 2} R_2((t-x)^2; x) = \frac{x(1-x)}{2} \quad (1.5)$$

and min is attained exactly when  $\omega_0 = \omega_1 = \omega_2$ .

It is natural to suppose the following

**Conjecture.** *Let  $n > 2$ . Then for all  $x \in (0, 1)$  it holds*

$$\min_{\omega_k, 0 \leq k \leq 2} R_n((t-x)^2; x) < \frac{x(1-x)}{n}. \quad (1.6)$$

As far as we know neither the proof of this Conjecture is available, nor the counterexample is proved, i.e. the min in (1.6) is attained if and only if all weights  $\omega_k$  are equal, which is the case  $n = 2$ . If the Conjecture is not true, this means that the classical Bernstein operator is best possible among all rational Bernstein operators  $R_n(f)$ . Concerning the value of the second moment we consider the infinite sequence of positive weight numbers  $\omega_0, \omega_1, \dots, \omega_n, \dots$  and the sequence of rational Bernstein operators  $\{R_n(f)\}_{n=1}^\infty$ . We define

$$\rho_{n,1} := \frac{\max_k |\omega_{k+1} - \omega_k|}{\min_k \omega_k}, \quad (1.7)$$

where the max and min is taken for  $0 \leq k \leq n$ . Also let

$$\rho_{n,2} := \frac{\max_k \omega_k}{\min_k \omega_k}. \quad (1.8)$$

Our second result is :

**Theorem 2.** *Let  $f \in C^2[0, 1]$ ,*

$$\rho_{n,1} \leq \frac{1}{n^\alpha}, \alpha > 1$$

*and  $\rho_{n,2} \leq A$  for all natural numbers  $n$ . Then the following uniform with respect to  $x \in [0, 1]$  convergence holds:*

$$\lim_{n \rightarrow \infty} n \cdot [R_n(f; x) - f(x) - \frac{1}{2}R_n((t-x)^2; x) \cdot f''(x)] = 0. \quad (1.9)$$

In Section 2 we prove Theorems 1 and 2. Some examples are included to show the impact of the choice of weights  $\omega_k$  on the value of the second moment and on the rate of the uniform convergence in Theorem 2. This paper was motivated by the talk, given by prof. H.Gonska at the International Conference "Constructive Theory of Functions" in Varna, 2005.

## 2 Proofs and examples

**Proof of Theorem 1.** Let us suppose that it is possible to define positive numbers  $\omega_0, \omega_1, \omega_2$  such that (1.6) is true for all  $x \in [0, 1]$ . Without losing of generality let  $\omega_0 = 1$ . Simple calculations show that the left side of (1.6) is equal to  $x(1-x)F(\omega_1, \omega_2, x)$  where

$$F(\omega_1, \omega_2, x) = \frac{x(1-x) + 2\omega_1(\frac{1}{2} - x)^2 + \omega_2x(1-x)}{(1-x)^2 + 2\omega_1x(1-x) + \omega_2x^2}.$$

Hence

$$F(\omega_1, \omega_2, x) < \frac{1}{2}$$

should be fulfilled for all  $x \in [0, 1]$ . Consequently

$$\begin{aligned} x^2(-1 + 2\omega_1 - \omega_2) + x(1 - 2\omega_1 + \omega_2) + \frac{\omega_1}{2} &< \\ &< \frac{1}{2} \cdot [x^2(1 - 2\omega_1 + \omega_2) + x(-2 + 2\omega_1) + 1]. \end{aligned}$$

The last is equivalent to

$$g(x) := 3x^2(1 - 2\omega_1 + \omega_2) + x(-4 - 2\omega_2 + 6\omega_1) + 1 - \omega_1 > 0. \quad (2.1)$$

When  $x \rightarrow 0$  we get

$$g(0) = 1 - \omega_1 > 0. \quad (2.2)$$

When  $x \rightarrow 1$  we get

$$g(1) = -\omega_1 + \omega_2 > 0. \quad (2.3)$$

Obviously the coefficient in front of  $x^2$  should be positive due to (2.2) and (2.3), while

$$1 - 2\omega_1 + \omega_2 = \omega_2 - \omega_1 + 1 - \omega_1.$$

Next we calculate the discriminant of the quadratic inequality (2.1)

$$\begin{aligned} \Delta &= (-2 - \omega_2 + 3\omega_1)^2 - 3(1 - \omega_1)(1 - 2\omega_1 + \omega_2) = \\ &= 3\omega_1^2 + \omega_2^2 - 3\omega_1\omega_2 - 3\omega_1 + \omega_2 + 1 := f(\omega_1, \omega_2). \end{aligned}$$

Our next step is to show that

$$\min_{\omega_1, \omega_2} f(\omega_1, \omega_2) = f(1, 1) = 0. \quad (2.4)$$

We calculate

$$\begin{aligned} \frac{\partial f}{\partial \omega_1} &= 6\omega_1 - 3\omega_2 - 3 = 0. \\ \frac{\partial f}{\partial \omega_2} &= 2\omega_2 - 3\omega_1 + 1 = 0. \end{aligned}$$

The solution of the last system is  $\omega_1 = \omega_2 = 1$ . Also

$$\frac{\partial^2 f}{\partial \omega_1^2} = 6, \quad \frac{\partial^2 f}{\partial \omega_2^2} = 2, \quad \frac{\partial^2 f}{\partial \omega_1 \partial \omega_2} = -3.$$

We verify that

$$\det \begin{pmatrix} \frac{\partial^2 f}{\partial \omega_1^2} & \frac{\partial^2 f}{\partial \omega_1 \partial \omega_2} \\ \frac{\partial^2 f}{\partial \omega_1 \partial \omega_2} & \frac{\partial^2 f}{\partial \omega_2^2} \end{pmatrix} > 0.$$

The proof of (2.4) is completed. Let  $x_1$  and  $x_2$  be the zeros of  $g(x) = 0$ . In order to have  $g(x) > 0$  for all  $x \in [0, 1]$  we have two possible cases:

$$x_1 < x_2 < 0,$$

or

$$1 < x_1 < x_2.$$

In the first case we should have  $\frac{x_1 + x_2}{2} < 0$ . Hence

$$-\frac{-4 - 2\omega_2 + 6\omega_1}{6(1 - 2\omega_1 + \omega_2)} < 0,$$

or equivalently

$$4 + 2\omega_2 - 6\omega_1 < 0.$$

But the last inequality is a contradiction to (2.3) and (2.2). Similarly we proceed in the second case. Now we would have

$$1 < \frac{x_1 + x_2}{2} = -\frac{-4 - 2\omega_2 + 6\omega_1}{6(1 - 2\omega_1 + \omega_2)}.$$

Consequently

$$\begin{aligned} 6(1 - 2\omega_1 + \omega_2) &< 4 + 2\omega_2 - 6\omega_1, \\ 6\omega_1 - 4\omega_2 - 2 &> 0. \end{aligned}$$

The last inequality again is a contradiction to (2.2) and (2.3).

Our supposition at the beginning of the proof is not valid. This completes the proof of Theorem 1.  $\square$

**Example 1** Let  $\omega_k = 1 - \frac{k}{n^\alpha}$ ,  $k = 0, 1, \dots, n$  and  $\alpha > 1$ ,  $n > 1$ . To calculate the first moment of  $R_n(f; x)$  we apply (1.2) and obtain

$$R_n((t - x); x) = x(1 - x) \frac{1}{N} \left( -\frac{1}{n^\alpha} \right), \quad (2.5)$$

where

$$N = \sum_{k=0}^n \omega_k p_{n,k}(x) = 1 - \frac{x}{n^{\alpha-1}}. \quad (2.6)$$

From Taylor formula it follows

$$t^2 = x^2 + (t - x)2x + (t - x)^2 \quad (2.7)$$

We apply the operator  $R_n(f)$  to the both sides of (2.7) and (2.5) yields

$$R_n((t - x)^2; x) = R_n(t^2; x) - x^2 + \frac{2x \cdot x(1 - x)}{1 - \frac{x}{n^{\alpha-1}}} \cdot \frac{1}{n^\alpha}. \quad (2.8)$$

To compute  $R_n(t^2; x)$  we proceed as follows:

$$\begin{aligned} R_n(t^2; x) &= \frac{\sum_{k=0}^n \left(1 - \frac{k}{n^\alpha}\right) \left(\frac{k}{n}\right)^2 \cdot p_{n,k}(x)}{1 - \frac{x}{n^{\alpha-1}}} = \\ &= \frac{x^2 + \frac{x(1-x)}{n} - \frac{1}{n^{\alpha-1}} \cdot B_n(t^3; x)}{1 - \frac{x}{n^{\alpha-1}}}. \end{aligned} \quad (2.9)$$

It is known that for all  $x \in [0, 1]$

$$B_n((t - x)^3; x) = \frac{x(1 - x)(1 - 2x)}{n^2}.$$



Therefore

$$\begin{aligned} B_n(t^3; x) - 3B_n(t^2; x)x + 3x^3 - x^3 &= \frac{x(1-x)(1-2x)}{n^2}. \\ B_n(t^3; x) - 3x(x^2 + \frac{x(1-x)}{n}) + 2x^3 &= \frac{x(1-x)(1-2x)}{n^2}. \\ B_n(t^3; x) &= x^3 + \frac{3x^2(1-x)}{n} + \frac{x(1-x)(1-2x)}{n^2}. \end{aligned}$$

From the last presentation, (2.8) and (2.9) we calculate

$$\begin{aligned} R_n((t-x)^2; x) &= \frac{1}{N} \cdot \left[ x^2 + \frac{x(1-x)}{n} - \frac{1}{n^{\alpha-1}} \cdot x^3 - \frac{3x^2(1-x)}{n^\alpha} - \right. \\ &\quad \left. - \frac{x(1-x)(1-2x)}{n^{\alpha+1}} - x^2 + \frac{x^3}{n^{\alpha-1}} + \frac{2x^2(1-x)}{n^\alpha} \right] = \\ &= \frac{1}{N} \cdot \frac{x(1-x)}{n} \cdot \left[ 1 - \frac{x}{n^{\alpha-1}} - \frac{1-2x}{n^\alpha} \right], \end{aligned}$$

with  $N$ , defined in (2.6). Hence

$$R_n((t-x)^2; x) = \frac{x(1-x)}{n} \cdot \left[ 1 + \frac{2x-1}{n^\alpha - xn} \right]. \quad (2.10)$$

From (2.10) we conclude that the statement in our Conjecture with this choice of the weights is true, but only for  $x \in (0, \frac{1}{2})$ . In a similar way we consider the next example.

**Example 2.** Let  $\omega_k = 1 + \frac{k}{n^\alpha}$ ,  $k = 0, 1, \dots, n$  and  $\alpha > 1$ ,  $n > 1$ . Then

$$R_n((t-x)^2; x) < \frac{x(1-x)}{n},$$

for  $x \in (\frac{1}{2}, 1)$ .

**Example 3.** Let  $\omega_k = 1 + \frac{(-1)^k}{n^\alpha}$ ,  $k = 0, 1, \dots, n$ ,  $\alpha > 1$ . Then

$$\begin{aligned} \rho_{n,1} &= \frac{2}{n^\alpha(1 - \frac{1}{n^\alpha})} = \frac{2}{n^\alpha - 1} \approx 0, \quad n \rightarrow \infty. \\ \rho_{n,2} &= \rho_{n,1} + 1 \approx 1. \end{aligned} \quad (2.11)$$

**Proof of Theorem2.** We apply the following quantitative variant of Voronovskaja's Theorem for linear positive operators  $L$ , established very recently by H.Gonska in [3]-see Example 4.3 on p.108:

$$|L(f; x) - f(x) - L((t-x)^2; x) \cdot f'(x) - \frac{1}{2}L((t-x)^2; x) \cdot f''(x)| \leq$$

$$\leq \frac{1}{2} \cdot L((t-x)^2; x) \cdot \tilde{\omega} \left( f''; \frac{1}{3} \cdot \frac{L(|t-x|^3; x)}{L((t-x)^2; x)} \right), \quad (2.12)$$

where  $\tilde{\omega}(f; \varepsilon)$  is the least concave majorant of the modulus of continuity of  $f$  and  $\varepsilon > 0$  is its argument. We replace  $L$  by  $R_n$  in (2.12) and observe that

$$\begin{aligned} \frac{R_n(|t-x|^3; x)}{R_n((t-x)^2; x)} &\leq \rho_{n,2} \cdot \frac{B_n(|t-x|^3; x)}{B_n((t-x)^2; x)} \leq \\ &\leq \rho_{n,2} \cdot 3 \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}}, \end{aligned} \quad (2.13)$$

for all  $x \in [0, 1]$ . In the last inequality we have applied Theorem 5.1 in [3]. We also have

$$R_n((t-x)^2; x) \leq \rho_{n,2} \cdot \frac{x(1-x)}{n}. \quad (2.14)$$

From (2.12) and assumptions made for  $\rho_{n,1}$ ,  $\rho_{n,2}$  we verify that

$$\begin{aligned} |R_n(f; x) - f(x) - \frac{1}{2} R_n((t-x)^2; x) \cdot f''(x)| &\leq \\ &\leq x(1-x) \frac{1}{n^\alpha} \cdot |f'(x)| + \frac{1}{2} A \frac{x(1-x)}{n} \cdot \tilde{\omega} \left( f''; A \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}} \right). \end{aligned} \quad (2.15)$$

We multiply the both sides of (2.15) by  $n$  and complete the proof of Theorem 2.  $\square$

**Corollary 1.** *Under the conditions made in Theorem 2 the following upper bound for the error of approximation by the rational Bernstein operator  $R_n(f)$  holds true for all  $f \in C^2[0, 1]$ :*

$$\|R_n(f) - f\|_{C[0,1]} \leq \frac{1}{4n^\alpha} \cdot \|f'\|_{C[0,1]} + \frac{A}{4n} \cdot \|f''\|_{C[0,1]}. \quad (2.16)$$

Similar estimates to (2.16) are fulfilled with weights  $\omega_k$ , described in Examples 1, 2 and 3. In all these examples the weight numbers satisfy the restrictions, formulated in Theorem 2.

At the end let us define two other variants of numbers  $\omega_k$ , namely

**Example 4.**

$$\omega_k = \left[ \frac{k}{n} \left( 1 - \frac{k}{n} \right) + \frac{\delta}{n} \right]^\alpha, \quad 0 \leq k \leq n, \quad (2.17)$$

for  $\alpha > 0$ ,  $\delta \geq 0$ .

**Example 5.**

$$\omega_k = \left| \frac{k}{n} - \frac{1}{2} \right|^\alpha + \frac{\delta}{n}, \quad 0 \leq k \leq n, \quad (2.18)$$

for  $\alpha > 0$ ,  $\delta \geq 0$ . Numerical experiments made for  $\alpha = 0.5, 1, 3, 4$  and  $n = 50$ ,  $\delta = 0, 0.5, 1$  show that close to the ends of  $[0, 1]$  we have (1.6) with  $\omega_k$  from (2.18) and when  $x$  is near to  $\frac{1}{2}$ , (1.6) holds true with  $\omega_k$  from (2.17). But neither (2.17) nor (2.18) are solutions of our Conjecture. So the statement of the Conjecture remains as an open problem-nonlinear optimization problem, according to the choice of weight numbers.

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## **PROPERTIES OF PARAMETER-VARYING DYNAMIC LINEAR SYSTEMS WITH PARAMETRIZED PERTURBATIONS. APPLICATIONS TO DELAYED DYNAMICS**

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**Abstract.** This paper deals with a unifying approach to the problems of computing the admissible sets of parametrical multi perturbations in appropriate bounded sets such that some fundamental properties of parameter-varying linear dynamic systems are maintained provided that the so-called (i.e. perturbation-free) nominal system possesses such properties. The sets of parametrical multi perturbations include any combinations of parametrical multi perturbations in the matrix of dynamics as well as in the control, output and input-output interconnection matrices which belong to some prescribed bounded domain in the complex space. The various properties which are investigated are controllability, observability, output controllability, stabilizability and detectability as well as the existence of minimal state-space realizations together with the associate existence or not of associate decoupling, transmission and invariant zeros. All the matrices of parameters including the nominal and the disturbed ones which parameterize the dynamic system may be real or complex. The obtained results are then applied to systems subject to a finite number of discrete internal delays and parametrical multi perturbations by comparing the state- space descriptions of such systems with the general descriptions previously investigated. In particular, the contributions of the delays to the spectral descriptions are assimilated to the contributions of a set of varying parameters in a domain for the general description.

**Keywords:** Parameter-varying systems, Controllability, Observability, Zeros, Multi- parametrical perturbations.

### **1. Introduction**

The problem of robust stability of dynamic systems has received important attention in the last two decades, [1-3]. The related investigations require in general ad-hoc mathematical tools from Mathematical and Functional Analysis, [1-4]. Recently, the notion of stability radius has been used for related investigations, [5-7]. The stability radius of a linear dynamic system is a positive real number which defines the minimum size, in terms of norm, of a parametrical perturbation, belonging to an admissible class, such that the resulting system becomes unstable or critically stable provided that the nominal (i.e. perturbation-free) system is stable. Such a characterization has been used successfully in [5-7] to investigate the maximum size of both structured and unstructured multi parametrical (in general, complex) perturbations so that positive systems are maintained stable provided that its nominal part is stable. Further advantages of focusing the robust stability problem in that way are that the robust stability of wide classes of parameter-varying dynamic

systems, including some described by functional equations, may be studied in a unified way, [5-9]. The purpose of this paper is to study the fundamental properties of controllability, observability, stabilizability and detectability of parameter-varying linear systems, in a way inspired in the ideas developed in [5-7]. However, some variations are that the parameter-varying systems under (in general complex) parametrical multi perturbations are not necessarily positive and that spectral stability radii are not involved since the problem at hand is not that of robust stabilization. The main idea is to maintain the Popov-Belevitch-Hautus matrix functions [1] for the investigated properties being full rank for all multi perturbations provided that the matrix of the nominal system is also full rank. The multi perturbations are of a given structured class on a certain domain where the varying parameters belong to. The worst case of the admissible perturbations establishes the robustness degree of the property. The extension to unstructured perturbations is not discussed since it is direct, and even more simple, than that for structured ones. The study is addressed in a unified way for all the properties. In particular, it has direct interest in realization theory since the size of the disturbances which maintain a minimal state-space realization of the system may be characterized, provided that the nominal realization is minimal. To this end, the best of the two worst cases of losing either controllability or observability by the perturbed system of a nominally controllable and observable system ensures that the state-space realization is still minimal. A direct extension is that if the nominal system is stabilizable and detectable, the best of the two worst cases of losing some of both properties for some multi perturbation in the given class still guarantees that any eventual zero-pole cancellation in the transfer matrix is stable. The technical mechanism employed to investigate the various properties is the construction of square auxiliary matrix functions which are symmetrical positive definite (or Hermitian in the complex case). If their associate minimum singular value becomes zero, or equivalently, if their determinants become zero for some perturbation while their counterparts of the nominal system are positive then the investigated property is lost. One takes advantage that the functions characterizing the singular values and the determinant of a complex continuous matrix function are continuous functions on the definition domain of such a matrix function. The results are easily extendable to parameter-varying dynamic linear internal and external delays under multi – perturbations. There is an important background on time-delay systems, [10-34] including models of neural networks including delays, [25-27]. Controllability, observability and stability are very basic properties of dynamic systems which make possible a wide range of applications in non-uniform sampling, classical and adaptive control, estimation etc. [30], [34], [37-29], [41-43]. Output controllability of the output vector is a property with a close sense to the standard controllability of the state vector, [44]. In particular, the study of stability of time-delay systems has received attention in [10-11], [15], [19-24], [30-34], the positivity and periodicity of the trajectory-solutions have being investigated in [13-14], [27-29], [32], [35-36] and the state-trajectory solutions under impulsive controls in [12-13], [15-17] including the case of singular systems, [13]. Sufficiency-type conditions for the robust characterization of those properties follows directly from the general study based on the fact that time-delay systems might be characterized as nD- systems [8-9], [21]. Refinements of the conditions either allowing to derive results dependent on the delay sizes or guaranteeing that the properties hold for some cases not included in the nD- system characterizations are also investigated.

This is addressed by further considering the spectral characterization of linear delay systems under quasi-polynomials.

## 2. Notation

Subsets of the complex and real fields  $\mathbf{R}$  and  $\mathbf{C}$  are:

$$\mathbf{R}_{+0} := \{r \in \mathbf{R} : \operatorname{Re}(r) \geq 0\}, \mathbf{R}_+ := \{r \in \mathbf{R} : \operatorname{Re}(r) > 0\}, \mathbf{R}_- := \{r \in \mathbf{R} : \operatorname{Re}(r) < 0\}$$

$$\mathbf{C}_{+0} := \{r \in \mathbf{C} : \operatorname{Re}(r) \geq 0\}, \mathbf{C}_+ := \{r \in \mathbf{C} : \operatorname{Re}(r) > 0\}, \mathbf{C}_- := \{r \in \mathbf{C} : \operatorname{Re}(r) < 0\}$$

which stand for nonnegative, positive and negative real numbers or complex numbers of nonnegative, positive and negative real abscissas, respectively.  $\mathbf{C}_I := \{z \in \mathbf{C} : |z| = 1\}$  is the unit circumference in the complex plane.

$C^{(q)}(\mathbf{R}_{+0}, \mathbf{K}^p)$  and  $PC^{(q)}(\mathbf{R}_{+0}, \mathbf{K}^p)$  with  $K = \mathbf{C}$  or  $K = \mathbf{R}$  denote, respectively, real or complex vector functions of class  $q \geq 0$  and those of class  $q-1$  (if  $q \geq 1$ ) whose  $q$ -th derivative is continuously differentiable with piecewise continuous  $q$ -th derivative on the definition domain  $\mathbf{R}_{+0}$ .  $PC^{(0)}(\mathbf{R}_{+0}, \mathbf{K}^p)$  denotes the class of piecewise continuous  $p$ -vector functions with domain  $\mathbf{R}_{+0}$  and range  $\mathbf{K}^p$ .

$I_n$  denotes the  $n$ -th identity matrix. The superscript  $''^*''$  denotes the transpose conjugate of a complex matrix resulting in the usual transpose for real matrices (denoted with the superscript  $''^T''$ ),  $\det M$  and  $\sigma(M) := \{\lambda \in \mathbf{C} : \det(\lambda I_n - M) = 0\}$  denote the determinant of the matrix  $M$  and its spectrum, respectively. Subsets of  $\sigma(M)$  are  $\sigma_+(M) := \{\lambda \in \sigma(M) : \operatorname{Re} \lambda > 0\}$ ,  $\sigma_{+0}(M) := \{\lambda \in \sigma(M) : \operatorname{Re} \lambda \geq 0\}$ ,  $\sigma_-(M) := \{\lambda \in \sigma(M) : \operatorname{Re} \lambda < 0\}$  and  $\sigma_{-0}(M) := \{\lambda \in \sigma(M) : \operatorname{Re} \lambda \leq 0\}$ .

The spectral radius and spectral abscissa of  $M$  are denoted, respectively, by  $\rho(M) := \{\max |\lambda| : \lambda \in \sigma(M)\}$  and  $\mu(M) := \{\max \operatorname{Re} \lambda : \lambda \in \sigma(M)\}$ . The singular values of  $M$  are the positive squares of the eigenvalues of any of the matrix products  $MM^*$  and  $M^*M$  provided that they exist which are real and satisfy  $\bar{\sigma}(M) \geq \underline{\sigma}(M) \geq 0$  where  $\bar{\sigma}(M)$  and  $\underline{\sigma}(M)$  are the maximum and minimum singular value of  $M$ , respectively. Note that at least one of the matrix products  $MM^*$  and  $M^*M$  always exist. The spectral (or  $\ell_2$ ) vector norm and associated induced matrix norm are denoted by  $\|\cdot\|_2$ .

The interior, boundary (frontier) and closure of a set  $K$  are denoted as  $K^0$ ,  $K^{Fr}$ , and  $c\ell K$ , respectively.

$\hat{f}(s) := \text{Lap}(f(t))$ ,  $f(t) := \text{Lap}^{-1}(f(s))$  is a pair of Laplace transform and Laplace anti-transform provided that such an anti-transform exists.

The Kronecker (or direct) product of the matrices  $A = (a_{ij})$  and  $B$  is denoted by  $A \otimes B = (a_{ij} B)$  and for such a matrix  $A$ ,  $\text{vec}(A) = (a_1^*, a_2^*, \dots, a_n^*)^*$  with  $a_i^*$  being the  $i$ -th row of  $A$ , i.e.  $A^* = (a_1, a_2, \dots, a_n)$ . The boundary (or frontier) of a set  $Q$  is denoted as  $Q^{Fr}$ .

The complex unit is  $i = \sqrt{-1}$ .

### 3. The parameter-varying system and associate fundamental properties: controllability, observability, minimal realizability, stabilizability and detectability

The parameter-varying linear time-invariant dynamic system to be considered is

$$\dot{x}(z, t) = (A(z^{(A)}) + \tilde{A}(z^{(A)}))x(z, t) + (B(z^{(B)}) + \tilde{B}(z^{(B)}))u(t) \quad (1)$$

$$y(z, t) = (C(z^{(C)}) + \tilde{C}(z^{(C)}))x(z, t) + (D(z^{(D)}) + \tilde{D}(z^{(D)}))u(t) \quad (2)$$

,  $\forall t \in \mathbf{R}_{+0} := \mathbf{R}_+ \cup \{0\}$ , with  $\mathbf{R}_+ := \{s \in \mathbf{R} : s > 0\}$ , subject to initial conditions

$x(z, 0) = x_0(z) \in \mathbf{C}^n$ , where  $X \subset \mathbf{C}^n$ ,  $Y \subset \mathbf{C}^p$  and  $U \subset \mathbf{C}^m$  are, respectively, the state, output and

input linear spaces, and  $x \in C^{(1)}(\mathbf{C}^{q \times \mathbf{R}_{+0}}, X)$ ,  $y \in C^{(0)}(\mathbf{C}^{q \times \mathbf{R}_{+0}}, Y)$  and

$u \in PC^{(0)}(\mathbf{C}^{q \times \mathbf{R}_{+0}}, U)$  are, the everywhere continuously time-differentiable  $n$ -vector state trajectory

solution, the piecewise continuous  $p$ -vector output-trajectory solution, and the piecewise continuous  $m$ -vector control input, respectively, with  $p \leq m < n$  and  $q := q_A + q_B + q_C + q_D + 4$ , and

$A: \mathbf{C}^{q_A+1} \rightarrow \mathbf{C}^{n \times n}$ ,  $B: \mathbf{C}^{q_B+1} \rightarrow \mathbf{C}^{n \times m}$ ,  $C: \mathbf{C}^{q_C+1} \rightarrow \mathbf{C}^{p \times n}$  and  $D: \mathbf{C}^{q_D+1} \rightarrow \mathbf{C}^{p \times m}$ ; and

$\tilde{A}: \mathbf{C}^{q_A+1} \rightarrow \mathbf{C}^{n \times n}$ ,  $\tilde{B}: \mathbf{C}^{q_B+1} \rightarrow \mathbf{C}^{n \times m}$ ,  $\tilde{C}: \mathbf{C}^{q_C+1} \rightarrow \mathbf{C}^{p \times n}$  and  $\tilde{D}: \mathbf{C}^{q_D+1} \rightarrow \mathbf{C}^{p \times m}$  are,

respectively, the (so-called) nominal and perturbation (complex-valued) matrices of dynamics, control, output and input-output interconnections whose parameter-varying arguments are defined by the respective complex-valued  $(q_A + 1)$ ,  $(q_B + 1)$ ,  $(q_C + 1)$  and  $(q_D + 1)$ -tuples:

$$z^{(A)} := (1, z_1^{(A)}, z_2^{(A)}, \dots, z_{q_A}^{(A)}) \in \{1\} \times \mathbf{C}^{q_A} \subset \mathbf{C}^{q_A+1}, \quad z^{(B)} := (1, z_1^{(B)}, z_2^{(B)}, \dots, z_{q_B}^{(B)}) \in \{1\} \times \mathbf{C}^{q_B} \subset \mathbf{C}^{q_B+1} \quad (3)$$

$$z^{(C)} := (1, z_1^{(C)}, z_2^{(C)}, \dots, z_{q_C}^{(C)}) \in \{1\} \times \mathbf{C}^{q_C} \subset \mathbf{C}^{q_C+1}, \quad z^{(D)} := (1, z_1^{(D)}, z_2^{(D)}, \dots, z_{q_D}^{(D)}) \in \{1\} \times \mathbf{C}^{q_D} \subset \mathbf{C}^{q_D+1} \quad (4)$$

The (so-called) nominal system is affine parameter-varying defined from (1)-(2) with

$$A(z^{(A)}) := \sum_{i=0}^{q_A} z_i^{(A)} A_i, \quad B(z^{(B)}) := \sum_{i=0}^{q_B} z_i^{(B)} B_i, \quad C(z^{(C)}) := \sum_{i=0}^{q_C} z_i^{(C)} C_i, \quad D(z^{(D)}) := \sum_{i=0}^{q_D} z_i^{(D)} D_i \quad (5)$$

where  $z_0^{(A)} = z_0^{(B)} = z_0^{(C)} = z_0^{(D)} = I$ ,  $\tilde{A}(z^{(A)}) = 0$ ,  $\tilde{B}(z^{(B)}) = 0$ ,  $\tilde{C}(z^{(C)}) = 0$  and  $\tilde{D}(z^{(D)}) = 0$ , and

$$A_i: \mathcal{C} \rightarrow \mathcal{C}^{n \times n} \quad (i \in \bar{q}_A^0), \quad B_i: \mathcal{C} \rightarrow \mathcal{C}^{n \times m} \quad (i \in \bar{q}_B^0), \quad C_i: \mathcal{C} \rightarrow \mathcal{C}^{p \times n} \quad (i \in \bar{q}_C^0), \quad D_i: \mathcal{C} \rightarrow \mathcal{C}^{p \times m} \quad (i \in \bar{q}_D^0) \quad (6)$$

with  $\bar{n}^0 := \bar{n} \cup \{0\}$ ,  $\bar{n} := \{1, 2, \dots, n\}$ . The nominal system is the (unperturbed) reference one to then establish the known nominal bounded domain (i.e. a connected open set)  $\mathcal{C}_{\alpha 0} \subset \mathcal{C}^q$  where the nominal system fulfills the various investigated properties for all  $z := (z^{(A)}, z^{(B)}, z^{(C)}, z^{(D)}) \in \mathcal{C}_{\alpha 0}$  and also the bounded domain  $\mathcal{C}_\alpha \subset \mathcal{C}_{\alpha 0}$  where a class of systems (1)-(2), eventually submitted to perturbations, still maintain the particular property under investigation kept by the nominal one on  $\mathcal{C}_{\alpha 0}$ . The class of systems (1)-(2), which include the nominal system as particular case, are defined via (3)-(6) for parametrical multi perturbations in a set  $P := P_A \times P_B \times P_C \times P_D \subset \mathcal{C}^{q_A+1} \times \mathcal{C}^{q_B+1} \times \mathcal{C}^{q_C+1} \times \mathcal{C}^{q_D+1} \equiv \mathcal{C}^q$  of the form:

$$\begin{aligned} \tilde{A}(z^{(A)}) &:= \sum_{i=0}^{q_A} z_i^{(A)} \tilde{A}_i = \sum_{i=0}^{q_A} \sum_{j=1}^{n_A} z_i^{(A)} D_{ij}^{(A)} \Delta_{ij}^{(A)} E^{(A)}, \quad \tilde{B}(z^{(B)}) := \sum_{i=0}^{q_B} z_i^{(B)} \tilde{B}_i = \sum_{i=0}^{q_B} \sum_{j=1}^{n_B} z_i^{(B)} D_{ij}^{(B)} \Delta_{ij}^{(B)} E^{(B)} \\ \tilde{C}(z^{(C)}) &:= \sum_{i=0}^{q_C} z_i^{(C)} \tilde{C}_i = \sum_{i=0}^{q_C} \sum_{j=1}^{n_C} z_i^{(C)} D_{ij}^{(C)} \Delta_{ij}^{(C)} E^{(C)}, \quad \tilde{D}(z^{(D)}) := \sum_{i=0}^{q_D} z_i^{(D)} \tilde{D}_i = \sum_{i=0}^{q_D} \sum_{j=1}^{n_D} z_i^{(D)} D_{ij}^{(D)} \Delta_{ij}^{(D)} E^{(D)} \end{aligned} \quad (7)$$

where  $\tilde{A}: \mathcal{C}^{q_A+1} \rightarrow P_A \subset \mathcal{C}^{n \times n}$ ,  $\tilde{B}: \mathcal{C}^{q_B+1} \rightarrow P_B \subset \mathcal{C}^{n \times m}$ ,  $\tilde{C}: \mathcal{C}^{q_C+1} \rightarrow P_C \subset \mathcal{C}^{p \times n}$  and  $\tilde{D}: \mathcal{C}^{q_D+1} \rightarrow P_D \subset \mathcal{C}^{p \times m}$  with

$$\tilde{A}_i: \mathcal{C} \rightarrow \mathcal{C}^{n \times n} \quad (i \in \bar{q}_A^0), \quad \tilde{B}_i: \mathcal{C} \rightarrow \mathcal{C}^{n \times m} \quad (i \in \bar{q}_B^0), \quad \tilde{C}_i: \mathcal{C} \rightarrow \mathcal{C}^{p \times n} \quad (i \in \bar{q}_C^0), \quad \tilde{D}_i: \mathcal{C} \rightarrow \mathcal{C}^{p \times m} \quad (i \in \bar{q}_D^0) \quad (8)$$

are defined by  $\tilde{A}_i := \sum_{j=1}^{n_A} D_{ij}^{(A)} \Delta_{ij}^{(A)} E^{(A)} \quad (i \in \bar{q}_A^0)$ ,  $\tilde{B}_i := \sum_{j=1}^{n_B} D_{ij}^{(B)} \Delta_{ij}^{(B)} E^{(B)} \quad (i \in \bar{q}_B^0)$ ,

$\tilde{C}_i := \sum_{j=1}^{n_C} D_{ij}^{(C)} \Delta_{ij}^{(C)} E^{(C)} \quad (i \in \bar{q}_C^0)$  and  $\tilde{D}_i := \sum_{j=1}^{n_D} D_{ij}^{(D)} \Delta_{ij}^{(D)} E^{(D)} \quad (i \in \bar{q}_D^0)$ , where

$$D_{ij}^{(A)}: \mathcal{C} \rightarrow \mathcal{C}^{n \times \ell_{ij}^{(A)}}, (i, j) \in \bar{q}_A^0 \times \bar{n}_A; \quad D_{ij}^{(B)}: \mathcal{C} \rightarrow \mathcal{C}^{n \times \ell_{ij}^{(B)}}, (i, j) \in \bar{q}_B^0 \times \bar{n}_B$$



$$D_{ij}^{(C)}: C \rightarrow C^{p \times \ell_{ij}^{(C)}}, \forall (i, j) \in \bar{q}_C^0 \times \bar{n}_C; \quad D_{ij}^{(D)}: C \rightarrow C^{p \times \ell_{ij}^{(D)}}, \forall (i, j) \in \bar{q}_D^0 \times \bar{n}_D \quad (9)$$

$$\begin{aligned} \Delta_{ij}^{(A)}: C \rightarrow C^{\ell_{ij}^{(A)} \times \ell^{(A)}}, \forall (i, j) \in \bar{q}_A^0 \times \bar{n}_A; \quad \Delta_{ij}^{(B)}: C \rightarrow C^{\ell_{ij}^{(B)} \times \ell^{(B)}}, \forall (i, j) \in \bar{q}_B^0 \times \bar{n}_B \\ \Delta_{ij}^{(C)}: C \rightarrow C^{\ell_{ij}^{(C)} \times \ell^{(C)}}, \forall (i, j) \in \bar{q}_C^0 \times \bar{n}_C; \quad \Delta_{ij}^{(D)}: C \rightarrow C^{\ell_{ij}^{(D)} \times \ell^{(D)}}, \forall (i, j) \in \bar{q}_D^0 \times \bar{n}_D \end{aligned} \quad (10)$$

$$E^{(A)}: C \rightarrow C^{\ell_f^{(A)} \times n}; \quad E^{(B)}: C \rightarrow C^{\ell_f^{(B)} \times m}; \quad E^{(C)}: C \rightarrow C^{\ell_f^{(C)} \times n}; \quad E^{(D)}: C \rightarrow C^{\ell_f^{(D)} \times m} \quad (11)$$

The parametrical multi perturbations in the dynamic system (1)-(2) are defined by the matrices (7), subject to (8)-(11), dependent on the argument  $z := \left( z^{(A)}, z^{(B)}, z^{(C)}, z^{(D)} \right)$  which takes values in some domain  $C_\alpha$  of  $C^q$ . The matrices (11) are scaling matrices common to all the output components being independent of the various subscripts (i, j). The matrices in (9) are also scaling matrices of the state versus state components, state versus input component, output versus state component s and output versus input components. The matrices in (10) are specific parametrical perturbations which become weighted by the contribution of the corresponding component of z in  $C_\alpha$  through the global parametrical perturbations (7)-(8). Note by direct inspection that if all the perturbation matrices in (10) are zero then, the dynamic system (1)-(2) becomes the nominal one. Note also that an extension of the parametrical perturbations consisting of considering the scaling matrices Eqs. 11 to be dependent on the indices (i, j) would not become more general than that given in view of the whole structure of the multi perturbations (7)-(8). The following matrices are defined for each system (1)-(2) (see [1-3], [8-9]):

### Definitions

**3.1** The spectral controllability matrix function is defined by:

$$Z_C(s, z^{(A)}, z^{(B)}) := \left( sI_n - A(z^{(A)}) - \tilde{A}(z^{(A)}); B(z^{(B)}) + \tilde{B}(z^{(B)}) \right)$$

with that of the nominal system being  $Z_{C0}(s, z^{(A)}, z^{(B)}) := \left( sI_n - A(z^{(A)}); B(z^{(B)}) \right)$ .

**3.2** The spectral observability matrix function is defined by:

$$Z_O(s, z^{(A)}, z^{(C)}) := \left( sI_n - A^*(z^{(A)}) - \tilde{A}^*(z^{(A)}); C^*(z^{(C)}) + \tilde{C}^*(z^{(C)}) \right)^*$$

with that of the nominal system being  $Z_{O0}(s, z^{(A)}, z^{(B)}) := \left( sI_n - A^*(z^{(A)}); C^*(z^{(C)}) \right)^*$ .

**3.3** The spectral output controllability matrix function is defined by:

$$Z_{OC}(s, z) := \left( C(z^{(C)}) + \tilde{C}(z^{(C)}) \right) \left( sI_n - A(z^{(A)}) - \tilde{A}(z^{(A)}) : B(z^{(B)}) + \tilde{B}(z^{(B)}) \right) + D(z^{(D)}) + \tilde{D}(z^{(D)})$$

with that of the nominal system being  $Z_{OC0}(s, z^{(A)}, z^{(B)}) := \left( sI_n - A(z^{(A)}) : B(z^{(B)}) \right)$ .

**3.4** The system matrix function is defined by :

$$S(s, z) := \begin{pmatrix} sI_n - A(z^{(A)}) - \tilde{A}(z^{(A)}) & : & -B(z^{(B)}) - \tilde{B}(z^{(B)}) \\ \dots\dots\dots & & \dots\dots\dots \\ C(z^{(C)}) + \tilde{C}(z^{(C)}) & : & D(z^{(D)}) + \tilde{D}(z^{(D)}) \end{pmatrix}$$

$$\text{with that of the nominal system being } S_0(s, z) := \begin{pmatrix} sI_n - A(z^{(A)}) & : & -B(z^{(B)}) \\ \dots\dots\dots & & \dots\dots\dots \\ C(z^{(C)}) & : & D(z^{(D)}) \end{pmatrix}. \quad \square$$

Note by direct inspection that these matrices depend on the nominal system and the parametrical perturbations as follows

$$Z_C(s, z^{(A)}, z^{(B)}) = Z_{C0}(s, z^{(A)}, z^{(B)}) + \left( -\tilde{A}(z^{(A)}) : \tilde{B}(z^{(B)}) \right) \quad (12)$$

$$Z_O(s, z^{(A)}, z^{(C)}) = Z_{O0}(s, z^{(A)}, z^{(C)}) + \left( -\tilde{A}^*(z^{(A)}) : \tilde{C}^*(z^{(C)}) \right)^* \quad (13)$$

$$Z_{OC}(s, z) = Z_{OC0}(s, z) + C(z^{(C)}) \left( -\tilde{A}(z^{(A)}) : \tilde{B}(z^{(B)}) \right) + \tilde{C}(z^{(C)}) \left( sI_n - A(z^{(A)}) - \tilde{A}(z^{(A)}) : B(z^{(B)}) + \tilde{B}(z^{(B)}) \right) + \tilde{D}(z^{(D)}) \quad (14)$$

$$S(s, z) := S_0(s, z) + \begin{pmatrix} sI_n - \tilde{A}(z^{(A)}) & : & -\tilde{B}(z^{(B)}) \\ \dots\dots\dots & & \dots\dots\dots \\ \tilde{C}(z^{(C)}) & : & \tilde{D}(z^{(D)}) \end{pmatrix} \quad (15)$$

Related to Definitions 3.1-3-4 are the following ones:

### Definitions

**3.5**  $s_0 \in \mathbf{C}$  is an input-decoupling zero of (1)-(2) for a given  $(z^{(A)}, z^{(B)}) \in \mathbf{C}^{q_A + q_B + 2}$  if  $\text{rank } Z_C(s_0, z^{(A)}, z^{(B)}) < n$ .

**3.6**  $s_0 \in \mathbf{C}$  is an output-decoupling zero of (1)-(2) for a given  $(z^{(A)}, z^{(C)}) \in \mathbf{C}^{q_A + q_C + 2}$  if  $\text{rank } Z_O(s_0, z^{(A)}, z^{(C)}) < n$ .

**3.7**  $s_0 \in \mathbf{C}$  is an input/output-decoupling zero of (1)-(2) for a given  $z \in \mathbf{C}^q$  if  $\max(\text{rank } Z_C(s_0, z^{(A)}, z^{(B)}), \text{rank } Z_O(s_0, z^{(A)}, z^{(C)})) < n$ .

**3.8**  $s_0 \in \mathbf{C}$  is an external input-decoupling zero of (1)-(2) for a given  $z \in \mathbf{C}^q$  if  $\text{rank } Z_{OC}(s_0, z) < p$ .

**3.9**  $s_0 \in \mathbf{C}$  is an invariant zero of (1)-(2) for a given  $z \in \mathbf{C}^q$  if  $\text{rank } S(s_0, z) < n + \min(m, p)$ .

**3.10**  $s_0 \in \mathbf{C}$  is a transmission zero of (1)-(2) for a given  $z \in \mathbf{C}^q$  if it is an invariant zero which is not an input-decoupling or output-decoupling zero; i.e.  $\text{rank } S(s_0, z) < n + \min(m, p)$  and  $\text{rank } Z_C(s_0, z^{(A)}, z^{(B)}) = \text{rank } Z_O(s_0, z^{(A)}, z^{(C)}) = n$ .  $\square$

It is well-known that the system (1)-(2) is controllable (respectively, observable) for a certain  $z \in \mathbf{C}^q$  if it has no input-decoupling zero (respectively, no output-decoupling zero). Invariant zeros which are not decoupling zeros are transmission zeros in the sense that if  $s_0 \in \mathbf{C}$  is a transmission zero for a certain  $z \in \mathbf{C}^q$  then  $y = 0$  if  $u(t) = K_u e^{s_0 t}$  for any  $K_u \in \mathbf{R}$  if  $x_0 = 0$  (input-output transmission blocking property) and

$$\hat{G}(s, z) = (C(z) + \tilde{C}(z)) \left( s I_n - A(z^{(A)}) - \tilde{A}(z^{(A)}) \right)^{-1} \left( B(z^{(B)}) + \tilde{B}(z^{(B)}) + (D(z^{(D)}) + \tilde{D}(z^{(D)})) \right)$$

is the transfer matrix of the system (1)-(2) defined as  $\hat{G}(s, z) := \hat{y}(s, z) / \hat{u}(s, z)$  for  $x_0 = 0$ . Note that decoupling zeros are poles of the system transfer matrix. The transmission zeros are zeros of  $\hat{G}(s, z)$  which are not poles of (1)-(2), i.e. which are not eigenvalues of  $A(z) + \tilde{A}(z)$ . Note also that if the system is controllable and observable all invariant zero, if any, is a transmission zero. Input/output decoupling zeros are poles cancelled by zeros in the transfer function so that they are not transmission zeros. Finally, note that input/output-decoupling zeros are invariant zeros since

$$\max(\text{rank } Z_C(s_0, z^{(A)}, z^{(B)}), \text{rank } Z_O(s_0, z^{(A)}, z^{(C)})) < n \Rightarrow \text{rank } S(s_0, z) < n + \min(m, p)$$

However, input-decoupling zeros (respectively, output-decoupling zeros) which are not output-decoupling zeros (respectively, input-decoupling zeros) are not invariant zeros since

$$\text{rank } Z_C(s_0, z^{(A)}, z^{(B)}) < n \text{ may imply } \text{rank } S(s_0, z) = n + m \text{ for } p > m$$

$$\text{rank } Z_O(s_0, z^{(A)}, z^{(C)}) < n \text{ may imply } \text{rank } S(s_0, z) = n + p \text{ for } m > p$$

The system (1)-(2) is said to be controllable if there is a control vector function on  $[0, T]$  such that the state takes any prescribed finite value at any finite prescribed time  $T$ . It is said to be observable if any bounded initial can be computed from measures of the output vector on any finite time interval  $[0, T]$ . Those properties coincide in the linear-time-invariant case with the respective properties of spectral controllability/spectral observability which hold if and only if the spectral controllability/spectral observability matrix functions are full rank  $\forall (s, z) \in \mathbf{C} \times \mathbf{C}_\alpha$  and adopt very interesting particular forms for the the case of positive dynamic systems, [37]. Thus, spectral controllability/ observability properties will

be referred to in the following simply as controllability/observability. Definitions 3.5 to 3.10 combined with Popov- Belevitch- Hautus controllability and observability tests [1] lead to the subsequent result which considers bounded sets where the varying parameters belong to defined by  $C_{\alpha}^{(A)} \subset \{I\} \times \mathbb{C}^{q_A}$ ,  $C_{\alpha}^{(B)} \subset \{I\} \times \mathbb{C}^{q_B}$ ,  $C_{\alpha}^{(C)} \subset \{I\} \times \mathbb{C}^{q_C}$  and  $C_{\alpha}^{(D)} \subset \{I\} \times \mathbb{C}^{q_D}$  :

**Theorem 3.11.** The following properties hold:

(i) The system (1)-(2) is controllable in a bounded domain  $C_{\alpha}^{(A,B)} := C_{\alpha}^{(A)} \times C_{\alpha}^{(B)} \subset \mathbb{C}^{q_A + q_B + 2}$ , if and only if it has no input-decoupling zero in  $C_{\alpha}^{(A,B)}$ , and equivalently, if and only if

$$\text{rank } Z_C(s, z^{(A)}, z^{(B)}) = n, \forall (s, z^{(A)}, z^{(B)}) \in \text{Im}(s_f) \times C_{\alpha}^{(A,B)} \subset \mathbb{C}^{q_A + q_B + 3}$$

where the discrete function  $s_f$  is defined as  $s_f : C_{\alpha}^{(A)} \rightarrow \sigma(A(z^{(A)})) + \tilde{A}(z^{(A)})$ .

(ii) The system (1)-(2) is observable in a bounded domain  $C_{\alpha}^{(A,C)} := C_{\alpha}^{(A)} \times C_{\alpha}^{(C)} \subset \mathbb{C}^{q_A + q_C + 2}$  if and only if it has no output-decoupling zero in  $C_{\alpha}^{(A,C)}$ , and equivalently, if and only if

$$\text{rank } Z_O(s, z^{(A)}, z^{(C)}) = n, \forall (s, z^{(A)}, z^{(C)}) \in \text{Im}(s_f) \times C_{\alpha}^{(A,C)} \subset \mathbb{C}^{q_A + q_C + 3}.$$

(iii) The system (1)-(2) is controllable and observable in a bounded domain  $C_{\alpha}^{(A,B,C)} := C_{\alpha}^{(A)} \times C_{\alpha}^{(B)} \times C_{\alpha}^{(C)} \subset \mathbb{C}^{q_A + q_B + q_C + 3}$  if and only if it has no input-decoupling zero in  $C_{\alpha}^{(A)} \times C_{\alpha}^{(B)}$  and no output-decoupling zero in  $C_{\alpha}^{(A)} \times C_{\alpha}^{(C)}$ , and equivalently, if and only if  $\text{rank } Z_C(s, z^{(A)}, z^{(B)}) = \text{rank } Z_O(s, z^{(A)}, z^{(C)}) = n$ ,  $\forall (s, z^{(A)}, z^{(B)}, z^{(C)}) \in \text{Im}(s_f) \times C_{\alpha}^{(A,B,C)} \subset \mathbb{C}^{q_A + q_B + q_C + 3}$ . As a result, The system (1)-(2) is controllable and observable in  $C_{\alpha}^{(A,B,C)} \Leftrightarrow \neg \exists$  a system (1)-(2) with  $\dim(x(t)) < n$  for some  $z \in C_{\alpha}$  and transfer matrix

$$\hat{G}(s, z) = (C(z) + \tilde{C}(z)) \left( s I_n - A(z^{(A)}) - \tilde{A}(z^{(A)}) \right)^{-1} (B(z^{(B)}) + \tilde{B}(z^{(B)})) + (D(z^{(D)}) + \tilde{D}(z^{(D)}))$$

(iv) The system (1)-(2) is output controllable in a bounded domain  $C_{\alpha} := C_{\alpha}^{(A)} \times C_{\alpha}^{(B)} \times C_{\alpha}^{(C)} \times C_{\alpha}^{(D)}$  if and only if it has no external input-decoupling zero in  $C_{\alpha}$ , and equivalently, if and only if  $\text{rank } Z_{OC}(s, z^{(A)}, z^{(B)}, z^{(C)}, z^{(D)}) = p$ ,  $\forall (s, z^{(A)}, z^{(B)}, z^{(C)}, z^{(D)}) \in \mathbb{C} \times C_{\alpha} \subset \mathbb{C}^{q_A + q_B + q_C + q_D + 5}$ .

If  $\text{rank} \left( C(z) + \tilde{C}(z) \right) = p$  in  $C_\alpha^{(A)} \times C_\alpha^{(B)} \times C_\alpha^{(C)}$  then the system (1)-(2) is output controllable if and only if  $\text{rank } Z_{OC} \left( s, z^{(A)}, z^{(B)}, z^{(C)} \right) = p$ ,  $\forall \left( s, z^{(A)}, z^{(B)}, z^{(C)} \right) \in \text{Im}(s_f) \times C_\alpha^{(A)} \times C_\alpha^{(B)} \times C_\alpha^{(C)} \subset \mathbf{C}^{q_A + q_B + q_C + 3}$

**Proof: (i)** The continuous vector function  $s_f : C_\alpha^{(A)} \rightarrow \sigma \left( A(z^{(A)}) + \tilde{A}(z^{(A)}) \right)$  exists on  $C_\alpha^{(A)}$  since the eigenvalues of the square matrix function  $A(z^{(A)}) + \tilde{A}(z^{(A)})$  exist and are bounded continuous functions on the bounded domain  $C_\alpha^{(A)}$ . Define the graph of  $s_f$  on its definition domain as  $G(s_f) := \left\{ (\omega, s_f(\omega)) : \omega \in C_\alpha^{(A)} \right\}$ . Taking Laplace transforms in (1) gives the equivalent algebraic linear equation

$$Z_C \left( s, z^{(A)}, z^{(B)} \right) \left( \hat{x}^* \left( s, z^{(A)}, z^{(B)} \right) : -\hat{u}^* \left( s, z^{(A)}, z^{(B)} \right) \right)^* = x_0$$

for any bounded initial conditions  $x(0, z^{(A)}, z^{(B)}) = x_0$ . Take a linear state-feedback control of Laplace transform

$$\hat{u} \left( s, z^{(A)}, z^{(B)} \right) = K \left( s, z^{(A)}, z^{(B)} \right) \hat{x} \left( s, z^{(A)}, z^{(B)} \right) \quad \text{for some}$$

$K : \mathbf{C} \times C_\alpha^{(A)} \times C_\alpha^{(B)} \subset \mathbf{C} \times \mathbf{C}^{q_A} \times \mathbf{C}^{q_B} \rightarrow \text{range}(K) \subset \mathbf{C}^{m \times n}$  is a matrix function of order  $m \times n$ .

Combining the above equations, one gets the linear algebraic system:

$$Z_C \left( s, z^{(A)}, z^{(B)} \right) \left( I_n : -K^* \left( s, z^{(A)}, z^{(B)} \right) \right)^* \hat{x} \left( s, z^{(A)}, z^{(B)} \right) = x \left( 0, z^{(A)}, z^{(B)} \right)$$

Since  $\text{rank } Z_C \left( s, z^{(A)}, z^{(B)} \right) = n$  then the generic rank of the square  $n$ -matrix function  $Z_C \left( s, z^{(A)}, z^{(B)} \right) \left( I_n : -K^* \left( s, z^{(A)}, z^{(B)} \right) \right)^*$ , considered as a function of the feedback matrix  $K \left( s, z^{(A)}, z^{(B)} \right)$ , is  $n$  everywhere in  $\mathbf{C} \times C_\alpha^{(A,B)}$  so that the above algebraic linear system may be full rank at any point of the domain of  $K \left( s, z^{(A)}, z^{(B)} \right)$  if its range is (pointwise) chosen appropriately. As a result,

$$\begin{aligned} \text{rank} \left[ Z_C \left( s, z^{(A)}, z^{(B)} \right) \left( I_n : -K^* \left( s, z^{(A)}, z^{(B)} \right) \right)^* \right] &< n \\ \Leftrightarrow \det \left( sI_n - A(z^{(A)}) - \tilde{A}(z^{(A)}) - (B(z^{(B)}) + \tilde{B}(z^{(B)})) K \left( s, z^{(A)}, z^{(B)} \right) \right) &= 0 \end{aligned}$$

at arbitrarily fixed complex points  $s(z^{(A)}, z^{(B)})$  in  $C_\alpha^{(A,B)}$  since any  $s \in \mathbf{C}$  may be chosen not to be an eigenvalue of the matrix

$$A(z^{(A)}) + \tilde{A}(z^{(A)}) + (B(z^{(B)}) + \tilde{B}(z^{(B)}))K(s, z^{(A)}, z^{(B)})$$

everywhere in  $C_{\alpha}^{(A,B)}$ . Since the eigenvalues are arbitrarily assignable by linear state- feedback the system (1)-(2) is controllable if  $\text{rank } Z_C(s, z^{(A)}, z^{(B)}) = n$  what proves sufficiency. To prove necessity, proceed by contradiction by assuming that  $\text{rank } Z_C(s, z^{(A)}, z^{(B)}) < n$ . Then, there exists a vector  $0 \neq q \in \mathbf{C}^n$  for each point in  $C_{\alpha}^{(A,B)}$  such that :

$$q^* Z_C(s, z^{(A)}, z^{(B)}) = 0 \Leftrightarrow q^* (sI_n - A(z^{(A)}) - \tilde{A}(z^{(A)})) = q^* (B(z^{(B)}) + \tilde{B}(z^{(B)})) = 0$$

i.e.  $q$  is a nonzero eigenvector of  $A(z^{(A)}) + \tilde{A}(z^{(A)})$ , with associate eigenvalue  $\lambda_q$ , which is orthogonal to the matrix  $B(z^{(B)}) + \tilde{B}(z^{(B)})$ . Direct calculation with (1)-(2) yields that a state trajectory solution satisfies  $\dot{x}(t, z^{(A)}, z^{(B)}) = \lambda_q x(t, z^{(A)}, z^{(B)})$  with associate state trajectory solution  $x(t, z^{(A)}, z^{(B)}) = e^{\lambda_q t} q$  irrespective of the control,  $\forall t \in \mathbf{R}_{+0}$ . It is obvious that such a trajectory cannot reach any point  $\bar{x} = x(T, z^{(A)}, z^{(B)}) \neq e^{\lambda_q T} q$  at any finite arbitrary time  $T > 0$ . Thus, the system is not controllable in  $C_{\alpha}^{(A,B)}$  which proves necessity. Thus, the system (1)-(2) is controllable for any  $z \in C_{\alpha}^{(A,B)}$  if and only if

$$\begin{aligned} \text{rank } Z_C(s, z^{(A)}, z^{(B)}) &= n, \forall (s, z^{(A)}, z^{(B)}) \in \mathbf{C} \times C_{\alpha}^{(A,B)} \\ \Leftrightarrow \text{rank } Z_C(s, z^{(A)}, z^{(B)}) &= n, \forall (s, z^{(A)}, z^{(B)}) \in G(s_f) \times C_{\alpha}^{(B)} \\ \Leftrightarrow \text{rank } Z_C(\omega, z^{(A)}, z^{(B)}) &= n, \forall (s, z^{(A)}, z^{(B)}) \in \text{Im}(s_f) \times C_{\alpha}^{(A,B)} \end{aligned}$$

since loss of rank in some point of  $\mathbf{C} \times C_{\alpha}^{(A,B)}$  is only possible for  $s \in \sigma(A(z^{(A)}) + \tilde{A}(z^{(A)}))$ , which is  $\text{Im}(s_f)$ , from Popov- Belevitch- Hautus rank controllability test. Equivalently, the system (1)-(2) is controllable if and only if it has no input-decoupling zero in  $C_{\alpha}^{(A,B)}$  from Definition 3.5.

(ii) The proof is similar from the Popov-Belevitch-Hautus observability rank test  $\text{rank } Z_O(s, z^{(A)}, z^{(C)}) = n, \forall (s, z^{(A)}, z^{(C)}) \in G(s_f) \times C_{\alpha}^{(C)}$ , and , equivalently ,  $\forall (s, z^{(A)}, z^{(C)}) \in \text{Im}(s_f) \times C_{\alpha}^{(A,C)}$  , since observability is a dual property to controllability through the replacements  $A \rightarrow A^*, B \rightarrow C^*$ .

(iii) The first part follows by combining the proofs of Properties (i)-(ii). The second part is now proven.

Assume that there is at least a transmission zero  $s_0 \in \mathbf{C}$  and  $S(s_0, z) \begin{pmatrix} \hat{x}^T(s), u^T(s) \end{pmatrix}^T = 0 \Rightarrow \text{rank } S(s_0, z) < n$  for some  $\begin{pmatrix} \hat{x}(s_0, z), \hat{u}^T(s_0, z) \end{pmatrix}^T \neq (0, 0)^T \in \mathbf{R}^{n+m}$  and some  $z \in C_\alpha$  since  $s_0 \in \mathbf{C}$  is neither an input-decoupling or and output decoupling zero of (1)-(2). Since  $s_0 \notin \sigma(A(z) + \tilde{A}(z))$

$$\det S(s_0, z) = \det \left( \text{diag} \left( s_0 I_n - A(z^{(A)}) - \tilde{A}(z^{(A)}) \quad : \quad I_n \right) \right. \\ \left. \times \det \begin{pmatrix} I_n & \vdots & -(s_0 I_n - A(z^{(A)}) - \tilde{A}(z^{(A)}))^{-1} B(z^{(B)}) + \tilde{B}(z^{(B)}) \\ \cdots & \cdots & \cdots \\ C(z^{(C)}) + \tilde{C}(z^{(C)}) & \vdots & D(z^{(D)}) + \tilde{D}(z^{(D)}) \end{pmatrix} \right) = 0$$

$$\Leftrightarrow C(z^{(C)}) + \tilde{C}(z^{(C)}) \text{Adj} \left( s_0 I_n - A(z^{(A)}) - \tilde{A}(z^{(A)}) \right) B(z^{(B)}) + \tilde{B}(z^{(B)}) + D(z^{(D)}) + \tilde{D}(z^{(D)}) = 0$$

$\Leftrightarrow \hat{y}(s_0, z) \equiv 0$  for some  $\hat{u}(s_0, z) \neq 0 \in \mathbf{R}^m$  and some  $z \in C_\alpha$  if  $x_0 = 0$  and the system possesses the input-output transmission blocking property at the transmission zero  $s = s_0$  for some  $z \in C_\alpha^{(A, B, C)}$ . Since  $s_0 \notin \sigma(A(z) + \tilde{A}(z))$ ,  $\hat{G}(s, z)$  has no zero-pole cancellation at  $s = s_0$  so that there is no system (1)-(2) with state dimension less than  $n$  which possess the input-output transmission blocking property for some  $z \in C_\alpha^{(A, B, C)}$ .

(iv) Its proof is similar to those of (i)-(ii) from the Popov- Belevitch- Hautus output controllability rank test, namely,  $\text{rank } Z_{OC} \begin{pmatrix} s, z^{(A)}, z^{(B)}, z^{(C)}, z^{(D)} \end{pmatrix} = p, \forall s \in \mathbf{C} \times C_\alpha$ .  $\square$

Note that Theorem 3.11 also holds if  $C_\alpha$  is a closed domain. In this case,  $G(s_f)$  is closed since there is a finite number of eigenvalues of each matrix  $A(z^{(A)}) + \tilde{A}(z^{(A)})$  in  $C_\alpha^{(A)}$ . As a result, all pairs in the set  $G(s_f)$  conform a closed set since  $C_\alpha^{(A)}$  is closed. The subsequent results is a direct consequence of Theorem 3.11. It is of interest to formulate the various properties for the system (1)-(2) under parametrical multi perturbations belonging to a certain domain in an easy testable form provided that they hold for the nominal system in an easy testable form. Note also that Theorem 3.11 is not directly applicable to time-varying parameters but to varying parameterizations within some appropriate domains.

**Corollary 3.12.** The following properties hold:

(i) The system (1)-(2) is controllable in  $C_\alpha^{(A, B)}$  if and only if

$$\det \hat{Z}_C \begin{pmatrix} s, z^{(A)}, z^{(B)} \end{pmatrix} > 0, \forall \begin{pmatrix} s, z^{(A)}, z^{(B)} \end{pmatrix} \in \mathbf{C} \times C_\alpha^{(A, B)}$$

$$\Leftrightarrow \inf \left( \underline{\sigma} \left( Z_C \left( s, z^{(A)}, z^{(B)} \right) \right) : \left( s, z^{(A)}, z^{(B)} \right) \in \mathbf{C} \times C_{\alpha}^{(A,B)} \right) > 0$$

where  $\hat{Z}_C \left( s, z^{(A)}, z^{(B)} \right) := Z_C \left( s, z^{(A)}, z^{(B)} \right) Z_C^* \left( s, z^{(A)}, z^{(B)} \right)$  and, equivalently, if and only if

$$\det \hat{Z}_C \left( s, z^{(B)} \right) > 0, \forall \left( s, z^{(B)} \right) \in G \left( s_f \right) \times C_{\alpha}^{(B)}$$

$$\Leftrightarrow \inf \left( \underline{\sigma} \left( Z_C \left( s, z^{(A)}, z^{(B)} \right) \right) : \left( s, z^{(B)} \right) \in G \left( s_f \right) \times C_{\alpha}^{(B)} \right) > 0.$$

(ii) The system (1)-(2) is observable in  $C_{\alpha}$  if and only if

$$\det \hat{Z}_O \left( s, z^{(A)}, z^{(C)} \right) > 0, \forall \left( s, z^{(A)}, z^{(C)} \right) \in \mathbf{C} \times C_{\alpha}^{(A,C)}$$

$$\Leftrightarrow \inf \left( \underline{\sigma} \left( Z_O \left( s, z^{(A)}, z^{(C)} \right) \right) : \left( s, z^{(A)}, z^{(C)} \right) \in \mathbf{C} \times C_{\alpha}^{(A,C)} \right) > 0$$

where  $\hat{Z}_O \left( s, z^{(A)}, z^{(C)} \right) := Z_O^* \left( s, z^{(A)}, z^{(C)} \right) Z_O \left( s, z^{(A)}, z^{(C)} \right)$  and, equivalently, if and only if

$$\det \hat{Z}_O \left( s, z^{(A)}, z^{(C)} \right) > 0, \forall \left( s, z^{(C)} \right) \in G \left( s_f \right) \times C_{\alpha}^{(C)}$$

$$\Leftrightarrow \inf \left( \underline{\sigma} \left( Z_O \left( s, z^{(A)}, z^{(C)} \right) \right) : \left( s, z^{(C)} \right) \in G \left( s_f \right) \times C_{\alpha}^{(C)} \right) > 0.$$

(iii) The system (1)-(2) is controllable and observable in  $C_{\alpha}^{(A,B,C)}$  if and only if

$$\det \hat{Z}_C \left( s, z^{(A)}, z^{(B)} \right), \det \hat{Z}_O \left( s, z^{(A)}, z^{(C)} \right) > 0 \Leftrightarrow \underline{\sigma} \left( Z_C \left( s, z^{(A)}, z^{(B)} \right) \right), \underline{\sigma} \left( Z_O \left( s, z^{(A)}, z^{(C)} \right) \right) > 0, \forall \left( s, z^{(B)}, z^{(C)} \right) \in G \left( s_f \right) \times C_{\alpha}^{(B,C)}.$$

(iv) The system (1)-(2) is output controllable in  $C_{\alpha}$  if and only if  $Z_{OC} \left( s, z \right)$  has no zero

singular value  $\forall z \in C_{\alpha}$  and, equivalently, if and only if  $\det \hat{Z}_{OC} \left( s, z \right) > 0, \forall \left( s, z \right) \in \mathbf{C} \times C_{\alpha}$ , where

$$\hat{Z}_{OC} \left( s, z \right) := Z_{OC} \left( s, z \right) Z_{OC}^* \left( s, z \right).$$

**Proof:** (i) Note that  $\hat{Z}_C \left( s, z^{(A)}, z^{(B)} \right)$  is a  $(n+m)$ -square matrix which is normal by construction. Thus, its eigenvalues are nonnegative and real being the squares of the singular values of  $Z_C \left( s, z^{(A)}, z^{(B)} \right)$ .

Consider again the function  $s_f : C_{\alpha}^{(A)} \rightarrow \sigma \left( A \left( z^{(A)} \right) + \tilde{A} \left( z^{(A)} \right) \right)$  of Theorem 3.11. Thus, the system (1)-(2) is controllable from Theorem 3.11(i) if and only if

$$\text{rank } Z_C \left( s, z^{(A)}, z^{(B)} \right) = n, \forall \left( s, z^{(A)}, z^{(B)} \right) \in \mathbf{C} \times C_{\alpha}^{(A,B)}$$

$$\Leftrightarrow \text{rank } Z_C \left( s, z^{(B)} \right) = n, \forall \left( s, z^{(B)} \right) \in G \left( s_f \right) \times C_{\alpha}^{(B)}$$

$$\Leftrightarrow \text{rank } \hat{Z}_C \left( s, z^{(A)}, z^{(B)} \right) = n, \forall \left( s, z^{(A)}, z^{(B)} \right) \in \mathbf{C} \times C_{\alpha}^{(A,B)}$$

$$\Leftrightarrow \det \hat{Z}_C \left( s, z^{(A)}, z^{(B)} \right) > 0, \forall \left( s, z^{(A)}, z^{(B)} \right) \in \mathbf{C} \times C_{\alpha}^{(A,B)}$$



$$\begin{aligned}
&\Leftrightarrow \text{rank } \hat{Z}_C(\omega, z^{(B)}) = n, \forall (\omega, z^{(B)}) \in G(s_f) \times C_\alpha^{(B)} \\
&\Leftrightarrow \det \hat{Z}_C(\omega, z^{(B)}) > 0, \forall (\omega, z^{(B)}) \in G(s_f) \times C_\alpha^{(B)} \\
&\Leftrightarrow \underline{\sigma}(Z_C(s, z^{(A)}, z^{(B)})) > 0, \forall (s, z^{(A)}, z^{(B)}) \in C \times C_\alpha^{(A, B)} \\
&\Leftrightarrow \underline{\sigma}(Z_C(\omega, z^{(B)})) > 0, \forall (\omega, z^{(B)}) \in G(s_f) \times C_\alpha^{(B)}
\end{aligned}$$

what proves Property (i). The proofs of (ii)-(iv) are similar from the parallel properties of Theorem 3.11 and are then omitted.  $\square$

**Remark 3.13.** Since controllability is lost in  $C_\alpha$  if and only if  $Z_C(s, z^{(B)})$  is rank defective for some  $(s, z^{(B)}) \in G(s_f) \times C_\alpha^{(B)}$  [Theorem 3.11 (i)], it turns out that controllability in  $C_\alpha$  holds if and only if  $\underline{\sigma}(Z_C(s, z^{(B)})) > 0, \forall (s, z^{(B)}) \in G(s_f) \times C_\alpha^{(B)}$  and, equivalently, if and only if  $\det \hat{Z}_C(s, z^{(B)}) > 0, \forall (s, z^{(B)}) \in G(s_f) \times C_\alpha^{(B)}$ . As a result, the tests of Corollary 3.12 (i) have only to be performed for  $s \in \sigma(A(z) + \tilde{A}(z))$  for each  $(z^{(A)}, z^{(B)}) \in C_\alpha^{(A, B)}$ . Similar considerations apply to Corollary 3.12 [(ii)-(iv)].  $\square$

Theorem 3.11 and Corollary 3.12 are directly extendable to stabilizability and detectability as follows. First, define  $s_f^{+0} : \text{Dom}(s_f^{+0}) \rightarrow \sigma(A(z^{(A)}) + \tilde{A}(z^{(A)})) \cap C_{+0}$  of graph

$$\begin{aligned}
G(s_f^{+0}) &:= \{(\omega, s_f^{+0}(\omega)) : \omega \in \text{Dom}(s_f^{+0})\} \text{ where} \\
C_\alpha^{(A)} \supset \text{Dom}(s_f^{+0}) &:= \{\omega \in C_\alpha^{(A)} : s_f^{+0}(\omega) \in C_{+0}\}
\end{aligned}$$

Note that  $G(s_f^{+0}) \subset G(s_f)$  since  $\text{Dom}(s_f^{+0}) \subset \text{Dom}(s_f) \equiv C_\alpha^{(A)}$  and there is a natural projection function  $p_f^{+0} : \text{Dom}(p_f^{+0}) \equiv \text{Im}(s_f) \rightarrow \text{Im}(s_f^{+0}) \subset \text{Im}(s_f)$ .

**Corollary 3.14.**

(i) The system (1)-(2) is stabilizable in a bounded domain  $C_\alpha$  if and only any of the following equivalent properties hold:

- (i.1) All the input-decoupling zeros in  $C_\alpha^{(A, B)}$ , if any, have negative real parts
- (i.2)  $\text{rank } Z_C(s, z^{(A)}, z^{(B)}) = n, \forall (s, z^{(A)}, z^{(B)}) \in \text{Im}(s_f^{+0}) \times C_\alpha^{(A, B)}$
- (i.3)  $\text{rank } Z_C(s, z^{(A)}, z^{(B)}) = n, \forall (s, z^{(A)}, z^{(B)}) \in C_{+0} \times C_\alpha^{(A, B)}$

$$(i.4) \det \hat{Z}_C(s, z^{(B)}) > 0, \forall (s, z^{(B)}) \in G(s_f^{+0}) \times C_\alpha^{(B)}$$

$$(i.5) \underline{\sigma}(Z_C(G(s, z^{(B)}))) > 0, \forall (s, z^{(B)}) \in G(s_f^{+0}) \times C_\alpha^{(B)}$$

(ii) The system (1)-(2) is detectable in  $C_\alpha$  if and only any of the following equivalent properties hold:

(ii.1) All the output-decoupling zeros in  $C_\alpha^{(A,C)}$ , if any, have negative real parts

$$(ii.2) \text{rank } Z_O(s, z^{(A)}, z^{(C)}) = n, \forall (s, z^{(A)}, z^{(C)}) \in C_{+0} \times C_\alpha^{(A,C)}.$$

$$(ii.3) \text{rank } Z_O(s, z^{(A)}, z^{(C)}) = n, \forall (s, z^{(A)}, z^{(C)}) \in \text{Im}(s_f^{+0}) \times C_\alpha^{(A,C)}.$$

$$(ii.4) \det \hat{Z}_O(s, z^{(C)}) > 0, \forall (s, z^{(C)}) \in G(s_f^{+0}) \times C_\alpha^{(C)}$$

$$(ii.5) \underline{\sigma}(Z_C(G(s, z^{(C)}))) > 0, \forall (s, z^{(C)}) \in G(s_f^{+0}) \times C_\alpha^{(C)} \quad \square$$

**Remarks 3.15.** The system (1)-(2) is said to be stabilizable if there is no input-decoupling zero in  $C_{0+}$ . If the system (1)-(2) is stabilizable then there exists some state-feedback control law  $u: \mathbf{R}_{0+} \times X \rightarrow U$  such that the system (1)-(2) is globally asymptotically Lyapunov's stable under such a law. The system (1)-(2) is said to be detectable if there is no output-decoupling zero in  $C_{+0}$ . From Theorem 3.12 and Corollaries 3.12 and 3.14, it turns out that if the system (1)-(2) is controllable (respectively, observable) in  $C_\alpha$  then it is stabilizable (respectively, detectable) in  $C_\alpha$ . Also, if the unforced system (1)-(2) is globally asymptotically stable in the Lyapunov's sense on  $C_\alpha$  (i.e. the matrix  $A(z^{(A)})_+ \tilde{A}(z^{(A)})$  has all its eigenvalues with negative real parts for all  $z^{(A)} \in C_\alpha^{(A)}$ ) then it is also stabilizable (even if it is not controllable) and detectable (even if it is not observable) since the spectral controllability, respectively, observability matrices are jointly full rank on  $C_{+0} \times C_\alpha^{(A,B)}$ , respectively,  $C_{+0} \times C_\alpha^{(A,B)}$ .

□

**Remarks 3.16.** Note that the conditions implying the corresponding determinants or minimum singular values to be positive to guarantee each of the controllability, stabilizability, observability and detectability properties in Theorem 3.11 and Corollaries 3.13 and 3.14 is sufficient in a bounded domain. However, the boundedness of determinants and maximum singular values is also needed to guarantee each property in an unbounded domain.

□

#### 4. Maintenance of the properties from those of the nominal system

In this section, attention is paid to the normal matrix  $\hat{Z}_C(s, z^{(A)}, z^{(B)}) = Z_C(s, z^{(A)}, z^{(B)}) Z_C^*(s, z^{(A)}, z^{(B)})$  to obtain conditions for maintaining or loosing controllability under parametrical multi perturbations of a certain size provided that the nominal system is controllable. In the analysis, it is taken advantage of the fact that such a matrix is square and nonsingular if the system is controllable. Furthermore, if the matrix  $(A(z^{(A)}) + \tilde{A}(z^{(A)}); B(z^{(B)}) + \tilde{B}(z^{(B)}))$  is Hermitian, so that  $(sI_n - (A(z^{(A)}) + \tilde{A}(z^{(A)})); B(z^{(B)}) + \tilde{B}(z^{(B)}))$  is also Hermitian, then the eventual loss of rank of  $\hat{Z}_C(s, z^{(A)}, z^{(B)})$  for some  $(z^{(A)}, z^{(B)})$  only occurs for  $s \in \mathbf{C}$  being some singular value of  $Z_C(s, z^{(A)}, z^{(B)})$ , i.e. for some eigenvalue of  $A(z^{(A)}) + \tilde{A}(z^{(A)})$ . In the general case, the loss of rank can occur also for eigenvalues of  $A^*(z^{(A)}) + \tilde{A}^*(z^{(A)})$ . A close discussion is directly applicable to stabilizability and direct extensions, by modifying accordingly the matrices, are also applicable to observability, detectability and output controllability. Using (12),

$$\begin{aligned} \hat{Z}_C(s, z^{(A)}, z^{(B)}) &= \hat{Z}_{C0}(s, z^{(A)}, z^{(B)}) \\ &\times \left( I_n + (Z_{C0}(s, z^{(A)}, z^{(B)}) Z_{C0}^*(s, z^{(A)}, z^{(B)}))^{-1} \left( (-\tilde{A}(z^{(A)}); \tilde{B}(z^{(B)})) (-\tilde{A}(z^{(A)}); \tilde{B}(z^{(B)}))^* \right. \right. \\ &\quad \left. \left. + Z_{C0}(s, z^{(A)}, z^{(B)}) (-\tilde{A}(z^{(A)}); \tilde{B}(z^{(B)}))^* + (-\tilde{A}(z^{(A)}); \tilde{B}(z^{(B)})) Z_{C0}^*(s, z^{(A)}, z^{(B)}) \right) \right) \end{aligned} \quad (16)$$

provided that  $\hat{Z}_{C0}(s, z^{(A)}, z^{(B)}) := Z_{C0}(s, z^{(A)}, z^{(B)}) Z_{C0}^*(s, z^{(A)}, z^{(B)})$  is nonsingular, i.e. its minimum singular value is positive, that is, there is no zero-input decoupling zero of  $Z_{C0}(s, z^{(A)}, z^{(B)})$ . Expanding the first identity of (7), one gets:

$$(-\tilde{A}(z^{(A)}); \tilde{B}(z^{(B)})) = (-z^{(A)} I_n; z^{(B)} I_n) \Delta^{(AB)} \quad (17)$$

with

$$\begin{aligned} \Delta^{(AB)} &:= \text{diag} \left( \sum_{j=1}^{n_A} \text{diag} \left( D_{0j}^{(A)}; D_{1j}^{(A)}; \dots; D_{q_A j}^{(A)} \right) \left( \Delta_{0j}^{(A)*}; \Delta_{1j}^{(A)*}; \dots; \Delta_{q_A j}^{(A)*} \right)^* E^{(A)} \right. \\ &\quad \left. ; \sum_{j=1}^{n_B} \text{diag} \left( D_{0j}^{(B)}; D_{1j}^{(B)}; \dots; D_{q_B j}^{(B)} \right) \left( \Delta_{0j}^{(B)*}; \Delta_{1j}^{(B)*}; \dots; \Delta_{q_B j}^{(B)*} \right)^* E^{(B)} \right) \end{aligned} \quad (18)$$

Eq. 16 may be rewritten under (17)-(18) as

$$\begin{aligned} \hat{Z}_C(s, z^{(A)}, z^{(B)}) &= \hat{Z}_{C0}(s, z^{(A)}, z^{(B)}) \left( I_n + \hat{Z}_{C0}(s, z^{(A)}, z^{(B)})^{-1} \right. \\ &\quad \left. \times \left( Z_{C0}(s, z^{(A)}, z^{(B)}) \Delta^{(AB)*} (-z^{(A)} I_n; z^{(B)} I_n)^* + \Delta^{(AB)} Z_{C0}^*(s, z^{(A)}, z^{(B)}) \right) \right) \end{aligned}$$

$$+ \left( -z^{(A)} I_n : z^{(B)} I_n \right) \Delta^{(AB)} \Delta^{(AB)*} \left( -z^{(A)} I_n : z^{(B)} I_n \right)^* \quad (19)$$

A direct result follows:

**Theorem 4.1.** The following properties hold:

(i) Assume that the nominal system (1)-(2) is controllable in a bounded domain  $C_{C0} \subset \mathcal{C}^{q_A + q_B + 2}$ . Then, there exist parametrical multi perturbations  $(-\tilde{A}(z^{(A)}); \tilde{B}(z^{(B)}))$  satisfying (17)-(18), defined by maps  $C_C \subset C_{C0} \rightarrow P_A \times P_B \subset \mathcal{C}^{n \times (n+m)}$  such that the perturbed systems (1)-(2) are also controllable, and equivalently they have no input-decoupling zeros in  $C$ , within each given bounded domain  $C_C \subset C_{C0}$ . All those multi perturbations are subject to a computable norm upper-bound depending on  $C_C$ .

(ii) Assume that there is some nonzero vector  $v_Q \in \mathcal{C}^{n^2}$  which is some linear combination of the columns of the matrix

$$Q(z^{(A)}, z^{(B)}) := \left( -\sum_{i=0}^{q_A} \sum_{j=1}^{n_A} z_i^{(A)} D_{ij}^{(A)} \otimes E^{(A)*} : \sum_{i=0}^{q_B} \sum_{j=1}^{n_B} z_i^{(B)} D_{ij}^{(B)} \otimes E^{(B)*} \right)$$

for some given  $C_C \subset C_{C0}$ . Then, there is a parametrical multi perturbation within the class (7) which violates the norm upper-bound referred to in Property (i) such that the system is uncontrollable in  $C_C$ .

(iii) Assume that the nominal system (1)-(2) is controllable in  $C_{C0} \equiv \mathcal{C}^{q_A + q_B + 2}$  and that the scaling matrices  $E^{(A)}$  and  $E^{(B)}$  are both full rank. Then, there exist parametrical multi perturbations  $(-\tilde{A}(z^{(A)}); \tilde{B}(z^{(B)}))$  satisfying (17)-(18), defined by maps  $\mathcal{C}^{q_A + q_B + 2} \rightarrow P_A \times P_B \subset \mathcal{C}^{n \times (n+m)}$  such that the perturbed systems (1)-(2) is not controllable in  $\mathcal{C}^{q_A + q_B + 2}$ .

(iv) Properties (i) and (iii) also hold “mutatis-mutandis” for stabilizability with the modification that the perturbed system is stabilizable in Property (i) if and only if it has no input-decoupling zeros in  $C_{+0}$ .

(v) Properties (i) -(iii) also hold for observability by replacing  $\mathcal{C}^{q_A + q_B + 2} \rightarrow \mathcal{C}^{q_A + q_C + 2}$ ,

$$Q(z^{(A)}, z^{(B)}) \rightarrow Q'(z^{(A)}, z^{(B)}) := \left( -\sum_{i=0}^{q_A} \sum_{j=1}^{n_A} z_i^{(A)*} D_{ij}^{(A)*} \otimes E^{(A)} : \sum_{i=0}^{q_C} \sum_{j=1}^{n_C} z_i^{(C)} D_{ij}^{(C)*} \otimes E^{(B)} \right)$$

$(-\tilde{A}(z^{(A)}); \tilde{B}(z^{(B)})) \rightarrow (-\tilde{A}(z^{(A)})^*; \tilde{C}(z^{(C)})^*)$  and the modification that the perturbed system is observable in Property (i) if and only if it has no output-decoupling zeros in  $C$ . Properties (i) and (iii) also hold “mutatis-mutandis” for detectability with the changes  $\mathcal{C}^{q_A + q_B + 2} \rightarrow \mathcal{C}^{q_A + q_C + 2}$ ,

$Q(z^{(A)}, z^{(B)}) \rightarrow Q'(z^{(A)}, z^{(B)}), (-\tilde{A}(z^{(A)}); \tilde{B}(z^{(B)})) \rightarrow (-\tilde{A}(z^{(A)})^*; \tilde{C}(z^{(C)})^*)$  and  $C \rightarrow C_{+0}$  in Property (i).

**Proof:** (i) Since the nominal system (1)-(2) is controllable in  $C_{C0}$ ,

$$0 < \varepsilon_{C0} \leq \sup \left( \left\| \hat{Z}_{C0} \left( s, z^{(A)}, z^{(B)} \right) \right\|_2 : s \in \sigma \left( A \left( z^{(A)} \right) \right), \left( z^{(A)}, z^{(B)} \right) \in C_{C0} \right) \\ = \sup \left( \lambda_{\max} \left( \hat{Z}_{C0} \left( s, z^{(A)}, z^{(B)} \right) \right) : s \in \sigma \left( A \left( z^{(A)} \right) \right), \left( z^{(A)}, z^{(B)} \right) \in C_{C0}^{\text{Fr}} \right) \leq \delta_{C0} < \infty \quad (20)$$

where  $\hat{Z}_{C0} \left( s, z^{(A)}, z^{(B)} \right) := Z_{C0} \left( s, z^{(A)}, z^{(B)} \right) Z_{C0}^* \left( s, z^{(A)}, z^{(B)} \right)$ . Inequalities (20) hold since the maximum eigenvalue is bounded positive real and equalizes the  $\ell_2$ -norm because  $\hat{Z}_{C0} \left( s, z^{(A)}, z^{(B)} \right)$  is (at least) positive semidefinite in  $C_{C0}$  and since the maximum eigenvalue, as being a continuous function on its definition domain  $C_{C0}$ , reaches its maximum on the boundary of such a domain. Equation (20) implies that

$$0 < \delta_{C0}^{-1} \leq \sup \left( \left\| \hat{Z}_{C0}^{-1} \left( s, z^{(A)}, z^{(B)} \right) \right\|_2 : s \in \sigma \left( A \left( z^{(A)} \right) \right), \left( z^{(A)}, z^{(B)} \right) \in C_{C0}^{\text{Fr}} \right) \leq \varepsilon_{C0}^{-1} < \infty \quad (21)$$

Since the nominal system (1)-(2) is controllable in  $C_{C0}$ , so that  $\hat{Z}_{C0} \left( s, z^{(A)}, z^{(B)} \right)$  is positive definite in  $C \times C_{C0}$ , then  $\hat{Z}_C \left( s, z^{(A)}, z^{(B)} \right)$  is positive definite in  $C \times C_C \subset C \times C_{C0}$  from (19)-(21) with the domain

$C_C$  defined such that  $1 > \varepsilon_{C0}^{-1} (2\delta_{C0} + \delta)$  and

$\sup \left( \left\| \left( -z^{(A)} I_n : z^{(B)} I_n \right) \Delta^{(AB)} \right\|_2 : \left( z^{(A)}, z^{(B)} \right) \in C_C \right) < \delta$  for each  $C_C \subset C_{C0}$  from Banach's

Perturbation Lemma, [40]. Thus,  $C_C$  exists defined by  $\cup \text{Dom} \left( \Delta^{(AB)} \right)$  of all the parametrical multi perturbations  $\Delta^{(AB)} \in P_{\Delta AB} \cap C^{2n \times (n+m)}$  with:

$$P_{\Delta AB} := \left\{ \Delta^{(AB)} : \text{Dom} \left( \Delta^{(AB)} \right) \rightarrow C^{2n \times (n+m)} : \sup \left( \left\| \left( -z^{(A)} I_n : z^{(B)} I_n \right) \right\| \left\| \Delta^{(AB)} \right\|_2 \right. \right. \\ \left. \left. : \left( z^{(A)}, z^{(B)} \right) \in C_C \right) < \delta := \sqrt{\delta_{C0}^2 + \varepsilon_{C0}} - \delta_{C0} \right\}$$

where  $P_{\Delta AB}$  depends on  $C_C$  and is formed by all the parametrical perturbations  $\Delta^{(AB)} \in P_{\Delta AB}$  which satisfy:

$$\bar{\sigma} \left( \Delta^{(AB)} \right) := \left\| \Delta^{(AB)} \right\|_2 < \delta / \sqrt{\sup \left( \left\| z^{(A)} \right\|_2^2 + \left\| z^{(B)} \right\|_2^2 : \left( z^{(A)} : z^{(B)} \right) \in C_c \right)}$$

Thus, the system is controllable within prefixed bounded open domain  $C_C \subset C_{C0}$  for all parametrical multi perturbations in the set:

$$P_A \times P_B := \left\{ \left( \tilde{A}(z^{(A)}) : \tilde{B}(z^{(B)}) \right) \equiv \left( z^{(A)} I_n : z^{(B)} I_n \right) \Delta^{(AB)} : \left( z^{(A)}, z^{(B)} \right) \in C_C, \Delta^{(AB)} \in P_{\Delta AB} \right\}$$

(ii) Parametrical perturbations  $\left( \tilde{A}(z^{(A)}) : \tilde{B}(z^{(B)}) \right)$  with the structure

$$\left( \sum_{j=1}^{n_A} D_{0j}^{(A)} \Delta_{0j}^{(A)} E^{(A)} : \sum_{j=1}^{n_B} D_{0j}^{(B)} \Delta_{0j}^{(B)} E^{(B)} \right) \text{ lie in } P_A \times P_B \text{ from (7). Note that the linear}$$

algebraic equation  $Q(z^{(A)}, z^{(B)})x_Q = v_Q$  has at least a solution for  $v_Q$  being of the given form and

$$x_Q := \left( \text{vec}^T \left( D_{ij}^{(A)} : (i, j) \in \bar{q}_A^0 \times \bar{n}_A \right), \text{vec}^T \left( D_{ij}^{(B)} : (i, j) \in \bar{q}_B^0 \times \bar{n}_B \right) \right)^T$$

Direct calculations in (19) yield that  $\det \hat{Z}_C(s, z^{(A)}, z^{(B)}) = 0$ , since

$$\text{rank } Z_C(s, z^{(A)}, z^{(B)}) = \text{rank} \left( Z_{C0}(s, z^{(A)}, z^{(B)}) + \left( -\tilde{A}(z^{(A)}) : \tilde{B}(z^{(B)}) \right) \right) < n \quad \text{for some}$$

$(s, (z^{(A)}, z^{(A)})) \in C \times C_C$  since there exists a nonzero  $v_Q$  such that

$$\left( Z_{C0}(s, z^{(A)}, z^{(B)}) \otimes I_n + Q(z^{(A)}, z^{(B)}) \otimes x_Q^* \right) v_Q = 0$$

and the proof of Property (ii) is complete.

(iii) Since the nominal system is controllable,  $0 < \varepsilon_1 \leq \det \hat{Z}_{C0}(s, z^{(A)}, z^{(B)}) \leq \varepsilon_2 < \infty$  so that

$$0 < \varepsilon_2^{-1} \leq \det \hat{Z}_{C0}(s, z^{(A)}, z^{(B)}) \leq \varepsilon_1^{-1} < \infty, \quad \forall (s, z^{(A)}, z^{(B)}) \in C \times C^{q_A + q_B + 2}. \text{ Then, from (19),}$$

it follows that:

1. There are infinitely many parametrical multi perturbations  $\left( -\tilde{A}(z^{(A)}) : \tilde{B}(z^{(B)}) \right)$ , defined by a map

$$d : C^{q_A + q_B + 2} \rightarrow P_A \times P_B \subset C^{n \times (n+m)} \quad \text{satisfying (17)-(18) and fulfilling that}$$

$$\text{rank} \left( -z^{(A)} I_n : z^{(B)} I_n \right) \Delta^{(AB)} = n. \text{ Note that such parametrical perturbations always exist from (17)-}$$

(18) since  $E^{(A)}$  and  $E^{(B)}$  are full rank.

2. There exist continuous functions  $g_1 : C^{q_A + q_B + 2} \times P_A \times P_B \rightarrow C$  and

$$g_2 : C^{q_A + q_B + 2} \times P_A \times P_B \rightarrow C_{+0} \text{ from (19) satisfying:}$$

$$g_i(z^{(A)}, z^{(B)}, \Delta^{(AB)}) = O \left( \left\| \Delta^{(AB)} \right\|^i \right) \text{ for } i=1, 2; \quad \forall (z^{(A)}, z^{(B)}) \in C^{q_A + q_B + 2}$$

$$g_2(z^{(A)}, z^{(B)}) \Delta^{(AB)} > 0 \text{ if } (z^{(A)}, z^{(B)}) \neq 0.$$

Then,

$$\det \hat{Z}_C(s, z^{(A)}, z^{(B)}) \geq \varepsilon^{-1} \left( I + \text{trace}(\varepsilon_3 \varepsilon^{-1}) + \left| o(\varepsilon_3 \varepsilon^{-1}) \right| \right) \\ \times \left\| (z^{(A)}, z^{(B)}) \right\| \left( \left\| (z^{(A)}, z^{(B)}) \right\| g_2(z^{(A)}, z^{(B)}) - \left| g_1(z^{(A)}, z^{(B)}) \right| \right) \rightarrow \infty$$

for some  $\varepsilon_3 \in \mathbf{R}$  as  $\mathbf{C}^{q_A + q_B + 2} \ni (z^{(A)}, z^{(B)}) \rightarrow \infty$  (the infinity point in  $\mathbf{C}^{q_A + q_B + 2}$ ) and then the perturbed system (1)-(2) is not controllable (see Remark 3.16) what is a contradiction. The proof of Property (iii) is complete.

(iv)-(v) Their proofs are similar to those of properties (i)-(iii) with the given modifications and the replacements  $\hat{Z}_{CO}(s, z^{(A)}, z^{(C)}) \rightarrow \hat{Z}_{OO}(s, z^{(A)}, z^{(C)})$ ,  $\hat{Z}_C(s, z^{(A)}, z^{(C)}) \rightarrow \hat{Z}_O(s, z^{(A)}, z^{(C)})$ .  $\square$

**Remarks 4.2.** The extension of Theorem 4.1 to output controllability is immediate by invoking Definition 3.3 and Theorem 3.11 (iv). Also, note that if the state-space realization of the nominal system (1)-(2) is minimal in a certain domain, then Theorem 4.1 provides testable conditions to guarantee that the realization is maintained minimal for a set of parametrical perturbations (17)-(18) in a certain domain included in the above one if controllability and observability hold jointly in such a domain.  $\square$

**Remarks 4.3.** The various properties might also be investigated by the nominal system defined at some  $z_0 = (z_0^{(A)}, z_0^{(B)}, z_0^{(C)}, z_0^{(D)}) \in \mathbf{C}^q$  where the corresponding property holds. For this purpose, the replacements below are used to apply Theorem 4.1:

$$\tilde{A}(z^{(A)}) \rightarrow \bar{\tilde{A}}(z^{(A)}):= \tilde{A}(z^{(A)}) + A(z^{(A)}) - A(z_0^{(A)}), \quad \tilde{B}(z^{(B)}) \rightarrow \bar{\tilde{B}}(z^{(B)}):= \tilde{B}(z^{(B)}) + B(z^{(B)}) - B(z_0^{(B)}) \\ \tilde{C}(z^{(C)}) \rightarrow \bar{\tilde{C}}(z^{(C)}):= \tilde{C}(z^{(C)}) + C(z^{(C)}) - C(z_0^{(C)}), \quad \tilde{D}(z^{(D)}) \rightarrow \bar{\tilde{D}}(z^{(D)}):= \tilde{D}(z^{(D)}) + D(z^{(D)}) - D(z_0^{(D)})$$

by redefining the parametrical perturbations in certain domains of  $\mathbf{C}^q$  containing  $z$  and  $z_0$  and  $z = (z^{(A)}, z^{(B)}, z^{(C)}, z^{(D)})$  as  $\bar{\tilde{A}}(z^{(A)})$ ,  $\bar{\tilde{B}}(z^{(B)})$ ,  $\bar{\tilde{C}}(z^{(C)})$  and  $\bar{\tilde{D}}(z^{(D)})$ . Then, domains where the studied property is kept from the nominal system may be obtained in this way.  $\square$

## 5. Application to time-delay systems with point internal and external delays

Now, consider the following extension of the dynamic system (1)-(2) including discrete (point) delays:

$$\dot{x}(z, t) = \left( A(z^{(A)}) + \tilde{A}(z^{(A)}) \right) x(z, t) + \left( B(z^{(B)}) + \tilde{B}(z^{(B)}) \right) u(t)$$

$$+ \sum_{j=1}^{\eta} \left( A_j \left( z^{(Ad)} \right) + \tilde{A}_j \left( z^{(Ad)} \right) \right) x \left( z, t - h_j \right) + \sum_{j=1}^{\kappa} \left( B_j \left( z^{(Bd)} \right) + \tilde{B}_j \left( z^{(Bd)} \right) \right) u \left( z, t - h_j \right) \quad (22)$$

$$y \left( z, t \right) = \left( C \left( z^{(C)} \right) + \tilde{C} \left( z^{(C)} \right) \right) x \left( z, t \right) + \left( D \left( z^{(D)} \right) + \tilde{D} \left( z^{(C)} \right) \right) u \left( t \right) \quad (23)$$

fully described by (1)-(2), subject to parametrical multi perturbations (17)-(18),  $\eta$  internal (i.e. in the state) pair-wise distinct point delays  $h_j \in [0, \bar{h}_j) \subset \mathbf{R}_{+0}$ ,  $\forall j \in \bar{\eta}$  and  $\kappa$  external (i.e. in the input) pair-wise distinct point delays  $h_j \in [0, \bar{h}_j) \subset \mathbf{R}_{+0}$ ,  $\forall j \in \bar{\kappa}$ . The intervals  $[0, \bar{h}_{(.)})$  and  $[0, \bar{h}'_{(.)})$  are the admissibility domains of the corresponding internal and external delays, respectively. The external delays could be considered to act on the output instead of on the input with no loss in generality. The initial conditions are defined by any bounded piecewise absolutely continuous vector function with eventual isolated discontinuities  $\phi: [-\bar{h}, 0] \rightarrow \mathbf{C}^n$  where  $\bar{h} := \max_{1 \leq j \leq \eta} (h_j)$ . If  $u \in PC^{(0)}(\mathbf{C}^{q \times \mathbf{R}_{+0}}, U)$  then a unique solution exists on  $\mathbf{R}_+$  for each given  $\phi: [-\bar{h}, 0] \rightarrow \mathbf{C}^n$ , [8], [10], [12]. To set an appropriate framework related to that of the preceding sections, consider  $\hat{q}$ -tuples  $z$  and  $\hat{z}_\phi$  in  $\mathbf{C}^{\hat{q}}$  with  $\hat{q} := q + q_{Ad} + q_{Bd} = q_A + q_B + q_C + q_D + 4 + q_{Ad} + q_{Bd}$ :

$$z := \left( z^{(A)}, z^{(B)}, z^{(C)}, z^{(D)}, z^{(Ad)}, z^{(Bd)} \right); \quad \hat{z}_{\rho\phi} := \left( z^{(A)}, z^{(B)}, z^{(C)}, z^{(D)}, \hat{z}_{\rho\phi}^{(Ad)}, \hat{z}_{\rho\phi}^{(Bd)} \right) \quad (24)$$

where

$$\hat{z}_{\rho\phi}^{(Ad)} := \left( z_1^{(Ad)} \rho_{1 \prec \phi_1}^{(Ad)}, z_1^{(Ad)} \rho_{2 \prec \phi_2}^{(Ad)}, \dots, z_1^{(Ad)} \rho_{\eta \prec \phi_\eta}^{(Ad)}, \dots, z_{q_{Ad}}^{(Ad)} \rho_{1 \prec \phi_1}^{(Ad)}, z_{q_{Ad}}^{(Ad)} \rho_{2 \prec \phi_2}^{(Ad)}, \dots, z_{q_{Ad}}^{(Ad)} \rho_{\eta \prec \phi_\eta}^{(Ad)} \right) \quad (25)$$

$$\hat{z}_{\rho'\phi'}^{(Bd)} := \left( z_1^{(Bd)} \rho_{1 \prec \phi'_1}^{(Bd)}, z_1^{(Bd)} \rho_{2 \prec \phi'_2}^{(Bd)}, \dots, z_1^{(Bd)} \rho_{\kappa \prec \phi'_\kappa}^{(Bd)}, \dots, z_{q_{Bd}}^{(Bd)} \rho_{1 \prec \phi'_1}^{(Bd)}, z_{q_{Bd}}^{(Bd)} \rho_{2 \prec \phi'_2}^{(Bd)}, \dots, z_{q_{Bd}}^{(Bd)} \rho_{\kappa \prec \phi'_\kappa}^{(Bd)} \right) \quad (26)$$

and  $\hat{q}$ -tuples being complex vector functions from  $\mathbf{C} \times \mathbf{C}^{\hat{q}}$  to  $\mathbf{C}^{\hat{q}}$

$$\bar{z}_{hh'}(s) := \left( z^{(A)}, z^{(B)}, z^{(C)}, z^{(D)}, \bar{z}_h^{(Ad)}(s), \bar{z}_{h'}^{(Bd)}(s) \right) \quad (27)$$

where  $h := (h_1, h_2, \dots, h_\eta)$  and  $h' := (h'_1, h'_2, \dots, h'_\kappa)$  are tuples formed with the sets of internal and external delays, respectively, and

$$\bar{z}_h^{(Ad)}(s) := \left( z_1^{(Ad)} e^{-h_1 s}, z_1^{(Ad)} e^{-h_2 s}, \dots, z_1^{(Ad)} e^{-h_\eta s}, \dots, z_{q_{Ad}}^{(Ad)} e^{-h_1 s}, z_{q_{Ad}}^{(Ad)} e^{-h_2 s}, \dots, z_{q_{Ad}}^{(Ad)} e^{-h_\eta s} \right) \quad (28)$$



$$\bar{z}_{\bar{h}}^{(Bd)}(s) := \left( z_1^{(Ad)} e^{-h_1' s}, z_1^{(Bd)} e^{-h_2' s}, \dots, z_1^{(Bd)} e^{-h_{\kappa'}' s}, \dots, z_{q_{Bd}}^{(Bd)} e^{-h_1' s}, z_{q_{Bd}}^{(Bd)} e^{-h_2' s}, \dots, z_{q_{Bd}}^{(Bd)} e^{-h_{\kappa'}' s} \right) \quad (29)$$

$\forall \varphi_j, \varphi_{\ell}' \in [0, 2\pi), \forall \rho_j, \rho_{\ell}' \in \mathbf{R}_+, (j \in \bar{\eta}, \ell \in \bar{\kappa})$  where  $\gamma_{\prec \theta} := \gamma (\cos \theta + i \sin \theta)$  is a circumference of radius  $\gamma$  centered at zero in the complex plane. The dynamic system (22)-(23) may be equivalently described through an algebraic linear system by taking Laplace transforms with zero initial conditions as follows:

$$\left( sI_n - A(z^{(A)}) + \tilde{A}(z^{(A)}) - \sum_{j=1}^{\eta} \left( A_j(z^{(Ad)}) + \tilde{A}_j(z^{(Ad)}) \right) e^{-h_j' s} \right) \hat{x}(z, s) - \left( B(z^{(B)}) + \tilde{B}(z^{(B)}) - \sum_{j=1}^{\kappa} \left( A_j(z^{(Ad)}) + \tilde{A}_j(z^{(Ad)}) \right) e^{-h_j' s} \right) \hat{u}(s) = 0 \quad (30)$$

$$\hat{y}(z, s) = \left( C(z^{(C)}) + \tilde{C}(z^{(C)}) \right) \hat{x}(z, s) + \left( D(z^{(D)}) + \tilde{D}(z^{(C)}) \right) \hat{u}(s) \quad (31)$$

Note by direct inspection that there exist surjective mapping of the set of tuples (28) to the set of tuples (26) and from the set of tuples (29) to the set of tuples (27) by considering them as functions from  $\mathbf{C}$  to  $[0, \bar{h}_1) \times \dots \times [0, \bar{h}_{\eta}) \times \mathbf{R}^{2\eta} \times \mathbf{R}_{+0} \times [0, 2\pi)$  and to  $[0, \bar{h}_1) \times \dots \times [0, \bar{h}_{\eta}) \times \mathbf{R}^{2\kappa} \times \mathbf{R}_{+0} \times [0, 2\pi)$ , respectively, by associating  $s = \sigma + i\omega \rightarrow \rho_k = e^{-h_k \sigma}, \varphi_k = h_k \omega; \forall k \in \bar{\eta}$  and  $s = \sigma + i\omega \rightarrow \rho_k' = e^{-h_k' \sigma}, \varphi_k' = h_k' \omega; \forall k \in \bar{\kappa}$ , respectively. However, those mappings are not one-to-one, in general, for all the admissible sets of delays, since the inverse maps:

$$[0, \bar{h}_1) \times \dots \times [0, \bar{h}_{\eta}) \times \mathbf{R}^{2\eta} \times \mathbf{R}_{+0} \times [0, 2\pi) \subset \mathbf{R}^{\eta+2} \rightarrow \mathbf{C} \text{ and } [0, \bar{h}_1) \times \dots \times [0, \bar{h}_{\kappa}') \times \mathbf{R}^{2\kappa} \times \mathbf{R}_{+0} \times [0, 2\pi) \subset \mathbf{R}^{\kappa+2} \rightarrow \mathbf{C}$$

do not have the same definition domain as the respective ranges of the original mappings. The first inverse mapping only exists if and only if  $\frac{\ln \rho_k}{h_k}$  are identical and real,  $\forall k \in \bar{\eta}$  and also if  $\frac{\varphi_k}{h_k}$  are identical, real

and belong to  $[0, 2\pi), \forall k \in \bar{\eta}$ . The second inverse mapping only exists if and only if  $\frac{\ln \rho_k'}{h_k'} = K \in \mathbf{R}$ ,

$\frac{\varphi_k'}{h_k'} = K' \in [0, 2\pi), \forall k \in \bar{\kappa}$ . The simultaneous existence of both inverse mappings require the fulfillment

of joint constraints:

$$\frac{\ln \rho_j}{h_j} = \frac{\ln \rho_k'}{h_k'} = K \in \mathbf{R}, \frac{\varphi_j}{h_j} = \frac{\varphi_k'}{h_k'} = K' \in [0, 2\pi); \forall j \in \bar{\eta}, \forall k \in \bar{\kappa} \quad (32)$$

**Remark 5.1.** The properties of controllability, observability, stabilizability and detectability of the system (30)-(31) in a domain may be directly tested by extending directly Theorem 3.11, Corollaries 3.12 and 3.14 and Theorem 4.1 under the subsequent guidelines. It turns out that if the constraints (32) are not invoked, only sufficient conditions may be obtained by extending the results of the preceding sections for each tested property by considering the tuples (24)-(26) in the tests. If, in addition, the constraints (32), associated with (27)-(29), are required for given sets of internal and external delays then necessary and sufficient conditions may be obtained by extending such tests from the delay-free case. If the tests fail only for certain sets of  $\hat{q}$ -tuples (24)-(26) in some given domain, which do not fulfill (32) for given sets of delays, then the system fulfills the tested property for that set of delays. The property is also lost for the sets of delays which do not fulfill the test for the tuples (24)-(26) which have a solution under the constraints (32). If the test does not fail for any tuple (24)-(26) in some domain then the system fulfills the tested property independent of the delays in such a domain. In summary, replace  $q \rightarrow \hat{q}$ ,  $z^{(A)} \rightarrow (z^{(A)}, \hat{z}_{\rho \varphi}^{(Ad)})$ ,  $z^{(B)} \rightarrow (z^{(B)}, \hat{z}_{\rho \varphi}^{(Bd)})$  and extend Definitions 3.1-3.4 to the system (30)-(31) to then generalize the various results in Theorem 3.11, Corollaries 3.12 and 3.14 and Theorem 4.1 to the system (30)-(31) subject to delays. Then,

1. If any investigated property (namely, controllability, observability, stabilizability or detectability) holds for all  $z$  (defined in (24)) in a domain then the property holds within such a domain independent of the delays, i.e. for all sets of  $\eta$  internal delays and  $\kappa$  external delays ranging from zero to infinity.
2. Assume that two sets of internal and external delays are given and assume also that the investigated property holds for all  $z$  except at isolated points in a certain domain. Then, if some of the constraints (32) fails for the given sets of delays for all those all points in the domain then the system possesses the investigated property for the given sets of delays. If no sets of delays are specified, then the investigated property except holds for all delays in the admissibility domain except for those where some of the joint constraints (32) fails.
3. If the investigated property fails at some  $z$  only for sets of delays which fulfill (32) then the system maintains such a property for all delays in their admissibility domains except for those which fulfill (32).

The following technical result is useful for testing the controllability of the time-delay nominal system. The perturbed system is guaranteed to maintain controllability if the nominal one is controllable by incorporating extended sufficiency-type conditions from “ad-hoc” extended versions of Theorem 3.11, Corollaries 3.12 and 3.14 and Theorem 4.1 (see Remark 5.1). For the remaining properties of observability, stabilizability and detectability, the result is extendable “mutatis-mutandis” with the corresponding changes.

**Theorem 5.2** Assume that  $C_{\alpha 0}^{(A,B,d)} := C_{\alpha 0}^{(A)} \times C_{\alpha 0}^{(B)} \times C_{\alpha 0}^{(Ad)} \times C_{\alpha 0}^{(Bd)} \subset C^{\hat{q}_C}$  is bounded where  $\hat{q}_C := q_A + q_B + q_{Ad} + q_{Bd} + 2$ . Then, the following properties hold:

- (i) The nominal system with delays (22)-(23) is controllable independent of the delays on  $C_{\alpha 0}^{(A,B,d)}$  if

$$\det(\hat{Z}_{C0}(\hat{\sigma}, z)) \geq \underline{\delta} > 0, \forall (\hat{\sigma}, z) \in \mathbf{R}^{\eta+\kappa+1} \times [0, 2\pi)^{\eta+\kappa+1} \times C_{\alpha 0}^{(A,B)}$$

where

$$\hat{\sigma} := (\sigma, \rho_1, \rho_2, \dots, \rho_\eta, \rho'_1, \rho'_2, \dots, \rho'_\kappa, \omega, \varphi_1, \varphi_2, \dots, \varphi_\eta, \varphi'_1, \varphi'_2, \dots, \varphi'_\kappa) \in \mathbf{R}^{\eta+\kappa+1} \times [0, 2\pi)^{\eta+\kappa+1}$$

$$\text{which satisfies all the constraints } \sigma = -\frac{\ln \rho_i}{h_i} = -\frac{\ln \rho'_j}{h'_j}, \quad \omega = \frac{\arccos \varphi_i}{h_i} = \frac{\arccos \varphi'_j}{h'_j},$$

$$\forall (i, j) \in \overline{\eta} \times \overline{\kappa} \text{ for any two sets of distinct nonnegative real numbers } \{h_1, h_2, \dots, h_\eta\}, \\ \{h'_1, h'_2, \dots, h'_\kappa\}.$$

(ii) The nominal system with delays (22)-(23) is controllable on  $C_{\alpha 0}^{(A,B,d)}$  for a set of  $\eta$  positive distinct internal delays  $\{h_1, h_2, \dots, h_\eta\}$  and a set of  $\kappa$  positive distinct external delays  $\{h'_1, h'_2, \dots, h'_\kappa\}$  provided that

$$\det(\hat{Z}_{C0}(\hat{\sigma}, z)) \geq \underline{\delta} > 0, \forall (\hat{\sigma}, z) \in \mathbf{R}^{\eta+\kappa+1} \times [0, 2\pi)^{\eta+\kappa+1} \times C_{\alpha 0}^{(A,B,d)}$$

where

$$\hat{\sigma} := (\sigma, \rho_1, \rho_2, \dots, \rho_\eta, \rho'_1, \rho'_2, \dots, \rho'_\kappa, \omega, \varphi_1, \varphi_2, \dots, \varphi_\eta, \varphi'_1, \varphi'_2, \dots, \varphi'_\kappa) \in \mathbf{R}^{\eta+\kappa+1} \times [0, 2\pi)^{\eta+\kappa+1}$$

$$\text{satisfies } \sigma = -\frac{\ln \rho_i}{h_i} = -\frac{\ln \rho'_j}{h'_j}, \quad \omega = \frac{\arccos \varphi_i}{h_i} = \frac{\arccos \varphi'_j}{h'_j}, \quad \forall (i, j) \in \overline{\eta} \times \overline{\kappa}.$$

(iii) Finite covers of  $C_{\alpha 0}^{(A,B,d)}$  can be constructed such that sufficient-type conditions of controllability of

Properties (i)-(ii) of the nominal system (22)-(23) on  $C_{\alpha 0}^{(A,B)}$  may be constructed involving a finite numbers of computations.

**Proof: (i)-(ii).** The proofs of (i)-(ii) follow directly from the structure of the delay system (22)-(23) and Remark 5.1

(iii) It is organized in a very technical way. Note that the matrix function  $\hat{Z}_{C0}(s, z^{(A)}, z^{(B)})$  is Hermitian (and thus normal) by construction even if the various matrix functions defining the system (1)-(2) are not Hermitian. Assume that  $C_R$  is a bounded open or closed circle of finite radius  $R$  centred at the origin of  $\mathbf{C}$  and that  $R_R^2$  is a circle of the same radius centred at the origin of  $\mathbf{R}^2$  which is open if  $C_R$  and closed if  $C_R$  is closed. Then, there is a natural mapping from  $C_R \times C_{\alpha 0}^{(A,B)}$  to  $R_R^2 \times C_{\alpha 0}^{(A,B,d)}$  associating each matrix  $\hat{Z}_{C0}(s, z^{(A)}, z^{(B)})$  in  $\mathbf{C} \times C_{\alpha 0}^{(A,B,d)}$  to a matrix  $\hat{Z}_{C0}(\sigma, \omega, z^{(A)}, z^{(B)})$  in  $R_R^2 \times C_{\alpha 0}^{(A,B,d)}$  where  $\sigma = \operatorname{Re} s$  and  $\omega = \operatorname{Im}(s)$ . Both matrices have the same spectrum which consists of a set of

$v \leq (n+m)$  real vector functions  $\lambda_i : R_R^2 \times C_{\alpha 0}^{(A,B,d)} \rightarrow R_{0+} (i \in \bar{v})$  of multiplicities  $v_i (i \in \bar{v})$

which satisfy  $\sum_{i=1}^v v_i = n+m$ . The numbers  $\lambda_i (i \in \bar{v})$ ,  $v$ ,  $v_i (i \in \bar{v})$  depend on each tuple in

$C_R \times C_{\alpha 0}^{(A,B)}$ , equivalently, on each tuple in  $R_R \times C_{\alpha 0}^{(A,B,d)}$ . If each element of the spectrum  $\lambda_i (i \in \bar{v})$

of multiplicity  $v_i (i \in \bar{v})$  is considered as  $v_i$  identical elements, then there is a bijective mapping from a

continuous vector function  $\lambda_C : C_R \times C_{\alpha 0}^{(A,B,d)} \rightarrow R_{+0}^{(n+m)}$  to another one  $\lambda_{R^2} : R_R^2 \times C_{\alpha 0}^{(A,B,d)} \rightarrow R_{+0}^{(n+m)}$ .

Since the images are identical, it is not made distinction between  $\lambda_C$  and  $\lambda_{R^2}$ , by using simply the

notation  $\lambda$  for both vector functions in their definition domains. It is obvious that controllability of the

nominal system (1)-(2) over  $C_{\alpha 0}^{(A,B,d)}$  holds if and only if  $\lambda : R_R^2 \times C_{\alpha 0}^{(A,B,d)} \rightarrow R_{+}^{(n+m)}$ . Now, if

$C_{\alpha 0}^{(A,B,d)}$  is simply connected, closed and bounded, then can be covered by a finite cover from Heine-Borel

covering theorem. First, assume that  $C_{\alpha 0}^{(A,B,d)}$  is connected, closed and bounded and  $R_R^2$  is closed then a

finite cover  $Co_{\alpha 0}^{(A,B,d)}$  exists for  $R_R^2 \times C_{\alpha 0}^{(A,B,d)}$ , since  $R_R^2 \times C_{\alpha 0}^{(A,B,d)}$  is bounded and closed.

$Co_{\alpha 0}^{(A,B,d)}$  is the union of finite covers for each of its  $L$  connected components  $R_R^2 \times C_{\alpha 0 \ell}^{(A,B,d)} (\ell \in \bar{L})$

with  $L < \infty$  since  $C_{\alpha 0}^{(A,B,d)}$  is bounded, Then,  $Co_{\alpha 0}^{(A,B,d)} = \bigcup_{\ell=1}^L \left( R_R^2 \times C_{\alpha 0 \ell}^{(A,B,d)} \right)$ . Subsequently,

assume that  $C_{\alpha 0}^{(A,B,d)}$  is not closed (for instance, open or semi-open) with identical remaining hypotheses as

above. Then, there is an open bounded set  $Co_{\alpha 0}^{(A,B,d)} \supset R_R^2 \times C_{\alpha 0}^{(A,B,d)} = \bigcup_{\ell=1}^L \left( R_R^2 \times C_{\alpha 0 \ell}^{(A,B,d)} \right)$ .

Thus, a denumerable cover exists for  $Co_{\alpha 0}^{(A,B,d)}$  from Lindelöf covering theorem but it still exists a

finite cover  $Co_{\alpha 0}^{(A,B,d)}$  of  $R_R^2 \times C_{\alpha 0}^{(A,B)}$  satisfying the set inclusion chain:

$$\begin{aligned} Co_{\alpha 0}^{(A,B,d)} &= \bigcup_{\ell=1}^L \left( R_R^2 \times C_{\alpha 0 \ell}^{(A,B,d)} \right) \supset c\ell Co_{\alpha 0}^{(A,B,d)} = \bigcup_{\ell=1}^L c\ell \left( R_R^2 \times C_{\alpha 0 \ell}^{(A,B,d)} \right) \\ &= \bigcup_{\ell=1}^L c\ell \left( R_R^2 \times C_{\alpha 0 \ell}^{(A,B,d)} \right) \supset Co_{\alpha 0}^{(A,B,d)} \supset \bigcup_{\ell=1}^L \left( R_R^2 \times C_{\alpha 0 \ell}^{(A,B,d)} \right) = R_R^2 \times C_{\alpha 0}^{(A,B,d)} \end{aligned}$$

from Heine-Borel covering theorem for  $c\ell Co_{\alpha 0}^{(A,B,d)}$  (which is bounded and closed) where

$C_{\alpha 0 \ell}^{(A,B,d)} \supset C_{\alpha 0 \ell}^{(A,B,d)} (\ell \in \bar{L})$  is a finite collection of open bounded sets. As a result, a finite cover

$Co_{\alpha 0}^{(A,B,d)}$  of  $R_R^2 \times C_{\alpha 0}^{(A,B)}$  always exists if  $C_{\alpha 0}^{(A,B,d)}$  is bounded and connected. The finite cover :

$$\hat{H}\left(C_{\alpha 0}^{(A,B)}\right):=\bigcup_{i_j \in \bar{L}_j, j \in \overline{q_A+q_B}} \hat{H}_{i_1, \dots, i_{q+2}}$$

for some prefixed small real  $\varepsilon_0 \in \mathbf{R}_+$  is a finite union of disjoint hyper-rectangles defined by

$$\hat{H}_{i_1, \dots, i_{q+2}}:=\left\{(\sigma, \omega, z) \in \mathbf{R}_R^2 \times C_{\alpha 0}^{(A,B,d)}: (\sigma, \omega) \in \bigcup_{(j,k) \in \bar{L}_\sigma \times \bar{L}_\omega} H_{jk}(z)\right. \\ \left. z=\left(z_1, z_2, \dots, z_{q_A+q_B}\right), -R_{AB}+\sum_{\ell=1}^{i_j-1} \varepsilon_\ell < z_j < -R_{AB}+\sum_{\ell=1}^{i_j} \varepsilon_\ell+\varepsilon_0, \forall j \in \overline{q_A+q_B}\right\}$$

$\forall i_j \in \bar{L}_j, \forall j \in \overline{q_A+q_B}$  where  $R_{AB}$  is bounded positive large real number, where

$$H_{jk}(z):=\left\{(\sigma, \omega, z) \in \mathbf{R}_R^2 \times C_{\alpha 0}^{(A,B,d)}\right. \\ \left.: -R_R+\sum_{\ell=1}^{j-1} \varepsilon_{\sigma \ell} < \sigma < -R_R+\sum_{\ell=1}^j \varepsilon_{\sigma \ell}+\varepsilon_0, -R_R+\sum_{\ell=1}^{k-1} \varepsilon_{\omega \ell} < \omega < -R_R+\sum_{\ell=1}^k \varepsilon_{\omega \ell}+\varepsilon_0\right\} \\ \forall z \in C_{\alpha 0}^{(A,B,d)}, \forall j \in \bar{L}_\sigma, \forall k \in \bar{L}_\omega. \text{ Then, } \bigcup_{(j,k) \in \bar{L}_\sigma \times \bar{L}_\omega} H_{jk}(z) \text{ is a finite cover composed of open}$$

rectangles over a large open square in  $\mathbf{R}^2$  for each  $z$  in  $C_{\alpha 0}^{(A,B,d)}$ . It is quite obvious that, since any

characteristic zeros of the nominal system (1)-(2) are at a finite distance from the origin if the characteristic equation has a principal term and the characteristic quasi-polynomial is monic in the Laplace argument  $s$  then

the stability may be tested for any  $z$  over the union of  $H_{jk}(z)$ , for  $j \in \bar{L}_\sigma < \infty$  and over  $\hat{H}\left(C_{\alpha 0}^{(A,B,d)}\right)$  for

any parameterization in  $C_{\alpha 0}^{(A,B,d)}$ . If the nominal system is controllable then the open rectangles  $H_{jk}(z)$

for any  $z=\left(z^{(A)}, z^{(B)}\right) \in C_{\alpha 0}^{(A,B,d)}$  are constructed as follows. Define:

$$f_0\left(\sigma, \omega, z^{(A)}, z^{(B)}\right):=\det \left(\hat{Z}_{CO}\left(\sigma, \omega, z^{(A)}, z^{(B)}\right)\right)$$

Since the above function is analytic everywhere in its definition domain then for given strictly increasing real

sequences  $\left\{\sigma_i\right\}_0^\infty, \left\{\omega_i\right\}_0^\infty, \left\{z_{ji}\right\}_0^\infty \left(j \in \overline{q_A+q_B}\right)$  and for any real numbers  $\sigma \in\left(\sigma_i, \sigma_{i+1}\right)$ ,

$\omega \in\left(\omega_i, \omega_{i+1}\right), z_j \in\left(z_{ji}, z_{j,i+1}\right), \forall j \in \overline{q_A+q_B}$  being the components of  $z=\left(z^{(A)}, z^{(B)}\right)$ :

$$\left|f_0\left(\sigma, \omega, z^{(A)}, z^{(B)}\right)\right|_i \geq\left|f_0\left(\sigma_i, \omega_i, z^{(Ai)}, z^{(Bi)}\right)\right|-\left(\sigma-\sigma_i\right)\left|f_{0 \sigma}^{\prime}\left(\sigma_i, \omega_i, z^{(A)}, z^{(B)}\right)\right| \\ -\left(\omega-\omega_i\right)\left|f_{0 \omega}^{\prime}\left(\sigma_i, \omega_i, z^{(A)}, z^{(B)}\right)\right|-\sum_{j=1}^{q_A+q_B}\left(z_j-z_{ji}\right)\left|f_{0 z_j}^{\prime}\left(\sigma, \omega, z^{(A)}, z^{(B)}\right)\right| \geq \delta_i>0$$

provided that the above strictly increasing sequences are chosen subject to

$$\left(\sigma_i, \omega_i, z_{1i}, \dots, z_{q_A+q_B, i}\right) \in \mathbf{R}_R^2 \times C_{\alpha 0}^{(A,B,d)}$$

$$\sigma_i < \sigma_{i+1} < \sigma_i + \varepsilon_i, \omega_i < \omega_{i+1} < \omega_i + \varepsilon_i, z_{ji} < z_{j,i+1} < z_{ji} + \varepsilon_i, \forall j \in \overline{q_A + q_B}$$

for any real  $\varepsilon_i$  fulfilling  $0 < \varepsilon_i \leq \frac{\delta_i}{(q_A + q_B + 2)M_i}$ , where

$$M_i := \max \left( \left| f'_{0\sigma}(\sigma, \omega, z^{(A)}, z^{(B)}) \right|, \left| f'_{0\omega}(\sigma_i, \omega_i, z^{(A)}, z^{(B)}) \right|, \left| f'_{0z_j}(\sigma, \omega, z^{(A)}, z^{(B)}) \right|, j \in \overline{q_A + q_B} \right)$$

which is finite since  $R_R^2 \times C_{\alpha 0}^{(A,B,d)}$  is bounded. Then

$$\hat{H}(C_{\alpha 0}^{(A,B,d)}) := \bigcup_{i_j \in \overline{L_j}, j \in \overline{q_A + q_B}} \hat{H}_{i_1, \dots, i_{q+2}} \supseteq R_R^2 \times C_{\alpha 0}^{(A,B,d)} \text{ is a finite cover of}$$

$R_R^2 \times C_{\alpha 0}^{(A,B,d)}$  provided that  $\hat{H}_{i_1, \dots, i_{q+2}}$  are hyper-rectangles defined by

$$\sigma_i < \sigma_{i+1} < \sigma_i + \varepsilon_i, \omega_i < \omega_{i+1} < \omega_i + \varepsilon_i, z_{ji} < z_{j,i+1} < z_{ji} + \varepsilon_i, \forall j \in \overline{q_A + q_B} \text{ and}$$

that the real sequence  $\{\delta_i\}_0^\infty$  is bounded and positive satisfying  $\delta_i \geq \underline{\delta}$  for some real  $\underline{\delta} > 0$ . As a result, the existence of such a finite cover implies the controllability of the nominal system (1)-(2). If the

construction of the cover satisfying the condition  $0 < \varepsilon_i \leq \frac{\delta_i}{(q_A + q_B + 2)M_i}$  with  $\delta_i \geq \underline{\delta} > 0$  is

impossible then the nominal system is not controllable.

**Remark 5.3.** Theorem 5.2 combined with Remark 5.1 yields direct sufficiency type conditions for controllability of the system (22)-(23) on some bounded  $C_{\alpha}^{(A,B,d)} \subset C_{\alpha 0}^{(A,B,d)}$  independent of or dependent on the delays when subject to parametrical multi-perturbations. Note that as alternative to tests on determinants, tests on the singular values of  $Z_{C0}(\hat{\sigma}, z)$ , or on the eigenvalues or matrix ranks of  $\hat{Z}_{C0}(\hat{\sigma}, z)$ , may be applied. Extensions involving input and output decoupling zeros are also direct but specific derivation is omitted by space reasons. On the other hand, the remaining properties may be investigated as well with simple modifications of Theorem 5.2. In particular,

(1) Stabilizability tests follow from Theorem 5.2 for  $\sigma \in \mathbf{R}_{+0}$  only. Output controllability tests follow by replacing  $Z_{C0}(\hat{\sigma}, z)$  with  $Z_{OC0}(\hat{\sigma}, z)$  in Theorem 5.2.

(2) Observability tests on some bounded set  $C_{\alpha 0}^{(A,C,d)}$  follow by replacing  $C_{\alpha 0}^{(A,B,d)}$  with  $C_{\alpha 0}^{(A,C,d)} := C_{\alpha 0}^{(A)} \times C_{\alpha 0}^{(C)} \times C_{\alpha 0}^{(Ad)} \times C_{\alpha 0}^{(Cd)}$  and then using  $\hat{Z}_{O0}(\hat{\sigma}, z)$  instead of  $\hat{Z}_{C0}(\hat{\sigma}, z)$  in Theorem 5.2.

(3) Detectability tests on some bounded set  $C_{\alpha 0}^{(A,C,d)}$  follow by replacing  $C_{\alpha 0}^{(A,B,d)}$  with  $C_{\alpha 0}^{(A,C,d)}$  and then using  $\hat{Z}_{O0}(\hat{\sigma}, z)$  instead of  $\hat{Z}_{C0}(\hat{\sigma}, z)$  in Theorem 5.2 for  $\sigma \in \mathbf{R}_{+0}$  only.

Theorem 5.2 can also be applied for testing the controllability nominal delay-free system (1)-(2). However, in this simple case the conditions may be investigated by testing a finite number of (in general) complex eigenvalues of  $\hat{Z}_{C0}(\hat{\sigma}, z)$ , or (real) singular values of  $Z_{C0}(\hat{\sigma}, z)$ . In this case, the construction of a finite subcover is easier than that involved in Theorem 5.2 since the above number of eigenvalues/singular values is finite for each point in a bounded set  $C_{\alpha 0}^{(A,B)}$  instead of a finite number of functions with infinitely many associated point eigenvalues.  $\square$

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# The Smaller Root Principle for Finding Roots of a Complex Number Function

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## Abstract

Using the second order Taylor expansion of a function  $f(z)$  in complex space, a polynomial  $p(z)$  of degree 2 results. We demonstrate that the smallest root of  $p(z)$  is a search direction that always decreases the modulus of  $f(z)$ . Using this property, a new algorithm is proposed. We prove that the algorithm either finds a root or the estimate goes at infinity, without creating any accumulation points. In the special case of a polynomial, the algorithm always finds a root. Numerical examples are also given.

*Keywords.* Root finding, Global convergence, Complex Numbers, Second Order Taylor Approximation, Minimum Modulus Root

## 1 Introduction

In this paper, we consider functions  $f(z)$  such that  $z$  and  $f(z)$  are complex numbers and  $f(z)$  has a Taylor expansion. For these functions we have the Minimum Modulus Theorem, which means that if an holomorphic functional  $f(z)$  has no root inside a neighbourhood of  $z_0$ , then the minimum of  $|f(z)|$  is attained on the boundary of that neighbourhood. This implies there is no "local minimum" of the modulus of  $f(z)$ . Therefore, it makes sense to try to reduce  $|f(z)|$  iteratively.

Let use a second order Taylor expansion as follows, where  $\lambda \in (0, 1]$ ,  $\text{Re}(z)$  is the real part of  $z$  and  $h$  is a number determined later:

$$f(\lambda h + z) = f(z) + \lambda f'(z)h + \frac{1}{2}\lambda^2 f''(z)h^2 + O(\lambda^3) \quad (1)$$

$$\overline{f(\lambda h + z)} = \overline{f(z)} + \lambda \overline{f'(z)h} + \frac{1}{2}\lambda^2 \overline{f''(z)h^2} + O(\lambda^3). \quad (2)$$

$$\begin{aligned} |f(\lambda h + z)|^2 &= |f(z)|^2 + 2\lambda \text{Re}(\overline{f(z)}f'(z)h) + \\ &\lambda^2 \left[ \text{Re}(\overline{f(z)}f''(z)h^2) + |f'(z)h|^2 \right] + O(\lambda^3). \end{aligned} \quad (3)$$

Let define the following polynomial:

$$p(z) = f(z) + f'(z)h + \frac{1}{2}f''(z)h^2 \quad (4)$$

Assuming  $f(z) \neq 0$  and  $f''(z) \neq 0$ , let  $h$  be the smaller root of the polynomial  $p(z)$ . We will demonstrate that we can always find  $\lambda$  such that  $|f(\lambda h + z)|^2 < |f(z)|^2$ , using equation (3), even if  $f'(z) = 0$ . Using this principle, the paper proposes a new algorithm and it gives numerical examples and convergence analysis.

Let briefly examine the literature. For polynomials, there are several methods that are effective [15]. For more general functions, approaches are based on, for example: Newton related methods, Brent Method [10, 15], Homotopy-Continuation methods [1], Partitionning methods [6, 15], Fixed point methods [2], Tensor methods [17]. [6] is a good review of methods. These methods have various requirements such as non nul first derivative, convexity, an initial estimate value that is sufficiently close to a root, approximation of the second order derivative. On the other hand, as a warning, McMullen [12, 13] has shown that there are no generally convergent purely iterative algorithms for solving polynomials of degrees 4 or greater, with rational mapping such as Newton method.

As far as we know, no methods use the smaller root principle or an approach related to the Minimum Modulus Theorem of complex analysis.

The paper is organized as follows: Section 2 presents the foundations of the algorithm, that is, the properties of the smaller root of a second order polynomial. Section 3 defines the proposed algorithm. Section 4 shows some numerical experiments. Section 5 gives results about convergence for the algorithm. Section 6 concludes.

## 2 The Key Smaller Root Property

First, a Lemma demonstrates the properties of the smaller root. Secondly, we explains how the Lemma is used to reduce  $|f(z)|$  using equation (3). To simplify notations in the following Lemma and referring to equation (4), let denote  $c = f(z), b = f'(z), a = f''(z), r_1 = h$ .

**Lemma 2.1.** (*Smaller Root Properties*)

Let  $p(z) = az^2 + bz + c$  where  $a$  and  $c$  are not null. Let  $r_1$  be a root of  $p(z)$  with smallest modulus, and  $r_2$  the other root. We have: if  $b=0$  then  $\operatorname{Re}(\bar{c} a r_1^2) < 0$ . If  $b \neq 0$  then  $\operatorname{Re}(\bar{c} b r_1) < 0$ . Furthermore, if  $|b|$  is sufficiently small then  $\operatorname{Re}(\bar{c} b r_1) < 0$  and  $\operatorname{Re}(\bar{c} a r_1^2) + |br_1|^2 < 0$ .

*Proof.* Case  $b = 0$ :

$$a r_1^2 = -c \quad (5)$$

$$\bar{c} a r_1^2 = -|c|^2 \quad (6)$$

$$\operatorname{Re}(\bar{c} a r_1^2) = -|c|^2 < 0. \quad (7)$$

Case  $b \neq 0$ :

First we have...

$$r_1 + r_2 = -b/a \quad (8)$$

$$r_1 r_2 = c/a \quad (9)$$

$$r_1 \neq 0, r_2 \neq 0 \quad (10)$$

$$\bar{c} b r_1 = -\overline{a r_1 r_2} a(r_1 + r_2) r_1 \quad (11)$$

$$\bar{c} b r_1 = -|a|^2 |r_1|^2 (r_1 \bar{r}_2 + |r_2|^2) \quad (12)$$

$$Re(\bar{c} b r_1) = -|a|^2 |r_1|^2 ( Re(r_1 \bar{r}_2) + |r_2|^2 ) \quad (13)$$

It remains to show that...

$$Re(r_1 \bar{r}_2) + |r_2|^2 > 0 \quad (14)$$

Since  $r_1$  is the smaller root, we have...

$$|r_2|^2 \geq |r_1| |r_2| \geq -Re(r_1 \bar{r}_2) \quad (15)$$

Assume that we have an equality in the equation (15), that is ...

$$|r_2|^2 = -Re(r_1 \bar{r}_2) \quad (16)$$

Define ...

$$r_1 = |r_1| e^{i\theta_1} \quad (17)$$

$$r_2 = |r_2| e^{i\theta_2} \quad (18)$$

Then, we have ...

$$r_1 \bar{r}_2 = |r_1| |r_2| e^{i(\theta_1 - \theta_2)} \quad (19)$$

$$-Re(r_1 \bar{r}_2) = -|r_1| |r_2| \cos(\theta_1 - \theta_2) = |r_2|^2 \quad (20)$$

$$-\cos(\theta_1 - \theta_2) = \frac{|r_2|}{|r_1|} \quad (21)$$

This means  $|r_1| = |r_2|$ ,  $\theta_1 = \theta_2 + \pi$  and  $r_1 = -r_2$  or  $r_1 + r_2 = 0$ , which contradicts equation (8) and the assumptions.

To resume, we cannot have equality in equation (15), which proves equation (14). From equation (14) and (13), we obtain the required result (if  $b \neq 0$ ):

$$\bar{c} b r_1 < 0 \quad (22)$$

The last part of the lemma follows from the fact that if  $b$  approaches 0, then  $r_1$  approaches  $\sqrt{-c/a}$ ,  $Re(\bar{c} a r_1^2)$  approaches  $-|c|^2 < 0$  and  $|br_1|$  approaches 0.  $\square$

Note that equation (15) can only be true if we choose the smaller root. The following theorem applies Lemma 2.1 to equation (3) and shows how  $|f(z)|$  is reduced.

**Theorem 2.2.** *Assume that  $f(z)$  is not null and also the second derivative is not null without consideration of the first derivative, then the smaller root of the second order Taylor approximation is a direction that can always reduce  $|f(z)|$  using equation (3).*

*Proof.* Let  $h$  be the smaller root of  $p(z)$  in equation (4). Using Lemma 1.2, with  $c = f(z)$ ,  $b = f'(z)$ ,  $a = \frac{1}{2} f''(z)$ , we obtain the following. In the case  $f'(z) \neq 0$ , the dominant term involving  $\lambda$  is negative as  $Re(\bar{f}(z) f'(z) h) < 0$ . In the case  $f'(z) = 0$ , the dominant term is negative also as  $Re(\bar{f}(z) f''(z) h^2) < 0$ . Therefore, in both cases,  $\exists \lambda \in (0, 1]$  such that  $|f(\lambda h + z)|^2 < |f(z)|^2$ .  $\square$

### 3 The Smaller Root Algorithm

Let  $m$  be arbitrary small positive number (the floating point precision of the machine). This algorithm uses  $z_k$  to represent the current estimate of a root. The



algorithm assumes that  $|f''(z_k)| > m$ . Theorem 2.2 guarantees that  $|f(z_k)|$  can be reduced.

#### The Smaller Root Algorithm

1. Choose an arbitrary number  $z_0$ ,  $k=0$
2. Define  $p(z)$  using the current estimate  $z_k$  in equation (4) and compute the smaller root  $h_k$
3. Do a line search on  $\lambda \in (0, 1]$  that minimizes  $|f(z_k + \lambda h_k)|^2$ . Denote  $\lambda_k$  the optimal value of  $\lambda$ . Set  $z_{k+1} = z_k + \lambda_k h_k$ .
4. Set  $k$  to  $k+1$ . If  $|f(z_k)|$  is sufficiently small, stop. Otherwise, goto 2.

## 4 Numerical Experiments

The following paragraphs describes a few numerical experiments. The image in figure1 shows a convergence map for a complex polynomial with 7 roots. A point in the image is used as the initial value of  $z_0$ . The color of the point corresponds to the color of the root to which the algorithm converges (a randomly selected color). In Hubbard [8], it is shown that such polynomials has several initial conditions which fail to converge to a solution with the Newton method (convergence maps are fractal in nature). With our method, there is no such failures (convergence map are filled images).

The next example is the following equation where  $p(z)$  and  $q(z)$  are polynomial of degree 4:

$$f(z) = p(z) + \sin(q(z)) \quad (23)$$



Figure 1: Convergence map for a typical polynomial: 7 roots

The values of the polynomials are (r: real part, i: imaginary part):

$$rp = \{0.1, 0.2, 0.3, 0.4, 0.5\} \quad (24)$$

$$ip = \{0.5, 0.4, 0.3, 0.2, 0.1\} \quad (25)$$

$$rq = \{0.3, 0.2, 0.1, 0.5, 0.4\} \quad (26)$$

$$iq = \{0.4, 0.5, 0.1, 0.2, 0.3\} \quad (27)$$

The following table shows some results with different initial points.

Initial Point	Iterations	Error
(0.1,0.1)	12	0.0000008
(1.3,-0.5)	11	0.0000002

Note that we have used a simple binary search, in these experiments.

## 5 Convergence Analysis of the Algorithm

The proof of convergence of the proposed algorithm is based on continuity arguments and we prove that the algorithm cannot create an accumulation point which is not a root. This leads to convergence Corollaries.

**Lemma 5.1.** *In the algorithm, we assume that  $|f''(z)| > m > 0$ ,  $f(z)$  has a Taylor expansion and the line search in the algorithm is exact (see Brent method in [15]). Also, assume that the algorithm produces an accumulation point  $u_*$ . Then,  $z_*$  is a root of  $f(z)$ .*

*Proof.* We proceed by contradiction. Assume  $z_*$  is not a root of  $f(z)$ , that is  $f(z_*) \neq 0$ .

Since  $u_*$  is an accumulation point, there is a subsequence of estimate  $u$  used in the algorithm that converges to  $u_*$ . This subsequence has necessarily another subsequence of estimates  $u$  for which the computation of the smaller root  $h$  in the algorithm uses the same sign. This subsequence is denoted as follows:

$$z_j, j = 0, \dots, \infty \quad (28)$$

First observe that, since the sequence  $z_j$  use the same sign, the smaller roots  $h_j$  associated to  $z_j$  are obtained by a continuous expression with respect to  $u$ . This means there exists a number  $h_*$  such that

$$\lim_{j \rightarrow \infty} h_j = h_* \quad (29)$$

Let define polynomials as follows:

$$p_j(z) = f(u_j) + f'(u_j) h + f''(u_j) h^2, j = 1, \dots, \infty \quad (30)$$

$$p_*(z) = f(u_*) + f'(u_*) h + f''(u_*) h^2 \quad (31)$$

Since the coefficients of  $p_j(z)$  and of  $p_*(z)$  are continuous function of  $u$ , we have that  $h_j$  is a smaller root of  $p_j(z)$  and  $h_*$  is a smaller root of  $p_*(z)$ . This implies  $h_*$  can be used to reduce  $|f(u_*)|$  with step 3 of the algorithm. That is, there exists  $\lambda_* \in (0, 1]$  and  $w_* = z_* + \lambda_* h_*$  such that  $|f(w_*)| < |f(z_*)|$ .

Define  $d$  as follows:  $d = |f(z_*) - |f(z_* + w_* h_*)| > 0$ . Next, define the function  $e(\alpha)$  as follows:

$$e(\alpha) = e(z, h, \lambda) = f(z + \lambda h), \text{ where } \alpha \text{ is the vector } (z, h, \lambda) \quad (32)$$

The function  $e(\alpha)$  is differentiable with respect to  $z$ ,  $h$  and  $\lambda$  and therefore continuous with respect to  $z$ ,  $h$  and  $\lambda$  or  $\alpha$ .

Also define  $\omega_j$  as follows, where  $j_0$  is the least positive integer such that  $\omega_* - \frac{1}{j_0} \in (0, 1]$ :

$$\omega_j = \omega_* - \frac{1}{j}, j = j_0, \dots, \infty \quad (33)$$

Note that ...

$$\lim_{j \rightarrow \infty} \omega_j = \omega_* \quad (34)$$

Now, define  $\alpha_j, j = 1, \dots, \infty$  and  $\beta$  as follows:

$$\alpha_j = (z_j, h_j, \omega_j) \quad (35)$$

$$\beta = (z_*, h_*, \omega_*) \quad (36)$$

Since  $e(\alpha)$  is continuous, we have:

$$\lim_{j \rightarrow \infty} \alpha_j = \beta \quad (37)$$

$$\lim_{j \rightarrow \infty} e(\alpha_j) = e(\beta) \quad (38)$$

$$\lim_{j \rightarrow \infty} |e(\alpha_j)| = |e(\beta)| = |f(z_* + w_* h_*)| < |f(z_*)| \quad (39)$$

Therefore,  $\exists N$  such that  $\forall j > N$  we have:

$$| |e(\alpha_j)| - |e(\beta)| | < d/2 \quad (40)$$

That means ...

$$|e(\alpha_j)| < |e(\beta)| + d/2 = |f(z_* + w_* h_*)| + \frac{|f(z_*)| - |f(z_* + w_* h_*)|}{2} = \frac{|f(z_*)| + |f(z_* + w_* h_*)|}{2} < |f(z_*)| \quad (41)$$

The last equation (41) is a contradiction for the following reason: first, we have  $|f(z_j)| > |f(z_{j+1})| > |f(z_{j+2})| > \dots > |f(z_*)|$  and we have found  $\omega_j \in (0, 1)$  for which  $e(\alpha_j) = |f(z_j + \omega_j h_j)| < |f(z_*)|$ , which contradicts the fact that the line search (on  $\lambda$  in the algorithm) finds the minimum of  $|f(z_j + \lambda h_j)|$ , that is the line search is disfunctional.

We conclude that if there is an accumulation point, then it must be a root.

□

**Theorem 5.2.** (*Convergence of the Smaller Root Algorithm*)

Assume  $|f''(z_k)| > m$  in the algorithm. The algorithm converges to a root, or  $|z_k|$  goes at infinity, always decreasing  $|f(z_k)|$  without creating an accumulation point.

*Proof.* If the algorithm creates an accumulation point, from Lemma 5.1 it must converge to a root. Otherwise,  $|z_k|$  must go at infinity without creating an accumulation point. Theorem 2.2 demonstrates that  $|f(z_k)|$  is always decreased

.

□

**Corollary 5.3.** (*The case of a polynomial*) Assume  $f(z)$  is a non constant polynomial and  $|f''(z_k)| > m$  in the algorithm. Then, the algorithm always finds a root.

*Proof.* As  $|z|$  grows to infinity,  $|f(z)|$  goes to infinity since the higher order term eventually dominates. Also, we have seen that for all  $k$  in the algorithm, we have  $|f(z_k)| < |f(z_0)|$ . This means  $|z_k|$  cannot go at infinity (since  $|f(z_k)|$  is bounded by  $|f(z_0)|$ ). From Theorem 5.2, we conclude that it must converge to a solution.

□

## 6 Concluding remarks

With one equation, a new algorithm was presented for root finding of a function. This algorithm requires the second order derivative to be significantly non nul throughout the algorithm. Some numerical experiments were given together with convergence results: the algorithm always reduces the modulus of the function and, if it creates an accumulation point, then a solution is found. Note that the function  $e^z$  has no root which means that finding a root might be impossible. However, in the case of a polynomial, the algorithm always finds a root.

As it was presented, the algorithm requires the second order derivative to be non null, to simplify proofing. A more general algorithm would require the first or second order derivative to be non null.

Obviously, there are many unanswered questions yet to be addressed: rate of convergence, handling of singularities, enumeration of roots when real roots are needed, etc. Most importantly, is this approach applicable to a system of equations?

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# An algorithmic technique for the investigation of the non-negativity (or positivity) of a homogeneous matrix pencil

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In the classical literature of matrix pencils theory, the dual pencils  $sF - G$  and  $F - \hat{s}G$  are identified by the homogeneous matrix pencil  $sF - \hat{s}G$ , where  $s, \hat{s}$  are indeterminates. In the present paper, we discuss and we provide a characterization of the nonnegativity (or positivity) of a given homogeneous pencil. An algorithmic approach for the analytic determination of all the nonnegative (or positive) homogeneous pencils into a relevant set of indeterminates  $s, \hat{s}$  is provided. This new approach can be easily transferred into a standard computational routine by using simple Matlab m-files. Some numerical examples are also concluded.

**Keywords:** Matrix Pencil Theory; Nonnegativity (Positivity); Algorithmic Approach

**AMS (classification):** 15A22; 65F30; 15A48

## 1. Introduction

Linear descriptor systems are very common in many practical situations in nature like, for instance, transmission problems, communications [4], population growth models (see for instance the famous Leslie model [12]) and other biological systems or economical and actuarial dynamic models, see for instance [1]-[5], [9]-[11], [14]-[16], [18] etc. Those systems may be continuous, discrete, distributed, and internal (i.e. in the state) or external (i.e. in the input or/and output).

Thus, the study of problems and structural properties of regular/singular and extended state space theory may be reduced to the study of linear descriptor differential (or difference) equations

$$F\dot{\underline{x}}(t) = G\underline{x}(t) \text{ and } F\underline{x}(t) = G\underline{\dot{x}}(t)$$

$$(or \ F\mathbf{x}_{k+1} = G\mathbf{x}_k \text{ and } F\mathbf{x}_k = G\mathbf{x}_{k+1}). \quad (1)$$

where  $\mathbf{x} \in \mathbb{R}^n$  is the state vector, and  $F, G \in \mathbb{R}^{m \times n}$  (or  $F, G \in \mathbb{R}^{n \times n}$ ) are real rectangular (or square with  $\det F = 0$ ) matrices. In those cases, the matrix pencil theory becomes a key tool. Furthermore, to entirely cover the needs, the matrix pencil theory has to be general enough with a geometric, a dynamic, a topological, an invariant and a computational dimension.

Now, let  $(F, G) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n}$  and  $(s, \hat{s})$  be a pair of indeterminates. The polynomial matrices  $sF - G$  or  $F - \hat{s}G$  ( $s, \hat{s} \in \mathbb{R}$ ) can be defined by the homogeneous matrix pencil

$$L(s, \hat{s}) \triangleq sF - \hat{s}G \quad (2)$$

(or equivalently by a pair  $(F, -G)$ ), since the existence of the important notion of duality, the so-called *elementary divisor type duality* or *integrator-differentiator type duality*, has already been demonstrated, see [7] and [8]. Thus,  $sF - G$  and  $F - \hat{s}G$  are related by the special type of bilinear transformation:  $s \rightarrow \frac{1}{\hat{s}}$  which clearly transforms the points  $0, \infty, a \neq 0$  of the compactified real plain ( $\mathbb{R} \cup \{\infty\}$ ) (or of the Riemann sphere) to the points  $\infty, 0, \frac{1}{a}$ , relatively.

This paper is devoted to the study of the nonnegativity (or positivity) property of the pencil (2), which is derived by the singular (or regular) linear descriptor systems subject to constant, in general, rectangular coefficient matrices. We would like to stress out that this work follows closely the thoughts of Uhlig for definite and semi-definite matrices in a *real symmetric matrix* pencil, see [17]. Thus, similar results to [18] are finally derived, using still elementary geometric framework. Although, in our case, we have to underline that relevant matrices  $F, G \in \mathbb{R}^{m \times n}$  have not any particular algebraic structure.

The paper is organized as follows. A notation section is considered at the end of this introductory section. Section 2 is devoted to the main results of the paper. Thus, an extensive discussion of the nonnegativity (or positivity) of the homogenous matrix pencils of systems (1) is considered. The results are new with some interest, since we obtain a characterization when a homogeneous system is nonnegative (or positive). Finally, section 3 provides an algorithm and discusses several examples.

**Definition 1** *The homogeneous matrix pencil, see [6]*

$$L(s, \hat{s}) \in \mathcal{L}_{m,n}(s, \hat{s}) \triangleq \{sF - \hat{s}G : F, G \in \mathbb{R}^{m \times n} \text{ and } s, \hat{s} \in \mathbb{R}\},$$

with  $F = [f_{ij}]_{\substack{i=1,2,\dots,m \\ j=1,2,\dots,n}} \in \mathbb{R}^{m \times n}$ ,  $G = [g_{ij}]_{\substack{i=1,2,\dots,m \\ j=1,2,\dots,n}} \in \mathbb{R}^{m \times n}$  and  $s, \hat{s} \in \mathbb{R}$  is called *nonnegative (or positive) (n-pencil or p-pencil)* if there exists

$$\Omega = \{(s, \hat{s}) : sF - \hat{s}G \in \mathcal{L}_{m,n}(s, \hat{s})\} \subseteq \mathbb{R}^2 \text{ such as } \forall (s, \hat{s}) \in \Omega,$$

$$L_{ij}(s, \hat{s}) = f_{ij}s - g_{ij}\hat{s} \geq (>) 0 \quad (3)$$

for every  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, n$ .

**Definition 2**  $\mathcal{L}_{m,n}(s, \hat{s}) \triangleq \{sF - \hat{s}G : F, G \in \mathbb{R}^{m \times n} \text{ and } s, \hat{s} \in \mathbb{R}\}$  is called nonnegative (or positive) set of homogeneous pencils if for every pair  $(s, \hat{s}) \in \Omega$ ,  $L(s, \hat{s}) \geq (> \mathbb{O})$ , where  $\mathbb{O}$  is the zero matrix.

## 2 Main results

In this section, the main results concerning the investigation of the nonnegativity (or positivity) of homogeneous matrix pencils are presented and fully discussed. The next Remark is important; since four different systems of inequalities derive which provide us with a deeper understanding of nonnegativity (or positivity).

Firstly, we denote the element  $(i, j)$  as the  $(i-1)n + j$ -element of the new system of equations. Consequently,

$$f_{ij} \triangleq f_{(i-1)n+j},$$

thus  $f_{11} \triangleq f_1, f_{12} \triangleq f_2, \dots, f_{mn} \triangleq f_{(m-1)n+n}$ . Similarly for the  $g$ . Thus, we obtain the system of inequalities

$$f_r s - g_r \hat{s} \geq 0, \text{ for } r = 1, 2, \dots, (m-1)n + n \quad (4)$$

**Remark 1** System 1 might be analyzed to the following (A) - (D) systems of inequalities:

$$\text{A) } f_r s - g_r \hat{s} \geq (>) 0 \text{ with } f_r \geq 0 \text{ and } g_r \leq 0, r = 1, 2, \dots, k.$$

$$\text{B) } f_r s - g_r \hat{s} \geq (>) 0 \text{ with } f_r \geq 0 \text{ and } g_r \leq 0, r = k+1, k+2, \dots, \rho.$$

$$\text{C) } f_r s - g_r \hat{s} \geq (>) 0 \text{ with } f_r \leq 0 \text{ and } g_r \geq 0, r = \rho+1, \rho+2, \dots, \tau.$$

$$\text{D) } f_r s - g_r \hat{s} \geq (>) 0 \text{ with } f_r \leq 0 \text{ and } g_r \geq 0, r = \tau+1, \tau+2, \dots, mn.$$

The Remark above is simple and rather straightforward when someone considers the notation. Moreover, without loss of generality, we can assume that the first  $k$ -equations follow (A), the second  $\rho - k$ -equations follow (B), the third  $\tau - \rho$ -equations follow (C) and the last  $(m-1)n + n - \tau$ -equations follow (D).

In addition, before we go further with the statement and the proof of Theorem 1, it should be stressed out that the case when  $g_r$  and  $f_r$  are simultaneously zero is not considered, since it fulfils our requirement of nonnegativity. Note that the following results are also true for positivity if the equality "=" into our results is excluded.

**Definition 3** Denote with  $\Omega$  the set where the inequality  $f_r s - g_r \hat{s} \geq 0$  holds.

**Theorem 1** If we consider that the homogeneous matrix pencil,  $sF - \hat{s}G$  is nonnegative, then the intersection  $\Omega_A \cap \Omega_B \cap \Omega_C \cap \Omega_D = \emptyset$ .

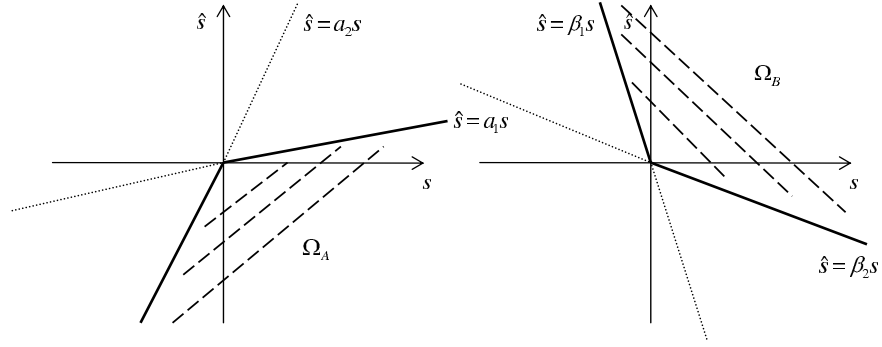


Figure 1: (A) system of inequalities holds. (B) system of inequalities holds.

*Proof.* For (A)-(D), consider the equality  $f_r s - g_r \hat{s} = 0$ , then we obtain

$$\hat{s} = \begin{cases} \frac{f_r}{g_r} s, & \text{if } g_r \neq 0 \\ 0, & \text{if } g_r = 0 \end{cases} \quad \text{and} \quad s = \begin{cases} \frac{g_r}{f_r} \hat{s}, & \text{if } f_r \neq 0 \\ 0, & \text{if } f_r = 0 \end{cases}$$

- For (A)

Denote that  $a_1 = \min_{r=1,2,\dots,k} \{f_r/g_r\}$  and  $a_2 = \max_{r=1,2,\dots,k} \{f_r/g_r\}$ , where  $f_r, g_r \geq 0$  for  $r = 1, 2, \dots, mn$ . (Note that  $a_2 = +\infty$  and  $\hat{s} = 0$  for  $g_r = 0$ .) Hence, the set  $\Omega_A$ , where the nonnegativity exists, is sketched in figure 1 (left part).

Thus,  $\Omega_A = \{(s, \hat{s}) : \text{if } s \geq 0 \text{ then } \hat{s} \leq a_1 s \text{ and if } s \leq 0 \text{ then } \hat{s} \leq a_2 s\}$ .

- For (B)

Denote that  $\beta_1 = \min_{r=k+1,k+2,\dots,\rho} \{f_r/g_r\}$  and  $\beta_2 = \max_{r=k+1,k+2,\dots,\rho} \{f_r/g_r\}$  where  $f_r \geq 0$  and  $g_r \leq 0$  for  $r = 1, 2, \dots, mn$ . (Note that  $\beta_1 = -\infty$ , and  $\hat{s} = 0$  when  $g_r = 0$ .) Hence, the set  $\Omega_B$ , where the nonnegativity exists, is sketched also in figure 1 (right part).

Thus,  $\Omega_B = \{(s, \hat{s}) : \text{if } s \geq 0 \text{ then } \hat{s} \geq \beta_2 s \text{ and if } s \leq 0 \text{ then } \hat{s} \geq \beta_1 s\}$ .

- For (C)

Denote that  $\gamma_1 = \min_{r=\rho+1,\rho+2,\dots,\tau} \{f_r/g_r\}$  and  $\gamma_2 = \max_{r=\rho+1,\rho+2,\dots,\tau} \{f_r/g_r\}$  where  $f_r \leq 0$  and  $g_r \geq 0$  for  $r = 1, 2, \dots, mn$ . (Note that  $\gamma_1 = -\infty$  and  $\hat{s} = 0$ , when  $g_r = 0$ .) Hence, the set  $\Omega_C$  where the nonnegativity exists is sketched in figure 2 (left part).

Thus,  $\Omega_C = \{(s, \hat{s}) : \text{if } s \geq 0 \text{ then } \hat{s} \leq \gamma_1 s \text{ and if } s \leq 0 \text{ then } \hat{s} \leq \gamma_2 s\}$ .

- For (D)

Denote that  $\delta_1 = \min_{r=\tau+1,\tau+2,\dots,(m-1)n+n} \{f_r/g_r\}$  and

$\delta_2 = \max_{r=\tau+1,\tau+2,\dots,(m-1)n+n} \{f_r/g_r\}$ , where  $f_r, g_r \geq 0$  for  $r = 1, 2, \dots, mn$ . (Note

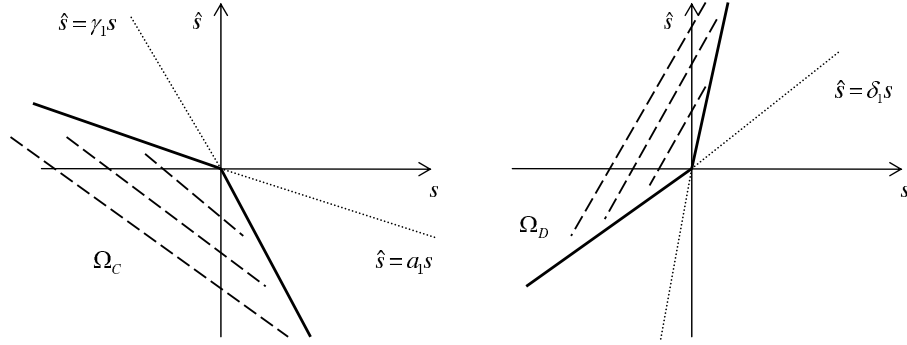


Figure 2: (C) system of inequalities holds. (D) system of inequalities holds.

that  $\delta_2 = +\infty$  and  $\hat{s} = 0$  when  $g_r = 0$ .) Hence, the set  $\Omega_D$ , where the nonnegativity exists, is sketched also in figure 2 (right part).

Thus,  $\Omega_D = \{(s, \hat{s}) : \text{if } s \geq 0 \text{ then } \hat{s} \geq \delta_2 s \text{ and if } s \leq 0 \text{ then } \hat{s} \geq \delta_1 s\}$ . Consequently, it is profound that  $\Omega_A \cap \Omega_B \cap \Omega_C \cap \Omega_D = \emptyset$ .  $\square$

**Corollary 1** If there exists a set  $\Omega$  for indeterminates  $s, \hat{s} \in \mathbb{R}$  where the homogeneous matrix pencil  $sF - \hat{s}G$  is nonnegative (or positive), then system (1) is not analyzed to (A) - (D) sub-systems of inequalities, simultaneously. *Proof.* It is a straightforward result of Theorem 1 and Definition 1.  $\square$

On the other hand, we can consider the results of the following Corollary which describes all the cases of the existence of nonnegativity or positivity.

**Corollary 2** There exists a set  $\Omega$  for indeterminates  $s, \hat{s} \in \mathbb{R}$  such as the homogeneous matrix pencil  $sF - \hat{s}G$  is nonnegative (or positive), when system (1) is analyzed to one of the following possible combinations of systems of inequalities.

- (1)  $ABC$ , (2)  $ABD$ , (3)  $ACD$ , (4)  $BCD$ , (5)  $AB$ , (6)  $AC$ , (7)  $AD$  (8)  $BC$ ,  
(9)  $BD$ , (10)  $CD$ , (11)  $A$ , (12)  $B$ , (13)  $C$  and (14)  $D$ .

In the next lines, we examine each case separately.

◊ (1) System (1) is analyzed to (A), (B) and (C) simultaneously.

Thus, we consider that system (A) has  $k$  inequalities, system (B) has  $\rho - k$  inequalities, and system (C) has  $mn - \rho$  inequalities. There exists solution if  $\beta_2 < \gamma_1$ , see figure 3 (left part).

Then the solution is in the set

$$\Omega_{ABC} = \{(s, \hat{s}) : \text{if } s \geq 0 \text{ and } \beta_2 s \leq \hat{s} \leq \gamma_1 s\}.$$

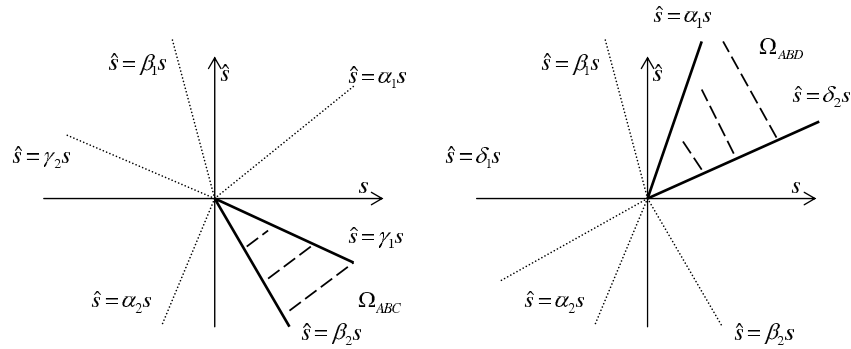


Figure 3: ABC system of inequalities holds. ABD system of inequalities holds.

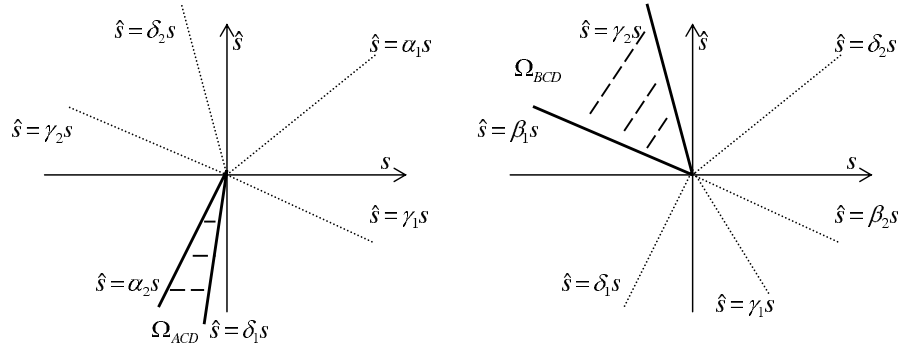


Figure 4: ACD system of inequalities holds. BCD system of inequalities holds.

◇ (2) System (1) is analyzed to (A), (B) and (D) simultaneously.

Thus, we consider that system (A) has  $k$  inequalities, system (B) has  $\rho - k$  inequalities, and system (D) has  $mn - \rho$  inequalities.

There exists solution if  $\delta_2 < \alpha_1$ , see figure 3 (right part).

Then the solution is in the set

$$\Omega_{ABD} = \{(s, \hat{s}) : \text{if } s \geq 0 \text{ and } \delta_2 s \leq \hat{s} \leq \alpha_1 s\}.$$

◇ (3) System (1) is analyzed to (A), (C) and (D) simultaneously.

Thus, we consider that system (A) has  $k$  inequalities, system (C) has  $\tau - k$  inequalities, and system (D) has  $mn - \tau$  inequalities. There exists solution if  $\alpha_2 < \delta_1$ , see figure 4 (left part).

Then the solution is in the set

$$\Omega_{ACD} = \{(s, \hat{s}) : \text{if } s \geq 0 \text{ and } \delta_1 s \leq \hat{s} \leq \alpha_2 s\}.$$

◇ (4) System (1) is analyzed to (B), (C) and (D) simultaneously.

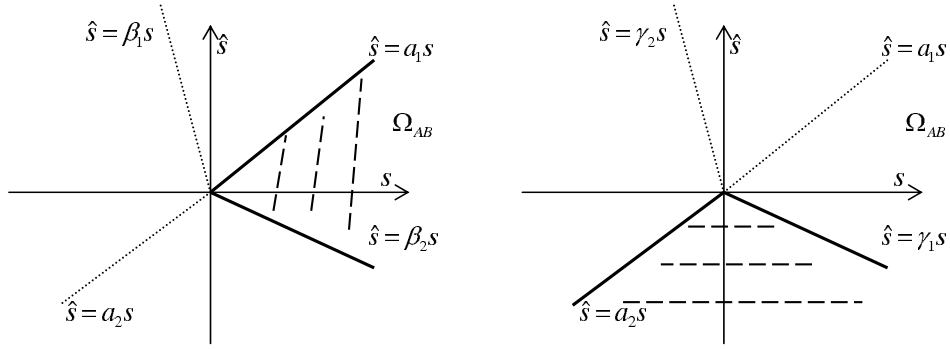


Figure 5: AB system of inequalities holds. AC system of inequalities holds.

Thus, we consider that system (B) has  $\rho$  inequalities, system (C) has  $\tau - \rho$  inequalities, and system (D) has  $mn - \tau$  inequalities. There exists solution if  $\gamma_2 < \beta_1$ , see figure 4 (right part). Then the solution is in the set

$$\Omega_{BCD} = \{(s, \hat{s}) : \text{if } s \leq 0 \text{ and } \beta_1 s \leq \hat{s} \leq \gamma_2 s\}.$$

◇ (5) System (1) is analyzed to (A) and (B) simultaneously.

Thus, we consider that system (A) has  $k$  inequalities, and system (B) has  $mn - k$  inequalities. There exists solution, see figure 5 (left part).

Then the solution is in the set

$$\Omega_{AB} = \{(s, \hat{s}) : \text{if } s \geq 0 \text{ and } \beta_2 s \leq \hat{s} \leq a_1 s\}.$$

◇ (6) System (1) is analyzed to (A) and (C) simultaneously.

Thus, we consider that system (A) has  $k$  inequalities, and system (C) has  $mn - k$  inequalities. There exists solution, see figure 5 (right part). Then the solution is in the set

$$\Omega_{AC} = \left\{ (s, \hat{s}) : \text{if } \hat{s} \leq 0 \text{ and } \frac{\hat{s}}{a_2} \leq s \leq \frac{\hat{s}}{\gamma_1} \right\}.$$

◇ (7) System (1) is analyzed to (A) and (D) simultaneously.

Thus, we consider that system (A) has  $k$  inequalities, and system (D) has  $mn - k$  inequalities. There exists solution if  $\delta_2 < \alpha_1$ , and the solution is in the set (see figure 3 right part)

$$\Omega_{AD} = \{(s, \hat{s}) : \text{if } s \geq 0 \text{ and } \delta_2 s \leq \hat{s} \leq \alpha_1 s\} \equiv \Omega_{ABD}.$$

◇ (8) System (1) is analyzed to (B) and (C) simultaneously.

Thus, we consider that system (B) has  $\rho$  inequalities, and system (C) has  $mn - \rho$



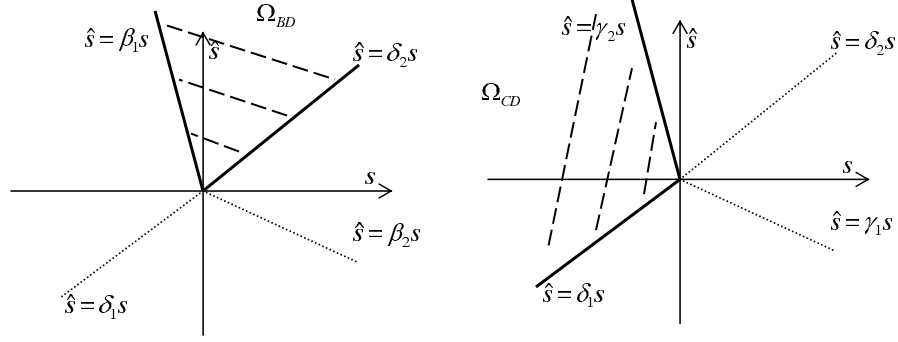


Figure 6: BD system of inequalities holds. CD system of inequalities holds.

inequalities. There exists solution if , and the solution is in the set (see figure 3 left part)

$$\Omega_{BC} = \{(s, \hat{s}) : \text{if } s \geq 0 \text{ and } \beta_2 s \leq \hat{s} \leq \gamma_1 s\} \equiv \Omega_{ABC}.$$

◇ (9) System (1) is analyzed to (B) and (D) simultaneously.

Thus, we consider that system (B) has  $\rho$  inequalities, and system (D) has  $mn - \rho$  inequalities. There exists solution, see figure 6 (left part).

Then the solution is in the set

$$\Omega_{BD} = \left\{ (s, \hat{s}) : \text{if } \hat{s} \geq 0 \text{ and } \frac{\hat{s}}{\beta_1} \leq s \leq \frac{\hat{s}}{\delta_2} \right\}.$$

◇ (10) System (1) is analyzed to (C) and (D) simultaneously.

Thus, we consider that subsystem (C) has  $\tau$  inequalities, and subsystem (D) has  $mn - \tau$  inequalities. There exists solution, see figure 6 (right part). Then the solution is in the set

$$\Omega_{CD} = \{(s, \hat{s}) : \text{if } s \leq 0 \text{ and } \delta_1 s \leq \hat{s} \leq \gamma_2 s\}.$$

Finally, for the other for cases, see  $\Omega_A$ ,  $\Omega_B$ ,  $\Omega_C$  and  $\Omega_D$ , respectively.

### 3 The algorithm for checking the nonnegativity of homogeneous matrix pencil

In this section, we summarize the results of section 2 into the following algorithm which is very useful in practice.

#### Algorithm

**Step 1:** Inputs: Give the singular system, i.e. matrices  $F, G$ .

**Step 2:** Considering the elements of matrices from step 1, we denote the number of systems of inequalities:

$$\begin{array}{ll} ABC \text{ -- } \rightarrow \text{ Go to Step 3,} & ABD \text{ -- } \rightarrow \text{ Go to Step 4,} \\ ACD \text{ -- } \rightarrow \text{ Go to Step 5,} & BCD \text{ -- } \rightarrow \text{ Go to Step 6,} \\ AB \text{ -- } \rightarrow \text{ Go to Step 7,} & AC \text{ -- } \rightarrow \text{ Go to Step 8,} \\ AD \text{ -- } \rightarrow \text{ Go to Step 9,} & BC \text{ -- } \rightarrow \text{ Go to Step 10,} \\ BD \text{ -- } \rightarrow \text{ Go to Step 11,} & CD \text{ -- } \rightarrow \text{ Go to Step 12,} \\ A \text{ -- } \rightarrow \text{ Go to Step 13,} & B \text{ -- } \rightarrow \text{ Go to Step 14,} \\ C \text{ -- } \rightarrow \text{ Go to Step 15,} & D \text{ -- } \rightarrow \text{ Go to Step 16.} \\ ABCD \text{ -- } \rightarrow \text{ Go to Step 17} \end{array}$$

**Step 3:** (ABC) If condition  $\beta_2 \leq \gamma_1$  is satisfied, then the homogeneous matrix pencil  $sF - \hat{s}G$  is nonnegative (or positive) for every  $(s, \hat{s}) \in \Omega_{ABC}$ , else if go to *step 18*.

**Step 4:** (ABD) If condition  $\delta_2 \leq \alpha_1$  is satisfied, then the homogeneous matrix pencil  $sF - \hat{s}G$  is nonnegative (or positive) for every  $(s, \hat{s}) \in \Omega_{ABD}$ , else if go to *step 18*.

**Step 5:** (ACD) If condition  $\alpha_2 \leq \delta_1$  is satisfied, then the homogeneous matrix pencil  $sF - \hat{s}G$  is nonnegative (or positive) for every  $(s, \hat{s}) \in \Omega_{ACD}$ , else go to *step 18*.

**Step 6:** (BCD) If condition  $\gamma_2 \leq \beta_1$  is satisfied, then the homogeneous matrix pencil  $sF - \hat{s}G$  is nonnegative (or positive) for every  $(s, \hat{s}) \in \Omega_{BCD}$ , else go to *step 18*.

**Step 7:** (AB) The homogeneous matrix pencil  $sF - \hat{s}G$  is nonnegative (or positive) for every  $(s, \hat{s}) \in \Omega_{AB}$ , else go to *step 18*.

**Step 8:** (AC) The homogeneous matrix pencil  $sF - \hat{s}G$  is nonnegative (or positive) for every  $(s, \hat{s}) \in \Omega_{AC}$ , else go to *step 18*.

**Step 9:** (AD) If condition  $\delta_2 \leq \alpha_1$  is satisfied, then the homogeneous matrix pencil  $sF - \hat{s}G$  is nonnegative (or positive) for every  $(s, \hat{s}) \in \Omega_{AD}$ , else go to *step 18*.

**Step 10:** (BC) If condition  $\beta_2 \leq \gamma_1$  is satisfied, then the homogeneous matrix pencil  $sF - \hat{s}G$  is nonnegative (or positive) for every  $(s, \hat{s}) \in \Omega_{BC}$ , else go to *step 18*.

**Step 11:** (BD) The homogeneous matrix pencil  $sF - \hat{s}G$  is nonnegative (or positive) for every  $(s, \hat{s}) \in \Omega_{BD}$ , else go to *step 18*.

**Step 12:** (CD) The homogeneous matrix pencil  $sF - \hat{s}G$  is nonnegative (or positive) for every  $(s, \hat{s}) \in \Omega_{CD}$ , else go to *step 18*.

**Step 13:** (A) The homogeneous matrix pencil  $sF - \hat{s}G$  is nonnegative (or positive) for every  $(s, \hat{s}) \in \Omega_A$ , else go to *step 18*.

**Step 14:** (B) The homogeneous matrix pencil  $sF - \hat{s}G$  is nonnegative (or positive) for every  $(s, \hat{s}) \in \Omega_B$ , else go to *step 18*.

**Step 15:** (C) The homogeneous matrix pencil  $sF - \hat{s}G$  is nonnegative (or positive) for every  $(s, \hat{s}) \in \Omega_C$ , else go to *step 18*.

**Step 16:** (D) The homogeneous matrix pencil  $sF - \hat{s}G$  is nonnegative (or positive) for every  $(s, \hat{s}) \in \Omega_D$ , else go to *step 18*.

**Step 17:** (ABCD) The homogeneous matrix pencil  $sF - \hat{s}G$  is not nonnegative (or positive) for every  $(s, \hat{s}) \in \mathbb{R} \times \mathbb{R}$ .

**Step 18:** The homogeneous matrix pencil  $sF - \hat{s}G$  is not nonnegative (or positive) for every  $(s, \hat{s}) \in \mathbb{R} \times \mathbb{R}$ .

#### End Algorithm

In order to understand better the results of that paper, some numerical examples are considered.

**Example 3.1** a) Consider the pencil  $(F, G)$ , where

$$F = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

It is profound that  $\det F = 0$ , and the system of inequality ABC is derived. Considering the results of section 2 and Algorithm above, we should check out whether or not the condition  $\beta_2 \leq \gamma_1$  is true. Analytically,

$$\beta_2 = \max_{f_r \geq 0, g_r \leq 0} \{f_r/g_r\} = \max \left\{ \frac{1}{-1}, \frac{1}{-1} \right\} = -1$$

and

$$\gamma_1 = \min_{f_r \leq 0, g_r \geq 0} \{f_r/g_r\} = \min \left\{ \frac{-1}{1}, \frac{-2}{1}, \frac{-2}{1} \right\} = -2$$

Thus, since  $\beta_2 > \gamma_1$  the homogeneous matrix pencil  $L(s, \hat{s}) = sF - \hat{s}G$  can not contain a nonnegative (or positive) matrix.

b) Consider now a slightly different to previous pencil  $(F, G)$ , where

$$F = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

It is profound that also  $\det F = 0$ , and the system of inequality ABC is derived. Considering the results of section 2 and Algorithm above, we obtain

$$\beta_2 = \max_{f_r \geq 0, g_r \leq 0} \{f_r/g_r\} = -1$$

and

$$\gamma_1 = \min_{f_r \leq 0, g_r \geq 0} \{f_r/g_r\} = \min \left\{ \frac{-1}{1}, \frac{-1}{1}, \frac{-1}{1} \right\} = -1.$$

Thus, the homogeneous matrix pencil  $L(s, \hat{s}) = sF - \hat{s}G$  can contain a nonnegative for every  $s = -\hat{s}$ , i.e.

$$s(F + G) = s \left( \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right) = s \begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix} \in L(s, \hat{s}),$$

for  $s > 0$ .

c) Now, we consider the pencil  $(F, G)$ , where

$$F = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 2 & -1 \\ 3 & -1 & 1 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

It is profound that  $\det F = 0$ , and the system of inequality ABC is derived. Considering the results of section 2 and Algorithm above, we obtain

$$\beta_2 = \max_{f_r \geq 0, g_r \leq 0} \{f_r/g_r\} = \max \left\{ \frac{3}{-1}, \frac{2}{-1} \right\} = -2$$

and

$$\gamma_1 = \min_{f_r \leq 0, g_r \geq 0} \{f_r/g_r\} = -1.$$

Thus, since the restriction above is true, the homogeneous matrix pencil  $L(s, \hat{s}) = sF - \hat{s}G$  is contain a nonnegative pencil for every

$$(s, \hat{s}) \in \Omega_{ABC} = \{(s, \hat{s}) : if s \geq 0 \text{ and } -2s \leq \hat{s} \leq -s\},$$

i.e. for the pair  $(s, \hat{s} = -\frac{3}{2}s)$ , we have

$$s \left( F + \frac{3}{2}G \right) = s \left( \begin{bmatrix} 3 & -1 & 1 \\ 1 & 2 & -1 \\ 3 & -1 & 1 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right) = \frac{1}{2}s \begin{bmatrix} 1 & 1 & 5 \\ 5 & 1 & 1 \\ 9 & 1 & 5 \end{bmatrix} \in L(s, \hat{s}),$$

for  $s \geq 0$ .

Moreover, it can be also positive if  $s > 0$  (which is trivial).

d) Finally, we consider the pencil  $(F, G)$ , where

$$F = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

It is profound that  $\det F = 0$ , and the system of inequality ABCD is derived. Following the results of section 2, the homogeneous matrix pencil can not contain non-negative matrix into a set of  $(s, \hat{s})$ .

## 4 Conclusions - Further Results

In this paper, we present an algorithm to characterize the nonnegativity (or positivity) of a given homogeneous pencil. Following the thoughts of Uhlig [17] and by using still an elementary geometric framework, we obtain the desired results. It should be pointed out that quite similar question whether a given pencil of real symmetric matrices contains a positive definite matrix was treated by many mathematicians, such as Hestenes and Mcshane (1935), Finsles (1937), Albert (1938), Reid (1938), Dines (1941), Calabi (1964), Taussky (1967), Hestenes (1968), Berman (1970), see [17] for more details. Although, we should stress out that our matrices have not any particular algebraic structure. This new algorithmic approach can be transferred into a standard computational routine by using Matlab m-files. This task is one of our future plans.

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# Oscillation of first-order impulsive difference equations with continuous arguments

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## Abstract

In this paper, we present some integral criteria for the oscillation of solutions to the following impulsive delay difference equation involving continuous arguments:

$$\begin{cases} \Delta_{\rho}x(t) + \sum_{i \in \mathcal{I}} p_i(t)x(t - \tau_i) = 0 & \text{for } t \in [t_0, \infty) \setminus \{\theta_n\}_{n \in \mathbb{N}} \\ \Delta x(\theta_n) + q_n x(\theta_n) = 0 & \text{for } n \in \mathbb{N}, \end{cases}$$

where  $t_0 \in \mathbb{R}$ ,  $\rho \in (0, \infty)$ ,  $\mathcal{I}$  is a bounded beginning segment of  $\mathbb{N}$ ,  $p_i \in C([t_0, \infty), [0, \infty))$ ,  $\tau_i \in [0, \infty)$  for all  $i \in \mathcal{I}$ ,  $\{q_n\}_{n \in \mathbb{N}} \subset (-\infty, 1)$ ,  $\{\theta_n\}_{n \in \mathbb{N}} \subset [t_0, \infty)$  is the increasing unbounded sequence of impulse points,  $\Delta_{\rho}$  is the forward difference operator with the step size  $\rho$ , and  $\Delta$  is the jump of the solution at the specified impulse point.

## 1 Introduction

This paper is concerned with the oscillatory nature of solutions of the following difference equation with continuous arguments:

$$\begin{cases} \Delta_{\rho}x(t) + \sum_{i \in \mathcal{I}} p_i(t)x(t - \tau_i) = 0 & \text{for } t \in [t_0, \infty) \setminus \{\theta_n\}_{n \in \mathbb{N}} \\ \Delta x(\theta_n) + q_n x(\theta_n) = 0 & \text{for } n \in \mathbb{N}, \end{cases} \quad (1)$$

where  $t_0 \in \mathbb{R}$ ,  $\rho \in (0, \infty)$ ,  $\mathcal{I}$  is a bounded beginning segment of  $\mathbb{N}$ ,  $p_i \in C([t_0, \infty), [0, \infty))$ ,  $\tau_i \in [0, \infty)$  for all  $i \in \mathcal{I}$ ,  $\{q_n\}_{n \in \mathbb{N}} \subset (-\infty, 1)$  and  $\{\theta_n\}_{n \in \mathbb{N}} \subset (t_0, \infty)$  is the increasing unbounded sequence of impulse points. Here,  $\Delta_{\rho}x(t) := x(t + \rho) - x(t)$  for  $t \in [t_0, \infty)$  and  $\Delta x(\theta_n) := x(\theta_n^+) - x(\theta_n)$  for  $n \in \mathbb{N}$ , where  $x(\theta_n^+)$  denotes the right sided limit of  $x$  at the impulse point  $\theta_n$  for some  $n \in \mathbb{N}$ , and the left sided limits are defined similarly. It's a well-known fact that all solutions of (1) are oscillatory in the absence of a subsequence  $\{\theta_{n_k}\}_{k \in \mathbb{N}}$  such that  $\{q_{n_k}\}_{k \in \mathbb{N}} \subset [1, \infty)$ . In the sequel, for simplicity in the notation, we shall assume that the empty product is the unit.

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In recent papers [8, 9], Wei and Shen studied some oscillation and asymptotic properties of the impulsive difference equation

$$\begin{cases} \Delta_\rho x(t) + p(t)x(t - \tau) = 0 & \text{for } t \in [t_0, \infty) \setminus \{\theta_n\}_{n \in \mathbb{N}} \\ \Delta_\rho x(\theta_n) + q_n x(\theta_n) = 0 & \text{for } n \in \mathbb{N}, \end{cases} \quad (2)$$

where  $t_0, \rho, p, \tau$  (with  $\tau/\rho \in \mathbb{N}$ ),  $\{q_n\}_{n \in \mathbb{N}}$  and  $\{\theta_n\}_{n \in \mathbb{N}}$  are as mentioned for (1). The method in those papers are analogues to those applied for the discrete case (for usual difference equations), this is why the authors consider use the operator  $\Delta_\rho$  in their impulse condition. Different types of impulse conditions in (1) and (2) indicate that our method employed here will be completely different from the method in [8, 9] since our results will be more closer to those employed for differential equations.

We wish to quote the following result from [6], which will be used in the sequel to obtain comparison results with delay differential equations.

**Theorem A** ([6, Theorem 1]). *Assume that  $\rho \in (0, \infty)$ ,  $p_i \in C([t_0, \infty), [0, \infty))$  and  $\tau_i \in [0, \infty)$  for all  $i \in \mathcal{I}$ , where  $\mathcal{I}$  is a bounded beginning segment of  $\mathbb{N}$ . Then, every solution of the nonimpulsive delay difference equation*

$$\Delta_\rho x(t) + \sum_{i \in \mathcal{I}} p_i(t)x(t - \tau_i) = 0 \quad \text{for } t \in [t_0, \infty)$$

*is oscillatory if every solution of the following delay differential equation*

$$x'(t) + \frac{1}{\rho} \sum_{i \in \mathcal{I}} \min_{t-2\rho \leq \zeta \leq t-\rho} \{p_i(\zeta)\} x(t - \tau_i) = 0 \quad \text{for } t \in [t_0 + 2\rho, \infty)$$

*is oscillatory.*

Let  $\tau_{\max} := \max\{\tau_i : i \in \mathcal{I}\}$ . By a *solution* of (1), we mean a function  $x : [t_0 - \tau_{\max}, \infty) \rightarrow \mathbb{R}$  such that  $x$  is continuous on  $(\theta_n, \theta_{n+1})$  for all  $n \in \mathbb{N}$  and satisfies (1), and that  $x(\theta_n^\pm)$  exists as a finite constant with  $x(\theta_n^-) = x(\theta_n)$  for all  $n \in \mathbb{N}$ . From now on, to make the definition of the solution consistent, we shall assume that  $t \notin \{\theta_n\}_{n \in \mathbb{N}}$  implies  $t + \rho \notin \{\theta_n\}_{n \in \mathbb{N}}$  and  $t - \tau_i \notin \{\theta_n\}_{n \in \mathbb{N}}$  for all  $i \in \mathcal{I}$ . Together with the impulsive delay difference equation (1), it is customary to specify an *initial condition* of the form

$$x = \varphi \quad \text{on } [t_0 - \tau_{\max}, t_0 + \rho], \quad (3)$$

where the *initial function*  $\varphi$  is a prescribed continuous real-valued function on the interval  $[t_0 - \tau_{\max}, t_0 + \rho]$  satisfying the *consistency condition*

$$\Delta_\rho x(t_0) + \sum_{i \in \mathcal{I}} p_i(t_0)x(t_0 - \tau_i) = 0. \quad (4)$$

By the method of steps, one can easily conclude that (1) admits a unique solution  $x$  which satisfies the initial condition (3) and the consistency condition (4). For convenience, we denote this solution by  $x = x(t, t_0, \varphi)$ . As is customary, a solution  $x$  of (1) is called nonoscillatory if it is eventually of fixed sign, otherwise, it is called oscillatory.



## 2 Main results

From now on, for convenience in the results, we shall suppose that there exist positive constants  $\alpha$  and  $\beta_i$  for  $i \in \mathcal{I}$  such that  $\prod_{t-\rho \leq \theta_k < t} (1 - q_k) \equiv \alpha$  for all  $t \in [t_0 + \rho, \infty)$  and  $\prod_{t-\tau_i \leq \theta_k < t} (1 - q_k) \equiv \beta_i$  for all  $t \in [t_0 + \tau_i, \infty)$  and all  $i \in \mathcal{I}$ . Consider the following nonimpulsive delay difference equation:

$$\Delta_\rho x(t) + \sum_{i \in \mathcal{I}} \frac{\alpha^{\tau_i/\rho}}{\beta_i} p_i(t) x(t - \tau_i) = 0 \quad \text{for } t \in [t_0, \infty). \quad (5)$$

As is customary, a *solution* of the nonimpulsive equation (5) is a function  $x \in C([t_0 - \tau, \infty), \mathbb{R})$  satisfying (5) on  $[t_0, \infty)$ .

**Theorem 1.** *If  $y = y(t, t_0, \varphi)$  is a solution of (5), then  $x = x(t, t_0, \varphi)$  defined by*

$$x(t) := \frac{1}{\alpha^{t/\rho}} \left[ \prod_{t_0 \leq \theta_k < t} (1 - q_k) \right] y(t) \quad \text{for } t \in [t_0, \infty) \quad (6)$$

*is a solution of (1).*

*Proof.* Let  $y = y(t, t_0, \varphi)$  be the solution of (5), we shall prove that  $x$  defined by (6) satisfies (1). It is obvious that  $x$  is continuous on each interval  $(\theta_n, \theta_{n+1})$  for all  $n \in \mathbb{N}$ . From (6), we get

$$\begin{aligned} \Delta_\rho y(t) &= \frac{\alpha^{t/\rho+1}}{\prod_{t_0 \leq \theta_k < t+\rho} (1 - q_k)} x(t + \rho) - \frac{\alpha^{t/\rho}}{\prod_{t_0 \leq \theta_k < t} (1 - q_k)} x(t) \\ &= \frac{\alpha^{t/\rho}}{\prod_{t_0 \leq \theta_k < t} (1 - q_k)} \Delta_\rho x(t) \end{aligned} \quad (7)$$

and

$$\begin{aligned} y(t - \tau_i) &= \frac{\alpha^{t/\rho}}{\alpha^{\tau_i/\rho} \prod_{t_0 \leq \theta_k < t - \tau_i} (1 - q_k)} x(t - \tau_i) \\ &= \frac{\beta_i \alpha^{t/\rho}}{\alpha^{\tau_i/\rho} \prod_{t_0 \leq \theta_k < t} (1 - q_k)} x(t - \tau_i) \end{aligned} \quad (8)$$

for all  $t \in [t_0, \infty)$  and all  $i \in \mathcal{I}$ . Substituting (7) and (8) into (5) and canceling the positive term  $\alpha^{t/\rho} / \prod_{t_0 \leq \theta_k < t} (1 - q_k)$ , we see that  $x$  defined by (6) solves the former equation in (1). On the other hand, for all  $n \in \mathbb{N}$ , we have

$$x(\theta_n^+) = \lim_{t \rightarrow \theta_n^+} \left( \frac{1}{\alpha^{t/\rho}} \left[ \prod_{t_0 \leq \theta_k < t} (1 - q_k) \right] y(t) \right) = \frac{1}{\alpha^{\theta_n/\rho}} \left[ \prod_{t_0 \leq \theta_k \leq \theta_n} (1 - q_k) \right] y(\theta_n) = (1 - q_n) x(\theta_n),$$

which shows that  $x$  satisfies the latter equation in (1) too. The proof is therefore completed.  $\square$

The following result can be regarded as the converse part of Theorem 1.

**Theorem 2.** *If  $x = x(t, t_0, \varphi)$  is a solution of (1), then  $y = y(t, t_0, \varphi)$  defined by*

$$y(t) = \frac{\alpha^{t/\rho}}{\prod_{t_0 \leq \theta_k < t} (1 - q_k)} x(t) \quad \text{for } t \in [t_0, \infty) \quad (9)$$

*is a solution of (5).*

*Proof.* Let  $x = x(t, t_0, \varphi)$  be a solution of (1). Since  $y$  defined by (9) is continuous on the intervals  $(\theta_n, \theta_{n+1})$  with  $y(\theta_n^-) = y(\theta_n)$  for all  $n \in \mathbb{N}$ , it follows that

$$y(\theta_n^+) = \lim_{t \rightarrow \theta_n^+} \left( \frac{\alpha^{t/\rho}}{\prod_{t_0 \leq \theta_k < t} (1 - q_k)} x(t) \right) = \frac{\alpha^{\theta_n/\rho}}{\prod_{t_0 \leq \theta_k \leq \theta_n} (1 - q_k)} x(\theta_n^+) = y(\theta_n)$$

for all  $n \in \mathbb{N}$ , which implies that  $x$  is continuous on  $[t_0, \infty)$ . It is not hard to see that  $y$  solves (5). This completes the proof.  $\square$

**Theorem 3.** *Every solution of (1) oscillates if and only if every solution of (5) oscillates.*

*Proof.* The proof follows from Theorem 1, Theorem 2 and the fact that  $\{q_n\}_{n \in \mathbb{N}} \subset (-\infty, 1)$ .  $\square$

Using Theorem A and Theorem 3, we can give the following oscillation result.

**Theorem 4.** *Every solution of (1) oscillates if every solution of the following delay differential equation*

$$x'(t) + \sum_{i \in \mathcal{I}} \frac{\alpha^{\tau_i/\rho}}{\rho \beta_i} \min_{t-2\rho \leq \zeta \leq t-\rho} \{p_i(\zeta)\} x(t - \tau_i) = 0 \quad \text{for } t \in [t_0 + 2\rho, \infty)$$

*oscillates.*

As an immediate consequence of Theorem 4, we have the following corollary.

**Corollary 1** (See [1, 2]). *Assume that*

$$\liminf_{t \rightarrow \infty} \int_{t-\tau_{\min}}^t \sum_{i \in \mathcal{I}} \frac{\alpha^{\tau_i/\rho}}{\rho \beta_i} \min_{\eta-2\rho \leq \zeta \leq \eta-\rho} \{p_i(\zeta)\} d\eta \frac{1}{e}$$

*or*

$$\limsup_{t \rightarrow \infty} \int_{t-\tau_{\min}}^t \sum_{i \in \mathcal{I}} \frac{\alpha^{\tau_i/\rho}}{\rho \beta_i} \min_{\eta-2\rho \leq \zeta \leq \eta-\rho} \{p_i(\zeta)\} d\eta 1,$$

*where  $\tau_{\min} := \min\{\tau_i : i \in \mathcal{I}\}$ . Then, every solution of (1) oscillates.*

We give the following example as a simple application of Theorem 4.

**Example 1.** *Let  $\mathcal{I}$  be a bounded beginning segment of  $\mathbb{N}$ ,  $\rho(0, \infty)$ ,  $q \in (-\infty, 1)$ ,  $p_i \in (0, \infty)$  and  $\tau_i \in \mathbb{N}$  for all  $i \in \mathcal{I}$ , and consider the following autonomous difference equation*

$$\begin{cases} \Delta_\rho x(t) + \sum_{i \in \mathcal{I}} p_i x(t - \rho \tau_i) = 0 & \text{for } t \in [0, \infty) \setminus \rho\mathbb{N} \\ \Delta x(n) + qx(n) = 0 & \text{for } n \in \rho\mathbb{N}. \end{cases} \quad (10)$$

*Due to Theorem 4, since  $\alpha = (1 - q)$  and  $\beta_i = (1 - q)^{\tau_i}$  for all  $i \in \mathcal{I}$ , the associated differential equation with (10) is*

$$x'(t) + \sum_{i \in \mathcal{I}} \frac{p_i}{\rho} x(t - \rho \tau_i) = 0 \quad \text{for } t \in [2\rho, \infty). \quad (11)$$

*From [1, Theorem 2.2.1], we learn that if*

$$\sum_{i \in \mathcal{I}} p_i \tau_i \frac{1}{e},$$

*then every solution of (11) oscillates, which implies the same for (10).*

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## Non-polynomial spline method of a non-linear system of second-order boundary value problems

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### Abstract

The non-polynomial spline method is proposed to solve a non-linear system of the second-order boundary value problems (BVPs). Some numerical results are given to demonstrate the validity and applicability of the presented method. Results obtained by the method indicate the method is simple and effective.

*Keywords:* Second-order boundary value problems; non-polynomial spline method; non-linear system.

### 1. Introduction

We consider a non-linear system of second-order BVPs of the form [1,2,3,5,6]:

$$\left. \begin{aligned} u'' + a_1(x)u' + a_2(x)u + a_3(x)v' + a_4(x)v + H_1(x, u, v) &= f_1(x), \\ v'' + b_1(x)v' + b_2(x)v + b_3(x)u' + b_4(x)u + H_2(x, u, v) &= f_2(x), \end{aligned} \right\} \quad (1)$$

with the following boundary conditions

$$u(0) = u(1) = 0, v(0) = v(1) = 0 \quad (2)$$

where  $0 < x < 1$ ,  $H_1, H_2$  are nonlinear functions of  $u$  and  $v$ ,  $a_i(x), b_i(x), f_1(x)$ , and  $f_2(x)$ , are given functions, and  $a_i(x), b_i(x)$  are continuous,  $i = 1, 2, 3, 4$ .

The existence and approximations of the solutions to non-linear systems of second-order BVPs have investigated by many authors[1-6]. In [1] the

sinc-collocation method is presented for solving second-order systems. Their method consists of reducing the solution of Eq.(1) to a set of algebraic equations by expanding  $u(x)$  and  $v(x)$  as sinc functions with unknown coefficients. New method is presented to solve Eq.(1) used in the form of series in the reproducing kernel space in [2]. The variation iteration method is applied for the solution with the assumption that the solutions are unique in [3]. He's homotopy perturbation method (HPM) is proposed for the solution of systems in [5]. A new modification of the homotopy analysis method (HAM) is presented for solving systems of second-order BVPs in [6].

The section of this paper are organized as follows: In the next section we describe the basic formulation of the spline function required for our subsequent development. In section 3 the method are used to analysis to solution of problem (1) and (2). In section 4 some numerical result, that are illustrated using MATLAB 6.5, are given to clarify the method. Section 5 ends this paper with a brief conclusion. Note that we have computed the numerical results by MATLAB 6.5.

## 2. Spline method

We divide the interval  $[a, b]$  into  $n$  equal subintervals using the grid points

$$x_i = a + ih, i = 0, 1, 2, \dots, n,$$

with

$$a = x_0, x_n = b, h = (b - a)/n$$

where  $n$  is an arbitrary positive integer.

Let  $u(x)$  be the exact solution and  $u_i$  be an approximation to  $u(x_i)$  obtained by the non-polynomial cubic  $S_i(x)$  passing through the points  $(x_i, u_i)$  and  $(x_{i+1}, u_{i+1})$ , we do not only require that  $S_i(x)$  satisfies interpolatory conditions at  $x_i$  and  $x_{i+1}$ , but also the continuity of first derivative at the common nodes  $(x_i, u_i)$  are fulfilled. We write  $S_i(x)$  in the form:

$$S_i(x) = a_i + b_i(x - x_i) + c_i \sin \tau(x - x_i) + d_i \cos \tau(x - x_i), i = 0, 1, \dots, n - 1 \quad (3)$$

where  $a_i, b_i, c_i$  and  $d_i$  are constants and  $\tau$  is a free parameter.

A non-polynomial function  $S(x)$  of class  $C^2[a, b]$  interpolates  $u(x)$  at the grid points  $x_i, i = 0, 1, 2, \dots, n$ , depends on a parameter  $\tau$ , and reduces to ordinary cubic spline  $S(x)$  in  $[a, b]$  as  $\tau \rightarrow 0$ .

To derive expression for the coefficients of Eq. (3) in term of  $u_i, u_{i+1}, M_i$  and  $M_{i+1}$ , we first define:

$$S_i(x_i) = u_i, S_i(x_{i+1}) = u_{i+1}, S''(x_i) = M_i, S''(x_{i+1}) = M_{i+1}. \quad (4)$$

From algebraic manipulation, we get the following expression:

$$\left. \begin{aligned} a_i &= u_i + \frac{M_i}{\tau^2}, \\ b_i &= \frac{u_{i+1} - u_i}{h} + \frac{M_{i+1} - M_i}{\tau\theta}, \\ c_i &= \frac{M_i \cos\theta - M_{i+1}}{\tau^2 \sin\theta}, \\ d_i &= -\frac{M_i}{\tau^2}, \end{aligned} \right\} \quad (5)$$

where  $\theta = \tau h$  and  $i = 0, 1, 2, \dots, n-1$ .

Using the continuity of the first derivative at  $(x_i, u_i)$ , that is  $S'_{i-1}(x_i) = S'_i(x_i)$  we obtain the following relations for  $i=1, \dots, n-1$ .

$$\alpha M_{i+1} + 2\beta M_i + \alpha M_{i-1} = (1/h^2)(u_{i+1} - 2u_i + u_{i-1}) \quad (6a)$$

It is easy to see that  $v(x)$  is written as in the same manner

$$\alpha N_{i+1} + 2\beta N_i + \alpha N_{i-1} = (1/h^2)(v_{i+1} - 2v_i + v_{i-1}) \quad (6b)$$

where  $\alpha = (-1/\theta^2 + 1/\theta \sin \theta)$ ,  $\beta = (1/\theta^2 - \cos \theta / \theta \sin \theta)$  and  $\theta = \tau h$ .

The method is fourth-order convergent if  $1 - 2\alpha - 2\beta = 0$  and  $\alpha = 1/12$  [4].

### 3. Analysis of the method

To illustrate the application of the Spline method developed in the previous section we consider the non-linear system of second-order BVP that is given in Eq. (1). At the grid point  $(x_i, u_i)$ , the proposed non-linear system of second-order BVP in Eq. (1) may be discretized by

$$\left. \begin{aligned} u'' + a_1(x_i)u' + a_2(x_i)u + a_3(x_i)v' + a_4(x_i)v + H_1(x, u, v) &= f_1(x_i), \\ v'' + b_1(x_i)v' + b_2(x_i)v + b_3(x_i)u' + b_4(x_i)u + H_2(x, u, v) &= f_2(x_i). \end{aligned} \right\} \quad (7)$$

Substituting  $M_i = u''$  and  $N_i = v''$  in equation system (5):

$$\left. \begin{aligned} M_i + a_1(x_i)u'_i + a_2(x_i)u_i + a_3(x_i)v'_i + a_4(x_i)v_i + H_1(x_i, u_i, v_i) &= f_1(x_i), \\ N_i + b_1(x_i)v'_i + b_2(x_i)v_i + b_3(x_i)u'_i + b_4(x_i)u_i + H_2(x_i, u_i, v_i) &= f_2(x_i). \end{aligned} \right\} \quad (8)$$

Solving Eq. (8) for  $M_i$  and  $N_i$ , we get

$$\left. \begin{aligned} M_i &= -a_1(x_i)u'_i - a_2(x_i)u_i - a_3(x_i)v'_i - a_4(x_i)v_i - H_1(x_i, u_i, v_i) + f_1(x_i) \\ N_i &= -b_1(x_i)v'_i - b_2(x_i)v_i - b_3(x_i)u'_i - b_4(x_i)u_i - H_2(x_i, u_i, v_i) + f_2(x_i) \end{aligned} \right\} \quad (9)$$

The following approximations for the first-order derivative of  $u$  and  $v$  in Eq. (9) can be used

$$\left. \begin{aligned} u'_i &\cong \frac{u_{i+1} - u_{i-1}}{2h}, \\ u'_{i+1} &\cong \frac{3u_{i+1} - 4u_i + u_{i-1}}{2h}, \\ u'_{i-1} &\cong \frac{-u_{i+1} + 4u_i - 3u_{i-1}}{2h}, \\ v'_i &\cong \frac{v_{i+1} - v_{i-1}}{2h}, \\ v'_{i+1} &\cong \frac{3v_{i+1} - 4v_i + v_{i-1}}{2h}, \\ v'_{i-1} &\cong \frac{-v_{i+1} + 4v_i - 3v_{i-1}}{2h}. \end{aligned} \right\} \quad (10)$$

So Eq. (9) becomes

$$\left. \begin{aligned} M_i &= -a_1(x_i)\frac{u_{i+1} - u_{i-1}}{2h} - a_2(x_i)u_i - a_3(x_i)\frac{v_{i+1} - v_{i-1}}{2h} \\ &\quad - a_4(x_i)v_i - H_1(x_i, u_i, v_i) + f_1(x_i) \end{aligned} \right\} \quad (11a)$$

$$\left. \begin{aligned} M_{i+1} &= -a_1(x_{i+1})\frac{3u_{i+1} - 4u_i + u_{i-1}}{2h} - a_2(x_{i+1})u_i - a_3(x_{i+1})\frac{3v_{i+1} - 4v_i + v_{i-1}}{2h} \\ &\quad - a_4(x_{i+1})v_i - H_1(x_{i+1}, u_{i+1}, v_{i+1}) + f_1(x_{i+1}) \end{aligned} \right\} \quad (11b)$$

$$\left. \begin{aligned} M_{i-1} &= -a_1(x_{i-1})\frac{-u_{i+1} + 4u_i - 3u_{i-1}}{2h} - a_2(x_{i-1})u_i - a_3(x_{i-1})\frac{-v_{i+1} + 4v_i - 3v_{i-1}}{2h} \\ &\quad - a_4(x_{i-1})v_i - H_1(x_{i-1}, u_{i-1}, v_{i-1}) + f_1(x_{i-1}) \end{aligned} \right\} \quad (11c)$$

and

$$\left. \begin{aligned} N_i &= -b_1(x_i)\frac{v_{i+1} - v_{i-1}}{2h} - b_2(x_i)v_i - b_3(x_i)\frac{u_{i+1} - u_{i-1}}{2h} \\ &\quad - b_4(x_i)u_i - H_2(x_i, u_i, v_i) + f_2(x_i) \end{aligned} \right\} \quad (12a)$$

$$N_{i+1} = -b_1(x_{i+1})\frac{3v_i+1-4v_i+v_{i-1}}{2h} - b_2(x_{i+1})v_i - b_3(x_{i+1})\frac{3u_i+1-4u_i+u_{i-1}}{2h} - b_4(x_{i+1})u_i - H_2(x_{i+1}, u_{i+1}, v_{i+1}) + f_2(x_{i+1}) \quad (12b)$$

$$N_{i-1} = -b_1(x_{i-1})\frac{-v_i+1+4v_i-3v_{i-1}}{2h} - b_2(x_{i-1})v_i - b_3(x_{i-1})\frac{-u_{i+1}+4u_i-3u_{i-1}}{2h} - b_4(x_{i-1})u_i - H_2(x_{i-1}, u_{i-1}, v_{i-1}) + f_2(x_{i-1}) \quad (12c)$$

Substituting Eqs. (11a-11c)-(12a-12c) in Eqs. (6a) and (6b) respectively, we find the following  $2(n-1)$  linear algebraic equations in the  $2(n+1)$  unknowns for  $i = 0, 1, \dots, n$ .

$$\left\{ \begin{aligned} & \left[ \frac{\alpha a_1(x_{i-1})}{2h} - \frac{2\beta a_1(x_i)}{2h} - \frac{3\alpha a_1(x_{i+1})}{2h} - \alpha a_2(x_{i+1}) - \frac{1}{h^2} \right] u_{i+1} \\ & + \left[ \frac{-4\alpha a_1(x_{i-1})}{2h} - 2\beta a_2(x_i) + \frac{4\alpha a_1(x_{i+1})}{2h} + \frac{2}{h^2} \right] u_i \\ & + \left[ \frac{3\alpha a_1(x_{i-1})}{2h} - \alpha a_2(x_{i-1}) + \frac{2\beta a_1(x_i)}{2h} - \frac{\alpha a_1(x_{i+1})}{2h} - \frac{1}{h^2} \right] u_{i-1} \\ & + \left[ \frac{\alpha a_3(x_{i-1})}{2h} - \frac{2\beta a_3(x_i)}{2h} - \frac{3\alpha a_3(x_{i+1})}{2h} - \alpha a_4(x_{i+1}) \right] v_{i+1} \\ & + \left[ \frac{-4\alpha a_3(x_{i-1})}{2h} - 2\beta a_4(x_i) + \frac{4\alpha a_3(x_{i+1})}{2h} \right] v_i \\ & + \left[ \frac{3\alpha a_3(x_{i-1})}{2h} - \alpha a_4(x_{i-1}) + \frac{2\beta a_3(x_i)}{2h} - \frac{\alpha a_3(x_{i+1})}{2h} \right] v_{i-1} \\ & - \alpha H_1(x_{i-1}, u_{i-1}, v_{i-1}) - 2\beta H_1(x_i, u_i, v_i) - \alpha H_1(x_{i+1}, u_{i+1}, v_{i+1}) = \\ & - \alpha f_1(x_{i-1}) - 2\beta f_1(x_i) - \alpha f_1(x_{i+1}) \end{aligned} \right\} \quad (13)$$

and

$$\left\{ \begin{aligned} & \left[ \frac{\alpha b_1(x_{i-1})}{2h} - \frac{2\beta b_1(x_i)}{2h} - \frac{3\alpha b_1(x_{i+1})}{2h} - \alpha b_2(x_{i+1}) - \frac{1}{h^2} \right] v_{i+1} \\ & + \left[ \frac{-4\alpha b_1(x_{i-1})}{2h} - 2\beta b_2(x_i) + \frac{4\alpha b_1(x_{i+1})}{2h} + \frac{2}{h^2} \right] v_i \\ & + \left[ \frac{3\alpha b_1(x_{i-1})}{2h} - \alpha b_2(x_{i-1}) + \frac{2\beta b_1(x_i)}{2h} - \frac{\alpha b_1(x_{i+1})}{2h} - \frac{1}{h^2} \right] v_{i-1} \\ & + \left[ \frac{\alpha b_3(x_{i-1})}{2h} - \frac{2\beta b_3(x_i)}{2h} - \frac{3\alpha b_3(x_{i+1})}{2h} - \alpha b_4(x_{i+1}) \right] u_{i+1} \\ & + \left[ \frac{-4\alpha b_3(x_{i-1})}{2h} - 2\beta b_4(x_i) + \frac{4\alpha b_3(x_{i+1})}{2h} \right] u_i \\ & + \left[ \frac{3\alpha b_3(x_{i-1})}{2h} - \alpha b_4(x_{i-1}) + \frac{2\beta b_3(x_i)}{2h} - \frac{\alpha b_3(x_{i+1})}{2h} \right] u_{i-1} \\ & - \alpha H_2(x_{i-1}, u_{i-1}, v_{i-1}) - 2\beta H_2(x_i, u_i, v_i) - \alpha H_2(x_{i+1}, u_{i+1}, v_{i+1}) = \\ & - \alpha f_2(x_{i-1}) - 2\beta f_2(x_i) - \alpha f_2(x_{i+1}) \end{aligned} \right\} \quad (14)$$

We need four more equations. The four end conditions can be derivated as follows:

$$u_0 = 0, u_n = 0, v_0 = 0, v_n = 0 \quad (15)$$

This leads to the system



$$X_{1i} = \frac{\alpha a_1(x_{i-1})}{2h} - \frac{2\beta a_1(x_i)}{2h} - \frac{3\alpha a_1(x_{i+1})}{2h} - \alpha a_2(x_{i+1}) - \frac{1}{h^2} \quad (16)$$

$$Y_{1i} = \frac{-4\alpha a_1(x_{i-1})}{2h} - 2\beta a_2(x_i) + \frac{4\alpha a_1(x_{i+1})}{2h} + \frac{2}{h^2} \quad (17)$$

$$Z_{1i} = \frac{3\alpha a_1(x_{i-1})}{2h} - \alpha a_2(x_{i-1}) + \frac{2\beta a_1(x_i)}{2h} - \frac{\alpha a_1(x_{i+1})}{2h} - \frac{1}{h^2} \quad (18)$$

$$X_{2i} = \frac{\alpha a_3(x_{i-1})}{2h} - \frac{2\beta a_3(x_i)}{2h} - \frac{3\alpha a_3(x_{i+1})}{2h} - \alpha a_4(x_{i+1}) \quad (19)$$

$$Y_{2i} = \frac{-4\alpha a_3(x_{i-1})}{2h} - 2\beta a_4(x_i) + \frac{4\alpha a_3(x_{i+1})}{2h} \quad (20)$$

$$Z_{2i} = \frac{3\alpha a_3(x_{i-1})}{2h} - \alpha a_4(x_{i-1}) + \frac{2\beta a_3(x_i)}{2h} - \frac{\alpha a_3(x_{i+1})}{2h} \quad (21)$$

$$g_i = \frac{a_1(x_i)}{2h}, \quad h_i = a_2(x_i), \quad k_i = \frac{a_3(x_i)}{2h}, \quad l_i = a_4(x_i) \quad (22)$$

$$X_{1i} = \alpha g_{i-1} - 2\beta g_i - 3\alpha g_{i+1} - \alpha h_{i+1} - \frac{1}{h^2} \quad (23)$$

$$Y_{1i} = -4\alpha g_{i-1} + 4\alpha g_{i+1} - 2\beta h_i + \frac{2}{h^2} \quad (24)$$

$$Z_{1i} = 3\alpha g_{i-1} + 2\beta g_i - \alpha g_{i+1} - \alpha h_{i-1} - \frac{1}{h^2} \quad (25)$$

$$X_{2i} = \alpha k_{i-1} - 2\beta k_i - 3\alpha k_{i+1} - \alpha l_{i+1} \quad (26)$$

$$Y_{2i} = -4\alpha k_{i-1} + 4\alpha k_{i+1} - 2\beta l_i \quad (27)$$

$$Z_{2i} = 3\alpha k_{i-1} + 2\beta k_i - \alpha k_{i+1} - \alpha l_{i-1} \quad (28)$$

$$X_{3i} = \frac{\alpha b_3(x_{i-1})}{2h} - \frac{2\beta b_3(x_i)}{2h} - \frac{3\alpha b_3(x_{i+1})}{2h} - \alpha b_4(x_{i+1}) \quad (29)$$

$$Y_{3i} = \frac{-4\alpha b_3(x_{i-1})}{2h} - 2\beta b_4(x_i) + \frac{4\alpha b_3(x_{i+1})}{2h} \quad (30)$$

$$Z_{3i} = \frac{3\alpha b_3(x_{i-1})}{2h} - \alpha b_4(x_{i-1}) + \frac{2\beta b_3(x_i)}{2h} - \frac{\alpha b_3(x_{i+1})}{2h} \quad (31)$$

$$X_{4i} = \frac{\alpha b_1(x_{i-1})}{2h} - \frac{2\beta b_1(x_i)}{2h} - \frac{3\alpha b_1(x_{i+1})}{2h} - \alpha b_2(x_{i+1}) - \frac{1}{h^2} \quad (32)$$

$$Y_{4i} = \frac{-4\alpha b_1(x_{i-1})}{2h} - 2\beta b_2(x_i) + \frac{4\alpha b_1(x_{i+1})}{2h} + \frac{2}{h^2} \quad (33)$$

$$Z_{4i} = \frac{3\alpha b_1(x_{i-1})}{2h} - \alpha b_2(x_{i-1}) + \frac{2\beta b_1(x_i)}{2h} - \frac{\alpha b_1(x_{i+1})}{2h} - \frac{1}{h^2} \quad (34)$$

$$m_i = \frac{b_1(x_i)}{2h}, \quad p_i = b_2(x_i), \quad r_i = \frac{b_3(x_i)}{2h}, \quad s_i = b_4(x_i) \quad (35)$$

$$X_{3i} = \alpha r_{i-1} - 2\beta r_i - 3\alpha r_{i+1} - \alpha s_{i+1} \quad (36)$$

$$Y_{3i} = -4\alpha r_{i-1} + 4\alpha r_{i+1} - 2\beta s_i \quad (37)$$

$$Z_{3i} = 3\alpha r_{i-1} + 2\beta r_i - \alpha r_{i+1} - \alpha s_{i-1} \quad (38)$$

$$X_{4i} = \alpha m_{i-1} - 2\beta m_i - 3\alpha m_{i+1} - \alpha p_{i+1} - \frac{1}{h^2} \quad (39)$$

$$Y_{4i} = -4\alpha m_{i-1} + 4\alpha m_{i+1} - 2\beta p_i + \frac{2}{h^2} \quad (40)$$

$$Z_{4i} = 3\alpha m_{i-1} + 2\beta m_i - \alpha m_{i+1} - \alpha p_{i-1} - \frac{1}{h^2} \quad (41)$$

The method is described in matrix form in the following way for Eqs. (16)-(41):

$$A = \left[ \begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array} \right], \quad (42)$$

$$B = \left[ \begin{array}{c} 0 \\ -\alpha f_1(x_0) - 2\beta f_1(x_1) - \alpha f_1(x_2) \\ -\alpha f_1(x_1) - 2\beta f_1(x_2) - \alpha f_1(x_3) \\ \vdots \\ \vdots \\ \vdots \\ -\alpha f_1(x_{n-2}) - 2\beta f_1(x_{n-1}) - \alpha f_1(x_n) \\ 0 \\ 0 \\ -\alpha f_2(x_0) - 2\beta f_2(x_1) - \alpha f_2(x_2) \\ -\alpha f_2(x_1) - 2\beta f_2(x_2) - \alpha f_2(x_3) \\ \vdots \\ \vdots \\ \vdots \\ -\alpha f_2(x_{n-2}) - 2\beta f_2(x_{n-1}) - \alpha f_2(x_n) \\ 0 \end{array} \right], \quad (43)$$

$$H = \begin{bmatrix} 0 \\ -\alpha H_1(x_0, u_0, v_0) - 2\beta H_1(x_1, u_1, v_1) - \alpha H_1(x_2, u_2, v_2) \\ -\alpha H_1(x_1, u_1, v_1) - 2\beta H_1(x_2, u_2, v_2) - \alpha H_1(x_3, u_3, v_3) \\ \vdots \\ \vdots \\ -\alpha H_1(x_{n-2}, u_{n-2}, v_{n-2}) - 2\beta H_1(x_{n-1}, u_{n-1}, v_{n-1}) - \alpha H_1(x_n, u_n, v_n) \\ 0 \\ 0 \\ -\alpha H_2(x_0, u_0, v_0) - 2\beta H_2(x_1, u_1, v_1) - \alpha H_2(x_2, u_2, v_2) \\ -\alpha H_2(x_1, u_1, v_1) - 2\beta H_2(x_2, u_2, v_2) - \alpha H_2(x_3, u_3, v_3) \\ \vdots \\ \vdots \\ \vdots \\ -\alpha H_2(x_{n-2}, u_{n-2}, v_{n-2}) - 2\beta H_2(x_{n-1}, u_{n-1}, v_{n-1}) - \alpha H_2(x_n, u_n, v_n) \\ 0 \end{bmatrix}, \quad (44)$$

$$U = [u_0, u_1, \dots, u_n, v_0, v_1, \dots, v_n]'. \quad (45)$$

Here the four submatrices  $A_1, A_2, A_3$  and  $A_4$  are defined as

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ X_{11} & Y_{11} & Z_{11} & 0 & \dots & 0 & 0 \\ 0 & X_{12} & Y_{12} & Z_{12} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & X_{1(n-2)} & Y_{1(n-2)} & Z_{1(n-2)} \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 & 1 \end{bmatrix}, \quad (46)$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ X_{21} & Y_{21} & Z_{21} & 0 & \dots & 0 & 0 \\ 0 & X_{22} & Y_{22} & Z_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & X_{2(n-2)} & Y_{2(n-2)} & Z_{2(n-2)} \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 & 0 \end{bmatrix}, \quad (47)$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ X_{31} & Y_{31} & Z_{31} & 0 & \dots & 0 & 0 \\ 0 & X_{32} & Y_{32} & Z_{32} & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & 0 & X_{3(n-2)} & Y_{3(n-2)} & Z_{3(n-2)} \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \end{bmatrix}, \quad (48)$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ X_{41} & Y_{41} & Z_{41} & 0 & \dots & 0 & 0 \\ 0 & X_{42} & Y_{42} & Z_{42} & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & 0 & X_{4(n-2)} & Y_{4(n-2)} & Z_{4(n-2)} \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 & 1 \end{bmatrix}. \quad (49)$$

Finally the approximate solution is obtained by solving the nonlinear system using Levenberg-Marquardt optimization method [7] and Matlab 6.5.

$$AU + H = B. \quad (50)$$

#### 4. Numerical examples

In this section, to illustrate our methods we have solved two non-linear system of second-order BVP . All computations are done by using MATLAB 6.5.

##### Example 1.

Consider the following equations

$$\left. \begin{aligned} u''(x) - xv'(x) + u(x) &= f_1(x) \\ v''(x) + xu'(x) + u(x)v(x) &= f_2(x) \end{aligned} \right\} \quad (51)$$

subject to the boundary conditions

$$u(0) = u(1) = 0, v(0) = v(1) = 0, \quad (52)$$

where  $0 < x < 1$ ,  $f_1(x) = x^3 - 2x^2 + 6x$  and  $f_2(x) = x^2 - x$ .

The exact solutions of  $u(x)$  and  $v(x)$  are given as  $x^3 - x$  and  $x^2 - x$  respectively. The observed maximum absolute errors of  $u(x)$  and  $v(x)$  for  $n = 21$  (nodal points) are given in Table 1. The numerical results of  $u(x)$  and  $v(x)$  are also illustrated in Figures 1 and 2.

**Example 2.** We consider the following equations

$$\left. \begin{aligned} u''(x) + xu(x) + xu^2(x) &= f_1(x) \\ v''(x) + xu'(x) + v(x) &= f_2(x) \end{aligned} \right\} \quad (53)$$

subject to the boundary conditions

$$u(0) = u(1) = 0, v(0) = v(1) = 0, \quad (54)$$

where  $0 < x < 1$ ,  $f_1(x) = -\pi^2 \sin(\pi x) + x \sin(\pi x)^2 + x^4 - 3x^3 + 2x^2$  and  $f_2(x) = \pi x \cos(\pi x) + x^3 - 3x^2 + 8x - 6$

The exact solutions of  $u(x)$  and  $v(x)$  are given as  $\sin(\pi x)$  and  $x^3 - 3x^2 + 2x$  respectively. The observed maximum absolute errors of  $u(x)$  and  $v(x)$  for different values of  $n$  are given in Table 2. As one clearly observes from Table 2 the magnitude of the errors using higher nodal points ( $n = 61$ ) becomes smaller than the lower ones. The numerical results of  $u(x)$  and  $v(x)$  are also illustrated in Figures 3 and 4.

Table 1: The maximum absolute errors for  $u(x)$  and  $v(x)$  from example 1 .

$n$	$u$ of Abs.Error	$v$ of Abs.Error
11	1.956115354079246e-005	3.107643821670669e-004
21	4.851275161810165e-006	7.771611006257562e-005
41	1.210374921178925e-006	1.946985010448099e-005
61	5.380403360621955e-007	8.653049506895938e-006
121	1.345825387799593e-007	2.163365648760740e-006
211	4.394366903692770e-008	7.064318330030073e-007

Table 2: The maximum absolute errors for  $u(x)$  and  $v(x)$  for different values of nodal point from example 2 .

$n$	$u$ of Abs.Error	$v$ of Abs.Error
11	7.388978904299126e-005	6.124095382239458e-004
21	1.078529634501724e-005	1.593422934084987e-004
41	2.217826760020358e-006	4.012705321820853e-005
61	9.477969471483050e-007	1.787399659253230e-005
121	2.311524547327082e-007	4.471652837373386e-006
211	7.505851140621189e-008	1.460362707594864e-006

## 5. Conclusions

In this paper, the non-polynomial spline method is developed for the approximate solution of nonlinear system of the second-order boundary value problems. The numerical results obtained by using the method described in this study give acceptable results. We have concluded that numerical results converge to the exact solution when  $h$  goes to zero. The results illustrated in Figs. 1, 2, 3 and 4 showed that when  $n$  was increased, the maximum absolute error decreased. Use of spline method has show that it is an applicable new method for solving nonlinear system of BVPs.

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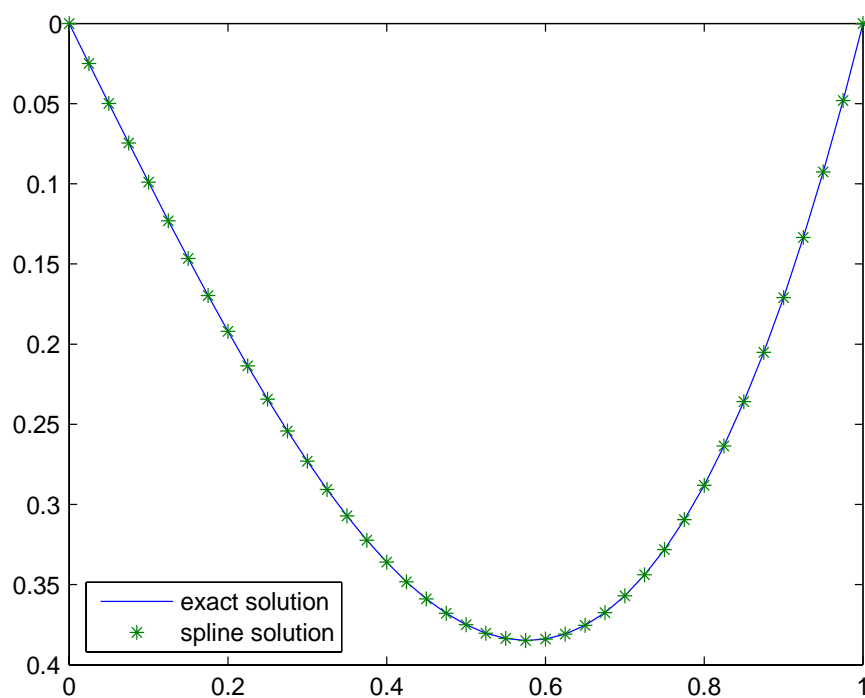


Figure 1: Results for  $u(x)$  in example 1 ( $n = 41$ ). Solid line is the exact solution  $u(x) = x^3 - x$ .



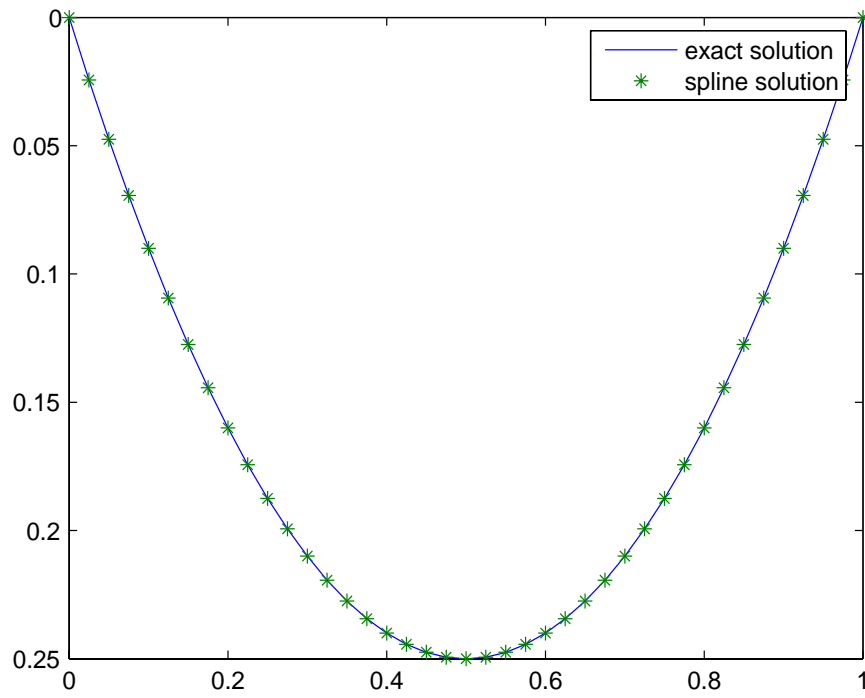


Figure 2: Results for  $v(x)$  in example 1 ( $n = 41$ ). Solid line is the exact solution  $v(x) = x^2 - x$ .

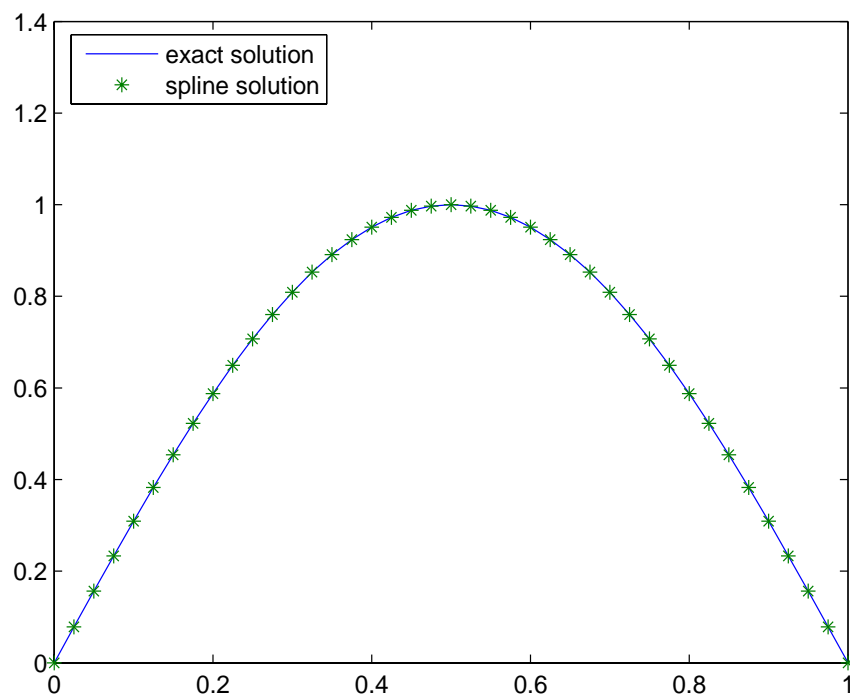


Figure 3: Results for  $u(x)$  in example 2 ( $n = 41$ ). Solid line is the exact solution  $u(x) = \sin(\pi x)$ .

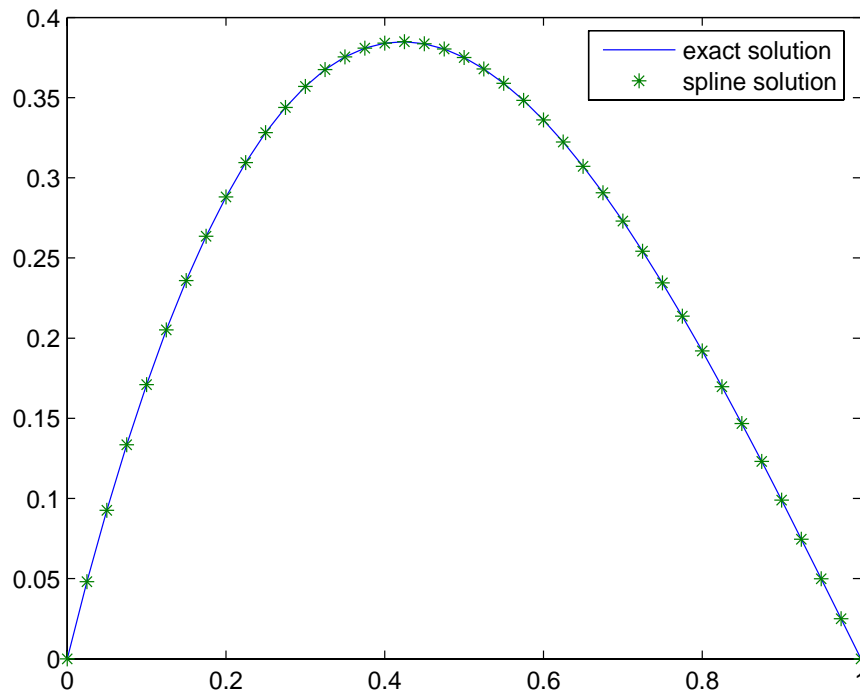


Figure 4: Results for  $v(x)$  in example 2 ( $n = 41$ ). Solid line is the exact solution  $v(x) = x^3 - 3x^2 + 2x$ .

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# GENERALIZED HYERS–ULAM STABILITY OF MULTI-DIMENSIONAL QUADRATIC EQUATIONS <sup>#</sup>

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ABSTRACT. For any fixed  $n \in \mathbb{N}$  with  $n \geq 2$ , we are going to investigate the general solution of the equation

$$2f\left(\sum_{i=1}^n x_i\right) + \sum_{i \neq j} f(x_i - x_j) = 2n \sum_{i=1}^n f(x_i),$$

in the class of all functions between quasi- $\beta$ -normed spaces, and then we are to prove the generalized Hyers–Ulam stability of the equation by using direct method.

## 1. INTRODUCTION

In 1940 and in 1964 S.M. Ulam [22] proposed the famous Ulam stability problem: “When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?” For very general functional equations, the concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation? If the answer is affirmative, we would say that the equation is stable.

In 1941, the first result concerning the stability of functional equations was presented by D.H. Hyers [9]. He has answered the question of Ulam for the case where  $G_1$  and  $G_2$  are Banach spaces. In 1978, Th.M. Rassias [18] provided a generalization of Hyers’ Theorem for approximately linear mappings by considering Cauchy difference to be unbounded. P. Găvruta [8] has generalized the Th.M. Rassias theorem by a general control function of the Cauchy difference.

Let  $E_1$  and  $E_2$  be vector spaces. A function  $f : E_1 \rightarrow E_2$  is called a quadratic function if and only if  $f$  is a solution function of the quadratic functional equation

$$(1.1) \quad f(x+y) + f(x-y) = 2f(x) + 2f(y).$$

It is well known that a function  $f$  between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function  $B$  such that  $f(x) =$

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$B(x, x)$  for all  $x$ , where the mapping  $B$  is given by  $B(x, y) = \frac{1}{4}(f(x+y) - f(x-y))$ . See [1, 11] for the details.

The Hyers–Ulam stability of the quadratic functional equation (1.1) was first proved by F. Skof [20] for functions  $f : E_1 \rightarrow E_2$ , where  $E_1$  is a normed space and  $E_2$  is a Banach space. P.W. Cholewa [4] demonstrated that Skof’s theorem is also valid if  $E_1$  is replaced by an abelian group. S. Czerwik [5] proved the Hyers–Ulam–Rassias stability of quadratic functional equation (1.1).

**Theorem 1.1.** *Let  $E_1$  and  $E_2$  be a real normed space and a real Banach space, respectively, and let  $r \neq 2$  be a positive constant. If a function  $f : E_1 \rightarrow E_2$  satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \epsilon(\|x\|^r + \|y\|^r)$$

*for some  $\epsilon > 0$  and for all  $x, y \in E_1$ , then there exists a unique quadratic function  $q : E_1 \rightarrow E_2$  such that*

$$\|f(x) - q(x)\| \leq \frac{2\epsilon}{|4 - 2^r|} \|x\|^r$$

*for all  $x \in E_1$ .*

Furthermore, according to the theorem of C. Borelli and G.L. Forti [3], we know the following generalization of stability theorem for quadratic functional equation.

**Theorem 1.2.** *Let  $G$  be an abelian group and  $E$  a Banach space, let  $f : G \rightarrow E$  be a mapping with  $f(0) = 0$  satisfying the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varphi(x, y)$$

*for all  $x, y \in G$ . Assume that one of the series*

$$\Phi(x, y) := \begin{cases} \sum_{k=0}^{\infty} \frac{1}{2^{2(k+1)}} \varphi(2^k x, 2^k y) < \infty \\ \sum_{k=0}^{\infty} 2^{2k} \varphi\left(\frac{x}{2^{(k+1)}}, \frac{y}{2^{(k+1)}}\right) < \infty \end{cases},$$

*then there exists a unique quadratic function  $Q : G \rightarrow E$  such that*

$$\|f(x) - Q(x)\| \leq \Phi(x, x)$$

*for all  $x \in G$ .*

During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers–Ulam stability to a number of functional equations (see [6, 10, 11, 12, 13]).

Now, we consider some basic concepts concerning quasi- $\beta$ -normed spaces and some preliminary results. We fix a real number  $\beta$  with  $0 < \beta \leq 1$  and let  $\mathbb{K}$  denote either  $\mathbb{R}$  or  $\mathbb{C}$ . Let  $X$  be a linear space over  $\mathbb{K}$ . A *quasi- $\beta$ -norm*  $\|\cdot\|$  is a real-valued function on  $X$  satisfying the following:

- (1)  $\|x\| \geq 0$  for all  $x \in X$  and  $\|x\| = 0$  if and only if  $x = 0$ .
- (2)  $\|\lambda x\| = |\lambda|^\beta \cdot \|x\|$  for all  $\lambda \in \mathbb{K}$  and all  $x \in X$ .
- (3) There is a constant  $K \geq 1$  such that  $\|x+y\| \leq K(\|x\| + \|y\|)$  for all  $x, y \in X$ .



The pair  $(X, \|\cdot\|)$  is called a *quasi- $\beta$ -normed space* if  $\|\cdot\|$  is a quasi- $\beta$ -norm on  $X$ . The smallest possible  $K$  is called the *modulus of concavity* of  $\|\cdot\|$ . A *quasi- $\beta$ -Banach space* is a complete quasi- $\beta$ -normed space. A quasi- $\beta$ -norm  $\|\cdot\|$  is called a  *$(\beta, p)$ -norm* ( $0 < p \leq 1$ ) if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all  $x, y \in X$ . In this case, a quasi- $\beta$ -Banach space is called a  *$(\beta, p)$ -Banach space*. We can refer to ([2, 19]) for the concept of quasi-normed spaces and  $p$ -Banach spaces. Given a  $p$ -norm, the formula  $d(x, y) := \|x - y\|^p$  gives us a translation invariant metric on  $X$ . By the Aoki–Rolewicz theorem [19] (see also [2]), each quasi-norm is equivalent to some  $p$ -norm. In [21], J. Tabor has investigated a version of the D.H. Hyers, Th.M. Rassias, Z. Gajda theorem (see [7, 18]) in quasi-Banach spaces. Recently, S. Lee and C. Park [14] have obtained stability results of isometric homomorphisms in quasi-Banach algebras and J. Rassias and H. Kim [17] have obtained stability results of general additive equations in quasi- $\beta$ -normed spaces.

Concerning the stability of quadratic equation in quasi- $\beta$ -normed spaces, we introduce a new quadratic functional equation

$$(1.2) \quad 2f\left(\sum_{i=1}^n x_i\right) + \sum_{i \neq j} f(x_i - x_j) = 2n \sum_{i=1}^n f(x_i),$$

for any fixed  $n \in \mathbb{N}$  with  $n \geq 2$ . In this paper, we are going to investigate the general solution of the equation (1.2) and then we are to prove the generalized Hyers–Ulam stability of the equation (1.2) for mappings from quasi- $\alpha$ -normed spaces to  $(\beta, p)$ -Banach spaces by using direct method.

## 2. GENERALIZED HYERS–ULAM STABILITY OF EQUATION (1.2)

First, we present the general solution of equation (1.2) in the class of all functions between vector spaces.

**Lemma 2.1.** *If vector spaces  $X$  and  $Y$  are common domain and range of the functions  $f$  in both the functional equations (1.1) and (1.2), then the functional equation (1.2) is equivalent to the functional equation (1.1).*

*Proof.* Suppose that a function  $f : X \rightarrow Y$  satisfies the equation (1.2) for all  $x_1, \dots, x_n \in X$ . If we replace  $x_1, \dots, x_n$  in (1.2) by 0, then we have

$$2f(0) + n(n-1)f(0) = 2n^2f(0).$$

Since  $n \geq 2$ ,  $f(0) = 0$ . Let  $x_1 = x$  and  $x_k = 0$  ( $k = 2, \dots, n$ ). Then

$$2f(x) + (n-1)f(x) + (n-1)f(-x) = 2nf(x).$$

Thus  $f(x) = f(-x)$  for all  $x \in X$ . Letting  $x_1 = x$ ,  $x_2 = y$  and  $x_k = 0$  for all  $k = 3, \dots, n$ , we have

$$\begin{aligned} 2f(x+y) + f(x-y) + (n-2)f(x) + f(y-x) + (n-2)f(y) \\ + f(-x)(n-2) + f(-y)(n-2) = 2nf(x) + 2nf(y) \end{aligned}$$

for all  $x, y \in X$ . By the evenness of  $f$ , we may conclude that

$$f(x+y) + f(x-y) = 2f(x) + 2f(y).$$

Thus  $f$  is quadratic. Conversely, if  $f$  is quadratic, then it is obvious that  $f$  satisfies the equation (1.2).  $\square$

Throughout this paper, let  $X$  be a quasi- $\alpha$ -normed space and let  $Y$  be a  $(\beta, p)$ -Banach space unless we give any specific reference. For notational convenience, given a mapping  $f : X \rightarrow Y$ , we define the difference operator  $Df : X^n \rightarrow Y$  of the equation (1.2) by

$$Df(x_1, \dots, x_n) := 2f\left(\sum_{i=1}^n x_i\right) + \sum_{i \neq j} f(x_i - x_j) - 2n \sum_{i=1}^n f(x_i), \quad n \geq 2$$

for all  $x_1, \dots, x_n \in X$ , which is called the approximate remainder of the functional equation (1.2) and acts as a perturbation of the equation. Let  $\varphi : X^n \rightarrow \mathbb{R}^+ := [0, \infty)$  be a mapping satisfying one of the conditions

$$(a) \quad \Phi_1(x_1, \dots, x_n) := \sum_{j=0}^{\infty} \frac{1}{n^{2jp\beta}} \varphi\left(n^j x_1, \dots, n^j x_n\right)^p < \infty,$$

$$(b) \quad \Phi_2(x_1, \dots, x_n) := \sum_{j=1}^{\infty} n^{2jp\beta} \varphi\left(\frac{x_1}{n^j}, \dots, \frac{x_n}{n^j}\right)^p < \infty$$

for all  $x_1, \dots, x_n \in X$ . Now, we are ready to investigate the generalized Hyers–Ulam stability problem for the functional equation (1.2).

**Theorem 2.2.** *Assume that a function  $f : X \rightarrow Y$  satisfies*

$$(2.1) \quad \|Df(x_1, \dots, x_n)\| \leq \varphi(x_1, \dots, x_n)$$

for all  $x_1, \dots, x_n \in X$  and  $\varphi$  satisfies the condition (a). Then there exists a unique quadratic function  $Q : X \rightarrow Y$  satisfying

$$(2.2) \quad \left\| f(x) - \frac{nf(0)}{2(n+1)} - Q(x) \right\| \leq \frac{1}{2^\beta n^{2\beta}} \sqrt[p]{\Phi_1(x, \dots, x)}$$

for all  $x \in X$ , where  $\|f(0)\| \leq \frac{\varphi(0, \dots, 0)}{(n-1)^\beta (n+2)^\beta}$ . The function  $Q$  is given by

$$Q(x) = \lim_{k \rightarrow \infty} \frac{f(n^k x)}{n^{2k}}$$

for all  $x \in X$ .

*Proof.* Letting  $x_1, \dots, x_n$  by 0 in (2.1), we get  $\|(n-1)(n+2)f(0)\| \leq \varphi(0, \dots, 0)$ , and so  $\|f(0)\| \leq \frac{\varphi(0, \dots, 0)}{(n-1)^\beta (n+2)^\beta}$ . Replacing  $x_k$  by  $x$  for all  $k = 1, \dots, n$  in (2.1), we obtain

$$(2.3) \quad \|2f(nx) + n(n-1)f(0) - 2n^2 f(x)\| \leq \varphi(x, \dots, x)$$

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for  $x \in X$ . Dividing (2.3) by  $\frac{1}{2^\beta n^{2\beta}}$ , we get

$$(2.4) \quad \left\| \frac{1}{n^2} \bar{f}(nx) - \bar{f}(x) \right\| \leq \frac{1}{2^\beta n^{2\beta}} \varphi(x, \dots, x)$$

for  $x \in X$  where  $\bar{f}(x) = f(x) - \frac{n}{2(n+1)}f(0)$  for  $x \in X$ . Thus using the formula (2.4) and triangle inequality we prove by induction that

$$(2.5) \quad \left\| \frac{1}{n^{2k}} \bar{f}(n^k x) - \bar{f}(x) \right\|^p \leq \sum_{j=0}^{k-1} \frac{1}{2^{p\beta} n^{2(j+1)p\beta}} \varphi(n^j x, \dots, n^j x)^p$$

for  $x \in X$  and for all  $k \in \mathbb{N}$ . Therefore we prove from the inequality (2.5) that for any integers  $m, k$  with  $m > k \geq 0$

$$(2.6) \quad \begin{aligned} & \left\| \frac{1}{n^{2m}} \bar{f}(n^m x) - \frac{1}{n^{2k}} \bar{f}(n^k x) \right\|^p \\ & \leq \frac{1}{n^{2kp\beta}} \sum_{j=0}^{m-k-1} \frac{1}{2^{p\beta} n^{2(j+1)p\beta}} \varphi(n^{j+k} x, \dots, n^{j+k} x)^p \\ & = \frac{1}{2^{p\beta} n^{2p\beta}} \sum_{j=k}^{m-1} \frac{1}{2^{p\beta} n^{2jp\beta}} \varphi(n^j x, \dots, n^j x)^p. \end{aligned}$$

Since the right hand side of (2.6) tends to zero as  $k \rightarrow \infty$ , the sequence  $\left\{ \frac{1}{n^{2k}} \bar{f}(n^k x) \right\}$  is Cauchy for all  $x \in X$  and thus converges by the completeness of  $Y$ . Define  $Q : X \rightarrow Y$  by

$$\begin{aligned} Q(x) &= \lim_{k \rightarrow \infty} \frac{1}{n^{2k}} \left( f(n^k x) - \frac{n}{2(n+1)} f(0) \right) \\ &= \lim_{k \rightarrow \infty} \frac{f(n^k x)}{n^{2k}}, \quad x \in X. \end{aligned}$$

Then, letting  $x_i := n^k x_i$  for all  $i = 1, \dots, n$  in (2.1), respectively, dividing both sides by  $n^{2kp\beta}$  and after then taking the limit as  $k \rightarrow \infty$  in the resulting inequality, we have

$$\begin{aligned} \|DQ(x_1, \dots, x_n)\|^p &= \lim_{k \rightarrow \infty} \left\| \frac{1}{n^{2k}} Df(n^k x_1, \dots, n^k x_n) \right\|^p \\ &\leq \lim_{k \rightarrow \infty} \frac{1}{n^{2kp\beta}} \varphi(n^k x_1, \dots, n^k x_n)^p = 0, \end{aligned}$$

which implies that

$$2Q\left(\sum_{i=1}^n x_i\right) + \sum_{i \neq j} Q(x_i - x_j) - 2n \sum_{i=1}^n Q(x_i) = 0$$

and so the function  $Q$  is quadratic by Lemma 2.1. Taking the limit in (2.5) as  $k \rightarrow \infty$ , we obtain the approximation (2.2) of  $f$  by the quadratic mapping  $Q$ .

To prove the uniqueness of the quadratic function  $Q$  subject to (2.2), let us assume that there exists a quadratic function  $Q' : X \rightarrow Y$  which satisfies the inequality (2.2).

Obviously, we obtain that

$$Q(x) = n^{-2k}Q(n^kx), \quad Q'(x) = n^{-2k}Q'(n^kx)$$

for all  $x \in X$ . Hence it follows from (2.2) that

$$\begin{aligned} & \|Q(x) - Q'(x)\|^p \\ &= \frac{1}{n^{2kp\beta}} \|Q(n^kx) - Q'(n^kx)\|^p \\ &\leq \frac{1}{n^{2kp\beta}} \left[ \left\| Q(n^kx) - f(n^kx) + \frac{nf(0)}{2(n+1)} \right\|^p + \left\| f(n^kx) - \frac{nf(0)}{2(n+1)} - Q'(n^kx) \right\|^p \right] \\ &\leq \frac{1}{n^{2kp\beta}} \frac{2}{2^{p\beta}n^{2p\beta}} \sum_{j=0}^{\infty} \frac{1}{n^{2jp\beta}} \varphi(n^{j+k}x, \dots, n^{j+k}x)^p \\ &= \frac{2}{2^{p\beta}n^{2p\beta}} \sum_{j=k}^{\infty} \frac{1}{n^{2jp\beta}} \varphi(n^jx, \dots, n^jx)^p \end{aligned}$$

for all  $k \in \mathbb{N}$ . Therefore letting  $k \rightarrow \infty$ , one has  $Q(x) - Q'(x) = 0$  for all  $x \in X$ , completing the proof of uniqueness.  $\square$

**Theorem 2.3.** Assume that a function  $f : X \rightarrow Y$  satisfies

$$\|Df(x_1, \dots, x_n)\| \leq \varphi(x_1, \dots, x_n)$$

for all  $x_1, \dots, x_n \in X$  and  $\varphi$  satisfies the condition (b). Then there exists a unique quadratic function  $Q : X \rightarrow Y$  satisfying

$$(2.7) \quad \|f(x) - Q(x)\| \leq \frac{1}{2^\beta n^{2\beta}} \sqrt[p]{\Phi_2(x, \dots, x)}$$

for all  $x \in X$ . The function  $Q$  is given by

$$Q(x) = \lim_{k \rightarrow \infty} n^{2k} f\left(\frac{x}{n^k}\right)$$

for all  $x \in X$ .

*Proof.* Since  $\sum_{j=0}^{\infty} n^{2jp\beta} \varphi(0, \dots, 0) < \infty$  by assumption and so  $\varphi(0, \dots, 0) = 0$ , we get  $f(0) = 0$  in this case. Replacing  $x$  by  $\frac{x}{n}$  in (2.3), we obtain

$$(2.8) \quad \|f(x) - n^2 f\left(\frac{x}{n}\right)\| \leq \frac{1}{2^\beta} \varphi\left(\frac{x}{n}, \dots, \frac{x}{n}\right)$$

for  $x \in X$ .

An induction argument together with (2.8) implies that

$$(2.9) \quad \left\| f(x) - n^{2k} f\left(\frac{x}{n^k}\right) \right\|^p \leq \frac{1}{2^{p\beta} n^{2p\beta}} \sum_{j=1}^k n^{2jp\beta} \varphi\left(\frac{x}{n^j}, \dots, \frac{x}{n^j}\right)^p$$

for  $x \in X$  and for all  $k \in \mathbb{N}$ .

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Therefore we prove from inequality (2.9) that for any integers  $m, k$  with  $m > k \geq 0$

$$\begin{aligned}
 (2.10) \quad & \left\| n^{2m} f\left(\frac{x}{n^m}\right) - n^{2k} f\left(\frac{x}{n^k}\right) \right\|^p \\
 &= n^{2kp\beta} \left\| n^{2(m-k)} f\left(\frac{x}{n^m}\right) - f\left(\frac{x}{n^k}\right) \right\|^p \\
 &\leq \frac{n^{2kp\beta}}{2^{p\beta} n^{2p\beta}} \sum_{j=1}^{m-k} n^{2jp\beta} \varphi\left(\frac{x}{n^{j+k}}, \dots, \frac{x}{n^{j+k}}\right)^p \\
 &= \frac{1}{2^{p\beta} n^{2p\beta}} \sum_{j=k+1}^m n^{2jp\beta} \varphi\left(\frac{x}{n^j}, \dots, \frac{x}{n^j}\right)^p
 \end{aligned}$$

for all  $x \in X$ . Since the right hand side of (2.10) tends to zero as  $k \rightarrow \infty$ , the sequence  $\{n^{2k} f(\frac{x}{n^k})\}$  is Cauchy for all  $x \in X$ , and thus converges by the completeness of  $Y$ . Define  $Q : X \rightarrow Y$  by

$$Q(x) = \lim_{k \rightarrow \infty} n^{2k} f\left(\frac{x}{n^k}\right)$$

for all  $x \in X$ . Replacing  $x_1, \dots, x_n$  in (2.1) by  $\frac{x_1}{n^k}, \dots, \frac{x_n}{n^k}$ , respectively, and multiplying both sides by  $n^{2kp\beta}$  and after then taking the limit as  $k \rightarrow \infty$  in the resulting inequality, we have

$$\begin{aligned}
 \|DQ(x_1, \dots, x_n)\|^p &= \lim_{k \rightarrow \infty} \|n^{2k} Df\left(\frac{x_1}{n^k}, \dots, \frac{x_n}{n^k}\right)\|^p \\
 &\leq \lim_{k \rightarrow \infty} n^{2kp\beta} \varphi\left(\frac{x_1}{n^k}, \dots, \frac{x_n}{n^k}\right)^p = 0,
 \end{aligned}$$

which implies that the function  $Q$  is quadratic by Lemma 2.1. Taking the limit in (2.9) as  $k \rightarrow \infty$ , we obtain the estimation (2.7) of  $f$  by the quadratic mapping  $Q$ .

To prove the uniqueness, let  $Q'$  be another quadratic function satisfying (2.7). Then it is easy to see that the following identities  $Q(x) = n^{2k} Q(\frac{x}{n^k})$  and  $Q'(x) = n^{2k} Q'(\frac{x}{n^k})$  hold for all  $x \in X$ . Thus we have

$$\begin{aligned}
 \|Q(x) - Q'(x)\|^p &= n^{2kp\beta} \left( \left\| Q\left(\frac{x}{n^k}\right) - f\left(\frac{x}{n^k}\right) \right\|^p + \left\| f\left(\frac{x}{n^k}\right) - Q'\left(\frac{x}{n^k}\right) \right\|^p \right) \\
 &\leq \frac{2n^{2kp\beta}}{2^{p\beta} n^{2p\beta}} \sum_{j=1}^{\infty} n^{2jp\beta} \varphi\left(\frac{x}{n^{j+k}}, \dots, \frac{x}{n^{j+k}}\right)^p \\
 &\leq \frac{1}{2^{p\beta} n^{2p\beta}} \sum_{j=k+1}^{\infty} n^{2jp\beta} \varphi\left(\frac{x}{n^j}, \dots, \frac{x}{n^j}\right)^p
 \end{aligned}$$

for all  $x \in X$  and all  $k \in \mathbb{N}$ . Therefore letting  $k \rightarrow \infty$ , one has  $Q(x) - Q'(x) = 0$  for all  $x \in X$ .  $\square$

As applications, we obtain the following corollaries concerning the stability of the equation (1.2).

**Corollary 2.4.** Let  $r$  be a real number with  $n^{2\beta} \neq n^{r\alpha}$  and let  $H : \overbrace{\mathbb{R}^+ \times \cdots \times \mathbb{R}^+}^{n\text{-times}} \rightarrow \mathbb{R}^+$  be a function such that  $H(tx_1, \dots, tx_n) \leq t^r H(x_1, \dots, x_n)$  for all  $t, x_1, \dots, x_n \in \mathbb{R}^+$ . Suppose that a function  $f : X \rightarrow Y$  satisfies the inequality

$$(2.11) \quad \|Df(x_1, \dots, x_n)\| \leq H(\|x_1\|, \dots, \|x_n\|)$$

for all  $x_1, \dots, x_n \in X$ . Then there exists a unique quadratic function  $Q : X \rightarrow Y$  which satisfies the inequality

$$\|f(x) - \frac{nf(0)}{2(n+1)} - Q(x)\| \leq \begin{cases} \frac{H(1, \dots, 1)\|x\|^r}{2^\beta \sqrt[n^{2p\beta} - n^{rp\alpha}]}, & \text{if } n^{2\beta} > n^{r\alpha} \\ \frac{H(1, \dots, 1)\|x\|^r}{2^\beta \sqrt[n^{rp\alpha} - n^{2p\beta}]}, & \text{if } n^{2\beta} < n^{r\alpha} \end{cases}$$

for all  $x \in X$ . The function  $Q$  is given by

$$Q(x) = \begin{cases} \lim_{k \rightarrow \infty} \frac{f(n^k x)}{n^{2k}}, & \text{if } n^{2\beta} > n^{r\alpha} \\ \lim_{k \rightarrow \infty} n^{2k} f\left(\frac{x}{n^k}\right), & \text{if } n^{2\beta} < n^{r\alpha} \end{cases}$$

for all  $x \in X$ , where  $f(0) = 0$  if  $r > 0$ .

*Proof.* If  $r > 0$ , we put  $x_1, \dots, x_n$  by 0 in (2.11) and we get  $f(0) = 0$  according to  $H(0, \dots, 0) = 0$ . Letting  $\varphi(x_1, \dots, x_n) := H(\|x_1\|, \dots, \|x_n\|)$  for all  $x_1, \dots, x_n \in X$  and then applying Theorem 2.2 and Theorem 2.3 we obtain easily the results.  $\square$

In the following corollary, we have a stability result of the equation (1.2) in the sense of Th. M. Rassias [18].

**Corollary 2.5.** Let  $r, \epsilon$  be real numbers such that  $\epsilon \geq 0$ ,  $n^{2\beta} \neq n^{r\alpha}$ . Assume that a function  $f : X \rightarrow Y$  satisfies the inequality

$$(2.12) \quad \|Df(x_1, \dots, x_n)\| \leq \epsilon(\|x_1\|^r + \cdots + \|x_n\|^r)$$

for all  $x_1, \dots, x_n \in X$  and  $X \setminus \{0\}$  if  $r < 0$ . Then there exists a unique quadratic function  $Q : X \rightarrow Y$  which satisfies the inequality

$$\|f(x) - \frac{nf(0)}{2(n+1)} - Q(x)\| \leq \begin{cases} \frac{n\epsilon\|x\|^r}{2^\beta \sqrt[n^{2p\beta} - n^{rp\alpha}]}, & \text{if } n^{2\beta} > n^{r\alpha} \\ \frac{n\epsilon\|x\|^r}{2^\beta \sqrt[n^{rp\alpha} - n^{2p\beta}]}, & \text{if } n^{2\beta} < n^{r\alpha} \end{cases}$$

for all  $x \in X$  and  $X \setminus \{0\}$  if  $r < 0$ . The function  $Q$  is given by

$$Q(x) = \begin{cases} \lim_{k \rightarrow \infty} \frac{f(n^k x)}{n^{2k}}, & \text{if } n^{2\beta} > n^{r\alpha} \\ \lim_{k \rightarrow \infty} n^{2k} f\left(\frac{x}{n^k}\right), & \text{if } n^{2\beta} < n^{r\alpha} \end{cases}$$

for all  $x \in X$  and  $X \setminus \{0\}$  if  $r < 0$ , where  $f(0) = 0$  if  $r > 0$ .

*Proof.* If  $r > 0$ , we put  $x_1, \dots, x_n$  by 0 in (2.12) and we get  $f(0) = 0$ . Letting  $\varphi(x_1, \dots, x_n) := \epsilon(\|x_1\|^r + \cdots + \|x_n\|^r)$  for all  $x_1, \dots, x_n \in X$  and then applying Theorem 2.2 and Theorem 2.3 we obtain easily the results.  $\square$

In the next corollary, we get a stability result of the equation (1.2) in the sense of J. M. Rassias [15, 16].

**Corollary 2.6.** Let  $\epsilon, r_1, \dots, r_n$  be real numbers such that  $\epsilon \geq 0$ ,  $n^{2\beta} \neq n^{r\alpha}$ , where  $r := \sum_i^n r_i$ . Suppose that a function  $f : X \rightarrow Y$  satisfies

$$\|Df(x_1, \dots, x_n)\| \leq \epsilon \|x_1\|^{r_1} \|x_2\|^{r_2} \cdots \|x_n\|^{r_n}$$

for all  $x_1, \dots, x_n \in X$  and  $X \setminus \{0\}$  if  $r_1, \dots, r_n < 0$ . Then there exists a unique quadratic function  $Q : X \rightarrow Y$  which satisfies the inequality

$$\left\| f(x) - \frac{nf(0)}{2(n+1)} - Q(x) \right\| \leq \begin{cases} \frac{\epsilon \|x\|^r}{2^\beta \sqrt[2p]{n^{2p\beta} - n^{rp\alpha}}}, & \text{if } n^{2\beta} > n^{r\alpha} \\ \frac{\epsilon \|x\|^r}{2^\beta \sqrt[2p]{n^{rp\alpha} - n^{2p\beta}}}, & \text{if } n^{2\beta} < n^{r\alpha} \end{cases}$$

for all  $x \in X$  and  $X \setminus \{0\}$  if  $r_1, \dots, r_n < 0$ , where  $f(0) = 0$  if  $r_1, \dots, r_n > 0$ .

*Proof.* We remark that  $\varphi(x_1, \dots, x_n) := \epsilon \|x_1\|^{r_1} \|x_2\|^{r_2} \cdots \|x_n\|^{r_n}$  satisfies the condition (a) or (b) for all  $x_1, \dots, x_n \in X$ . By Theorem 2.2 and Theorem 2.3, we get the results.  $\square$

As a result, we obtain the following Hyers–Ulam stability result of the equation (1.2).

**Corollary 2.7.** Assume that for some  $\theta \geq 0$  a function  $f : X \rightarrow Y$  satisfies the inequality

$$\|Df(x_1, \dots, x_n)\| \leq \theta$$

for all  $x_1, \dots, x_n \in X$ . Then there exists a unique quadratic function  $Q : X \rightarrow Y$  which satisfies the inequality

$$\left\| f(x) - \frac{n}{2(n+1)} f(0) - Q(x) \right\| \leq \frac{\theta}{2^\beta \sqrt[2p]{n^{2p\beta} - 1}}$$

for all  $x \in X$ .

*Proof.* If we put  $\varphi(x_1, \dots, x_n) := \theta$ , then  $\varphi$  satisfies the condition (a) and so we get the desired result.  $\square$

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## A note on generalized absolute Cesàro summability

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### Abstract

In this paper, a general theorem concerning the  $\varphi - |C, \alpha|_k$  summability factors of infinite series, has been proved.

## 1 Introduction

Let  $(\varphi_n)$  be a sequence of positive real numbers and let  $\sum a_n$  be a given infinite series with the sequence of partial sums  $(s_n)$ . We denote by  $t_n^\alpha$  n-th Cesàro means of order  $\alpha$ , with  $\alpha > -1$ , of the sequence  $(na_n)$ , i.e.,

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} v a_v, \quad (1)$$

where

$$A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \quad \text{and} \quad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0. \quad (2)$$

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The series  $\sum a_n$  is said to be summable  $|C, \alpha|_k$ ,  $k \geq 1$  and  $\alpha > -1$ , if (see [1])

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty, \quad (3)$$

and it is said to be summable  $\varphi - |C, \alpha|_k$ ,  $k \geq 1$  and  $\alpha > -1$ , if (see [3])

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} |t_n^\alpha|^k < \infty. \quad (4)$$

If we take  $\alpha = 1$ , then  $\varphi - |C, \alpha|_k$  summability reduces to  $\varphi - |C, 1|_k$  summability.

Özarslan [2] has proved the following theorem for  $\varphi - |C, 1|_k$  summability factors of infinite series.

**Theorem A([2]).** Let  $(\varphi_n)$  be a sequence of positive real numbers. If

$$\lambda_m = o(1) \quad \text{as } m \rightarrow \infty, \quad (5)$$

$$\sum_{n=1}^m n \log n |\Delta^2 \lambda_n| = O(1), \quad (6)$$

$$\sum_{v=1}^m \frac{\varphi_v^{k-1}}{v^k} |t_v|^k = O(\log m) \quad \text{as } m \rightarrow \infty, \quad (7)$$

$$\sum_{n=v}^m \frac{\varphi_n^{k-1}}{n^{k+1}} = O\left(\frac{\varphi_v^{k-1}}{v^k}\right), \quad (8)$$

then the series  $\sum a_n \lambda_n$  is summable  $\varphi - |C, 1|_k$ ,  $k \geq 1$ .

**2. The main result.** The aim of this paper is to generalize Theorem A for  $\varphi - |C, \alpha|_k$  summability. We shall prove the following theorem.

**Theorem.** Let  $(\varphi_n)$  be a sequence of positive real numbers and the conditions (5)-(6) of Theorem A are satisfied. If

$$\sum_{v=1}^m \frac{\varphi_v^{k-1}}{v^k} |t_v^\alpha|^k = O(\log m) \quad \text{as } m \rightarrow \infty, \quad (9)$$

$$\sum_{n=v}^{m+1} \frac{\varphi_n^{k-1}}{n^{k+\alpha}} = O\left(\frac{\varphi_v^{k-1}}{v^{k+\alpha-1}}\right), \quad (10)$$

then the series  $\sum a_n \lambda_n$  is summable  $\varphi - |C, \alpha|_k$ ,  $k \geq 1$  and  $0 < \alpha \leq 1$ .

It should be noted that if we take  $\alpha = 1$  in this theorem, then we get Theorem A. In this case condition (9) reduces to condition (7) and condition (10) reduces to condition (8).

**Proof of the Theorem.** Let  $T_n^\alpha$  be the  $n$ -th  $(C, \alpha)$  means of the sequence  $(na_n \lambda_n)$ , with  $0 < \alpha \leq 1$ . Then by (1), we have

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} v a_v \lambda_v. \quad (11)$$

Applying Abel's transformation, we get that

$$\begin{aligned} T_n^\alpha &= \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^v A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} v a_v \\ &= \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} A_v^\alpha \Delta \lambda_v t_v^\alpha + \lambda_n t_n^\alpha \\ &= T_{n,1}^\alpha + T_{n,2}^\alpha, \quad \text{say.} \end{aligned}$$

Since

$$|T_{n,1}^\alpha + T_{n,2}^\alpha|^k \leq 2^k (|T_{n,1}^\alpha|^k + |T_{n,2}^\alpha|^k),$$

to complete the proof of the Theorem, by (4) it is enough to show that

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k} |T_{n,r}^\alpha|^k < \infty, \quad \text{for } r = 1, 2. \quad (12)$$

Now, when  $k > 1$ , applying Hölder's inequality with indices  $k$  and  $k'$ , where  $\frac{1}{k} + \frac{1}{k'} = 1$ , we get that

$$\begin{aligned} \sum_{n=2}^m \frac{\varphi_n^{k-1}}{n^k} |T_{n,1}^\alpha|^k &= \sum_{n=2}^m \frac{\varphi_n^{k-1}}{n^k} \left| \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta \lambda_v t_v^\alpha A_v^\alpha \right|^k \\ &= O(1) \sum_{n=2}^m \frac{\varphi_n^{k-1}}{n^{k+\alpha k}} \left\{ \sum_{v=1}^{n-1} v^\alpha |\Delta \lambda_v| |t_v^\alpha| \right\}^k \end{aligned}$$

$$\begin{aligned}
&= O(1) \sum_{n=2}^m \frac{\varphi_n^{k-1}}{n^{k+\alpha}} \left\{ \sum_{v=1}^{n-1} v^\alpha |\Delta \lambda_v| |t_v^\alpha|^k \times \left\{ \sum_{v=1}^{n-1} v^\alpha |\Delta \lambda_v| \right\}^{k-1} \right\} \\
&= O(1) \sum_{n=2}^m \frac{\varphi_n^{k-1}}{n^{k+\alpha}} \left\{ \sum_{v=1}^n v^\alpha |\Delta \lambda_v| |t_v^\alpha|^k \right\} \\
&= O(1) \sum_{v=1}^m v^\alpha |\Delta \lambda_v| |t_v^\alpha|^k \sum_{n=v}^m \frac{\varphi_n^{k-1}}{n^{k+\alpha}} \\
&= O(1) \sum_{v=1}^m v |\Delta \lambda_v| \frac{\varphi_v^{k-1}}{v^k} |t_v^\alpha|^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta(v |\Delta \lambda_v|)| \left| \sum_{r=1}^v \frac{\varphi_r^{k-1}}{r^k} |t_r^\alpha|^k + m |\Delta \lambda_m| \sum_{r=1}^m \frac{\varphi_r^{k-1}}{r^k} |t_r^\alpha|^k \right| \\
&= O(1) \sum_{v=1}^{m-1} |\Delta \lambda_v| \log v + \sum_{v=1}^{m-1} v |\Delta^2 \lambda_v| \log v + m |\Delta \lambda_m| \log m \\
&= O(1) \quad \text{as } m \rightarrow \infty,
\end{aligned}$$

by virtue of hypotheses of the Theorem. Finally,

$$\begin{aligned}
\sum_{n=1}^m \frac{\varphi_n^{k-1}}{n^k} |T_{n,2}^\alpha|^k &= \sum_{n=1}^m \frac{\varphi_n^{k-1}}{n^k} |\lambda_n t_n^\alpha|^k \\
&= O(1) \sum_{n=1}^m \frac{\varphi_n^{k-1}}{n^k} |t_n^\alpha|^k \sum_{v=n}^\infty |\Delta \lambda_v| \\
&= O(1) \sum_{v=1}^\infty |\Delta \lambda_v| \sum_{n=1}^v \frac{\varphi_n^{k-1}}{n^k} |t_n^\alpha|^k \\
&= O(1) \sum_{v=1}^\infty |\Delta \lambda_v| \log v \\
&= O(1) \quad \text{as } m \rightarrow \infty,
\end{aligned}$$

by virtue of hypotheses of the Theorem. Therefore we get that

$$\sum_{n=1}^\infty \frac{\varphi_n^{k-1}}{n^k} |T_{n,r}^\alpha|^k = O(1) \quad \text{as } m \rightarrow \infty, \quad \text{for } r = 1, 2.$$

This completes the proof of the Theorem.

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# SOME EXTENSIONS OF OPTIMAL HARDY'S INEQUALITY USING ESTIMATES OF P-LAPLACIAN

AURELIA FLOREA AND IONEL ROVENTA

ABSTRACT. In this paper we establish some extensions of the Hardy's type inequalities using estimates of  $p$ -Laplacian. We use also Landau's inequalities.

## 1. PRELIMINARIES

Recall here the classical Hardy inequality, which asserts that

$$(1.1) \quad \int_{-\infty}^{\infty} \left( \frac{1}{2|x|} \int_{-|x|}^{|x|} f(t) dt \right)^p dx \leq \left( \frac{p}{p-1} \right)^p \int_{-\infty}^{\infty} f^p(x) dx, \quad 1 < p < \infty$$

for each positive function  $f$  defined on  $(-\infty, \infty)$ .

N. Levinson in [8] proved:

**Theorem 1.** *Let  $\Lambda$  be a nonnegative function,  $f \in C^2(0, \infty)$ . If  $\Lambda$  satisfies the condition*

$$(1.2) \quad \Lambda(t) \Lambda''(t) \geq \left(1 - \frac{1}{p}\right) (\Lambda'(t))^2, \quad p > 1, t \in (0, \infty)$$

*then*

$$(1.3) \quad \int_0^{\infty} \Lambda \left( \frac{1}{x} \int_0^x f(t) dt \right) dx \leq \left( \frac{p}{p-1} \right)^p \int_0^{\infty} \Lambda(f(x)) dx,$$

*for  $f \geq 0$  unless  $\Lambda(f) \equiv 0$ .*

**Example 1.** *Different choices of  $\Lambda$  may lead to interesting variants of the Hardy inequality. We present here three special cases.*

(a) The function  $\Lambda(t) = t^p$ ,  $p > 1$ , satisfies (1.2). In this case, (1.3) is the classical Hardy inequality (1.1).

(b) The function  $\Lambda(t) = t^{-q}$  with  $q > 0$ , satisfies (1.2) for every  $p > 1$ . Hence (1.3) has the form

$$\int_0^{\infty} \left( \frac{1}{x} \int_0^x f(t) dt \right)^{-q} dx \leq \left( \frac{p}{p-1} \right)^p \int_0^{\infty} f^{-q}(x) dx,$$

that is, the classical Hardy inequality still works for negative exponents.

(c) The function  $\Lambda(t) = e^{t^a}$  with  $0 < a \leq 1$ , satisfies (1.2), and thus we have

$$(1.4) \quad \int_0^{\infty} \exp \left( \frac{1}{x} \int_0^x f(t) dt \right)^a dx \leq \left( \frac{p}{p-1} \right)^p \int_0^{\infty} \exp(f^a(x)) dx.$$

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For  $a = 1$ ,  $f$  replaced by  $\ln f$  and  $p \rightarrow \infty$ , the inequality (1.4) becomes the well-known *Carleman-Knopp inequality*

$$(1.5) \quad \int_0^\infty \exp\left(\frac{1}{x} \int_0^x \ln f(t) dt\right) dx \leq e \int_0^\infty f(x) dx;$$

(1.5) can be considered as the limiting case, for  $p$  tending to infinity, of the classical Hardy inequality for  $f^{1/p}$ ,

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f^{1/p}(t) dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x) dx.$$

Indeed, the geometrical mean of  $f$ ,

$$\exp\left(\frac{1}{x} \int_0^x \ln f(t) dt\right),$$

satisfies

$$\lim_{p \rightarrow \infty} \left(\frac{1}{x} \int_0^x f^{1/p}(t) dt\right)^p = \exp\left(\frac{1}{x} \int_0^x \ln f(t) dt\right)$$

(see [5], p. 139).

**Remark 1.** We should easily infer that the function  $\Lambda$  from Theorem 1 is in fact convex. For some interesting estimates about such functions, see [9]. Moreover, a generalization of Nash equilibrium, an  $g$ -equilibrium has obtained in [10].

## 2. MAIN RESULT

Our aim is to obtain a Hardy Littlewood type inequality, using some estimates of  $p$ -Laplacian. As a consequence, we obtain a more general result than Theorem 1.

Consider the natural maximal domain for  $p$ -Laplacian in the Banach space  $L^p(0, \infty)$ . This is

$$D_p := \{f : [0, \infty) \rightarrow \mathbb{C} \mid |f|, |f'|^{p-2} f' \in AC[0, \infty), f \in L^p(0, \infty), \Delta_p f \in L^q(0, \infty)\}, p > 1,$$

where  $\Delta_p u := \operatorname{div}(|\nabla u(x)|^{p-2} \nabla u(x))$  and  $AC$  denotes the family of absolute continuous functions. In particular for  $p = 2$ ,  $D_2$  is the standard maximal domain of  $f''$  in the Hilbert space  $L^2(0, \infty)$ .

An interesting result for [1] take into account the following Hardy-Littlewood-type inequality.

**Theorem 2.** For every  $f \in D_p$ , there exists a constant  $K > 0$  such that

$$\|f'\|_p^p \leq K \|f\|_p \|\Delta_p f\|_q, \text{ where } \frac{1}{p} + \frac{1}{q} = 1, p, q > 1.$$

The inequality from Theorem 2 gives an estimate of the derivative  $f'$  in  $L^p(0, \infty)$ . As in the case of Hardy-Littlewood inequality an essential first step in establishing the inequality is showing that  $f'$  is indeed an element of this space. In the present case the required information is provided by the following lemma.

**Lemma 1.** Let  $f \in D_p$ . Then  $f' \in L_p(0, \infty)$ ; moreover,  $\lim_{x \rightarrow \infty} (|f'|^{p-2} f' \bar{f})(x) = 0$ .

*Proof.* See [1]. □

It is easy to infer the next two additional lemmas. For more details, see [1].

**Lemma 2.** Let  $g : [0, \infty) \rightarrow \mathbb{C}$  be a locally absolutely continuous function and  $k > 1$ . Then  $|g|^k : [0, \infty) \rightarrow [0, \infty)$  is locally absolutely continuous, and  $(|g|^k)' = k|g|^{k-2} \operatorname{Re}(\bar{g}g')$ .

**Lemma 3.** Let  $p > 1, k > 0$ . If  $a, b > 0$  satisfy  $a^p \leq k(a+b)b^{p-1}$ , then  $a \leq cb$ , where  $\frac{1}{c}$  is the unique zero of

$$f(x) := kx^{p-1}(x+1) - 1, (x > 0).$$

**Proof of Theorem 2.**

Consider  $f \in D_p$ . By Lemma 1,  $f' \in L^p(0, \infty)$  and  $\lim_{x \rightarrow \infty} (|f'|^{p-2} f' \bar{f})(x) = 0$ . Hence using the integration by parts we infer

$$\|f'\|_p^p = -(|f'|^{p-2} f' \bar{f})(0) - \int_0^\infty (\Delta_p f) \bar{f} \leq |f'(0)|^{p-1} |f(0)| + \|\Delta_p f\|_q \|f\|_p.$$

Thus we need only to estimate the first term on the right-hand side in terms of the second. By Lemma 2, we have, for  $x > 0$ ,

$$\begin{aligned} |f(0)|^p - |f(x)|^p &\leq p \int_0^x |f|^{p-2} \operatorname{Re}(\bar{f} f') \leq p \int_0^x |f|^{p-1} |f'| \\ &\leq p \left( \int_0^x |f|^{(p-1)q} \right)^{1/q} \left( \int_0^x |f'|^p \right)^{1/p} \leq p \|f\|_p^{p-1} \|f'\|_p, \end{aligned}$$

using Holder's inequality and  $(p-1)q = p$ . Considering that  $\lim_{x \rightarrow \infty} |f(x)|^p = 0$ , we conclude that

$$|f(0)|^p \leq p \|f\|_p^{p-1} \|f'\|_p.$$

Similarly, by Lemma 2,

$$\begin{aligned} |f'(0)|^p - |f'(x)|^p &= \int_0^x \frac{d}{dx} (|f'|^{p-2} f' |f|^q) \leq q \int_0^x |f'|^{p-2} |f|^{q-2} |\operatorname{Re}(|f'|^{p-2} \bar{f}' \Delta_p f)| \\ &= q \int_0^x |f'|^{(p-1)(q-2)+p-1} |\Delta_p f| = q \int_0^x |f'| |\Delta_p f| \leq q \|f'\|_q \|\Delta_p f\|_q, \end{aligned}$$

and as  $\lim_{x \rightarrow \infty} |f'(x)|^p = 0$  we conclude that

$$|f'(0)|^p \leq q \|f'\|_q \|\Delta_p f\|_q.$$

Therefore

$$\begin{aligned} (|f'(0)|^{p-1} |f(0)|)^p &\leq p \|f\|_p^{p-1} \|f'\|_p q^{p-1} \|f'\|_q^{p-1} \|\Delta_p f\|_q^{p-1} \\ &\leq p q^{p-1} \|f\|_p^{p-1} \|\Delta_p f\|_q^{p-1} (|f'(0)|^{p-1} |f(0)| + \|\Delta_p f\|_q \|f\|_p). \end{aligned}$$

Now apply Lemma 3 with  $a = |f'(0)|^{p-1} |f(0)|$ ,  $b = \|\Delta_p f\|_q \|f\|_p$  and  $k = p q^{p-1}$ , to obtain the desired inequality with  $K = c + 1$ .

The main result of this paper is in the following:

**Theorem 3.** Let  $\Lambda \in D_2$  be a nonnegative real valued function. Then there exists some  $p_0 > 1$  such that for all  $p > p_0$  we have

$$(2.1) \quad \int_0^\infty \Lambda \left( \frac{1}{x} \int_0^x f(t) dt \right) dx \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty \Lambda(f(x)) dx,$$

for  $f \geq 0$  unless  $\Lambda(f) \equiv 0$ .



*Proof.* The result from Theorem 2, in the case  $p = 2$ , asserts that  $\int_0^\infty (f'(t))^2 dt < \infty$ . In fact, we should infer that inequality 1.2 is satisfied for a suitable  $p = K > 0$ . If not, we obtain a contradiction with Theorem 2 and with the fact that  $f' \in L^2(0, \infty)$ .

Choosing  $p_0 = \frac{K}{K-1}$ , the function  $f$  satisfies the condition 1.2 for all  $p \geq p_0$ . In conclusion we should apply Theorem 1 and the proof is done.  $\square$

### 3. FURTHER RESULTS

Let us denote, for  $x \in \mathbb{R}^N$ , by  $B(x)$  the ball  $\{y \in \mathbb{R}^N : |y| \leq |x|\}$  and by  $|B(x)|$  its volume. In [2], it is shown that the  $N$ -dimensional Hardy operator,  $\mathcal{H}_N$  defined by

$$(\mathcal{H}_N f)(x) = \frac{1}{|B(x)|} \int_{B(x)} f(y) dy, \quad x \in \mathbb{R}^N$$

satisfies

$$(3.1) \quad \int_{\mathbb{R}^N} |(\mathcal{H}_N f)(x)|^p dx \leq \left(\frac{p}{p-1}\right)^p \int_{\mathbb{R}^N} |f(x)|^p dx, \quad 1 < p < \infty$$

the constant  $\left(\frac{p}{p-1}\right)^p$  being again the best possible.

If we shall consider the bounded intervals, then the Hardy's inequality can be deduced from the following lemma (see [9]):

**Lemma 4.** *Let  $0 < b < \infty$  and  $-\infty \leq a < c \leq \infty$ . If  $u$  is a positive convex function on  $(a, c)$ , then*

$$(3.2) \quad \int_0^b u\left(\frac{1}{x} \int_0^x h(t) dt\right) \frac{dx}{x} \leq \int_0^b u(h(x)) \left(1 - \frac{x}{b}\right) \frac{dx}{x}$$

for all integrable and positive functions  $h : (0, b) \rightarrow (a, c)$ .

**Corollary 1.** *For  $u(x) = x^p$ , the result of Lemma 1 can be put in the following form:*

$$(3.3) \quad \int_0^\alpha \left(\frac{1}{x} \int_0^x f(t) dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\alpha f^p(x) \left(1 - \left(\frac{x}{\alpha}\right)^{(p-1)/p}\right) dx$$

where  $\alpha = b^{p/(p-1)}$  and  $f(x) = h(x^{1-1/p})x^{-1/p}$ . This yields an analogue of Hardy's inequality for functions  $f \in L^p(0, \alpha)$ , where  $0 < \alpha < \infty$ , from which Hardy's inequality follows by letting  $\alpha \rightarrow \infty$ .

We prove now a generalization of Hardy's inequality considered on the bounded intervals:

**Theorem 4.** *Let  $\Lambda$  be a nonnegative function,  $f \in C^2(0, \infty)$ . If  $\Lambda$  satisfies the condition*

$$(3.4) \quad \Lambda(t) \Lambda''(t) \geq \left(1 - \frac{1}{p}\right) (\Lambda'(t))^2, \quad p > 1, t \in (0, \infty)$$

then

$$(3.5) \quad \int_0^\alpha \Lambda\left(\frac{1}{x} \int_0^x f(t) dt\right) dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\alpha \Lambda(f(x)) \left(1 - \left(\frac{x}{\alpha}\right)^{(p-1)/p}\right) dx,$$

for  $\alpha > 0$ ,  $f \geq 0$  unless  $\Lambda(f) \equiv 0$ .

*Proof.* If we denote  $F(t) = (\Lambda(t))^{1/p}$ , then by (3.4)  $F''(t) \geq 0$  and thus  $F(t)$  is convex. By Jensen's inequality,

$$(3.6) \quad F\left(\frac{1}{x} \int_0^x f(t) dt\right) \leq \frac{1}{x} \int_0^x F(f(t)) dt.$$

which yields

$$\int_0^\alpha F^p\left(\frac{1}{x} \int_0^x f(t) dt\right) dx \leq \int_0^\alpha \left(\frac{1}{x} \int_0^x F(f(t)) dt\right)^p dx.$$

By Hardy's inequality (3.3) applied to  $F(f(x))$  we have

$$(3.7) \quad \int_0^\alpha \left(\frac{1}{x} \int_0^x F(f(t)) dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\alpha (F^p(f(x))) \left(1 - \left(\frac{x}{\alpha}\right)^{(p-1)/p}\right) dx.$$

Now, (3.5) follows from (3.6) and (3.7) inequalities since  $\Lambda = F^p$ .  $\square$

**Remark 2.** The following Landau inequality is well known:

$$(3.8) \quad \|\Lambda'\|_\infty \leq 2\sqrt{\|\Lambda\|_\infty \|\Lambda''\|_\infty},$$

where  $\Lambda$  is a real function twice differentiable on  $I$ ,  $m(I) \geq 2\sqrt{\|\Lambda\|_\infty / \|\Lambda''\|_\infty}$ , ( $\|\Lambda\|_\infty = \sup_{x \in I} |\Lambda(x)|$  and  $m(I)$  is the length of  $I$ ). The constant 2 in the right-hand side of inequality (3.8) is the best possible.

**Remark 3.** If  $\Lambda : \mathbb{R} \rightarrow \mathbb{R}$  is a twice differentiable function on  $\mathbb{R}$ , then (3.8) should be replaced by

$$\|\Lambda'\|_\infty \leq \sqrt{2\|\Lambda\|_\infty \|\Lambda''\|_\infty}.$$

**Remark 4.** Let  $\Lambda : (0, \infty) \rightarrow \mathbb{R}$  be a twice differentiable nonnegative function such that

$$(3.9) \quad \Lambda(t) \Lambda''(t) \geq \left(1 - \frac{1}{p}\right) (\Lambda'(t))^2,$$

for some  $p > 1$ . Then

$$\|\Lambda'\|_\infty \leq \sqrt{\frac{p}{p-1}} \sqrt{\|\Lambda\|_\infty \|\Lambda''\|_\infty}$$

and Landau's inequality shows that (3.9) holds true at least for  $p \in (1, 4/3]$ .

The last remarks allow us to refer to the min-max inequalities, especially Ky-Fan's type inequalities. For, interesting extensions of Ky-Fan inequalities, see [10] and [11].

**Proposition 1.** If  $p \in (1, 4/3]$ , then for every function twice differentiable defined on  $(0, \infty)$  with  $\|\Lambda\|_\infty, \|\Lambda''\|_\infty < \infty$ , we have

$$\int_0^\infty \Lambda\left(\frac{1}{x} \int_0^x f(t) dt\right) dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty \Lambda(f(x)) dx,$$

for  $f \geq 0$  unless  $\Lambda(f) \equiv 0$ .

**Theorem 5.** Let  $\Lambda : \mathbb{R} \rightarrow \mathbb{R}$  be a twice differentiable nonnegative function, which satisfies the condition (3.9) for some  $p > 1$ . Then

$$(3.10) \quad \int_{-\infty}^{\infty} \Lambda \left( \frac{1}{2|x|} \int_{-|x|}^{|x|} f(t) dt \right) dx \leq \left( \frac{p}{p-1} \right)^p \int_{-\infty}^{\infty} \Lambda(f(x)) dx.$$

*Proof.* If we denote  $F(t) = (\Lambda(t))^{1/p}$ , then by (3.4)  $F''(t) \geq 0$  and hence  $F(t)$  is convex. Thus by Jensen's inequality

$$(3.11) \quad F \left( \frac{1}{2|x|} \int_{-|x|}^{|x|} f(t) dt \right) \leq \frac{1}{2|x|} \int_{-|x|}^{|x|} F(f(t)) dt.$$

and then

$$(3.12) \quad \int_{-\infty}^{\infty} F^p \left( \frac{1}{2|x|} \int_{-|x|}^{|x|} f(t) dt \right) dx \leq \int_{-\infty}^{\infty} \left( \frac{1}{2|x|} \int_{-|x|}^{|x|} F(f(t)) dt \right)^p dx.$$

If we write the inequality (1.1) for  $F(f(x))$  we have

$$(3.13) \quad \int_{-\infty}^{\infty} \left( \frac{1}{2|x|} \int_{-|x|}^{|x|} F(f(t)) dt \right)^p dx \leq \left( \frac{p}{p-1} \right)^p \int_{-\infty}^{\infty} F^p(f(x)) dx.$$

Now, (3.10) follows from (3.12) and (3.13) inequalities since  $\Lambda = F^p$ .  $\square$

**Proposition 2.** If  $p \in (1, 2]$ , then for every twice differentiable function  $\Lambda$  defined on  $\mathbb{R}$  with  $\|\Lambda\|_{\infty}, \|\Lambda''\|_{\infty} < \infty$ , we have

$$(3.14) \quad \int_{-\infty}^{\infty} \Lambda \left( \frac{1}{2|x|} \int_{-|x|}^{|x|} f(t) dt \right) dx \leq \left( \frac{p}{p-1} \right)^p \int_{-\infty}^{\infty} \Lambda(f(x)) dx,$$

for  $f \geq 0$  defined on  $(-\infty, \infty)$ , unless  $\Lambda(f) \equiv 0$ .

*Proof.* We shall use Remark 3 and the theorem above.  $\square$

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# GENERALIZED DERIVATIONS AND ITS STABILITY

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ABSTRACT. In this article, we are going to examine the generalized Hyers-Ulam stability and the superstability of generalized derivations corresponding to the Jensen type functional equation.

## 1. Introduction

The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. The first study of stability problems had been formulated by S.M. Ulam [20] during a talk in 1940: *Under what condition does there exists a homomorphism near an approximate homomorphism?* In the following year 1941, D.H. Hyers [8] was answered affirmatively the question of Ulam for Banach spaces, which states that if  $\varepsilon \geq 0$  and  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is a mapping with  $\mathcal{X}$  a normed space,  $\mathcal{Y}$  a Banach space such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \quad (1.1)$$

for all  $x, y \in \mathcal{X}$ , then there exists a unique additive mapping  $L : \mathcal{X} \rightarrow \mathcal{Y}$  such that

$$\|f(x) - L(x)\| \leq \varepsilon$$

for all  $x \in \mathcal{X}$ . This stability phenomenon is called the *Hyers-Ulam stability* of the additive functional equation  $f(x+y) = f(x) + f(y)$ . A generalized version of the theorem of Hyers for approximately additive mappings was given by T. Aoki [1] and for approximately linear mappings was presented by Th.M. Rassias [15] in 1978 by considering the case when the inequality (1.1) is unbounded. Due to that fact, the additive functional equation is said to have the *generalized Hyers-Ulam stability* or the *Hyers-Ulam-Rassias stability* property. Since then, a great deal of work has been done by a number of authors and the problems concerned with the generalizations and the applications of the stability to a number of functional equations have been developed as well. The first result on the stability of the Jensen functional equation  $2f(\frac{x+y}{2}) = f(x) + f(y)$  was given by Z. Kominek [12].

A linear mapping  $d$  from an algebra  $\mathcal{A}$  into itself is called a *generalized derivation* if the functional equation  $d(xyz) = d(xy)z - xd(y)z + xd(yz)$  is valid for all  $x, y, z \in \mathcal{A}$ . In addition, if  $\mathcal{A}$  has a unit element and  $y$  is a unit, then we will say that  $d$  is just an *extended derivation*. In fact, a generalized derivation on an algebra  $\mathcal{A}$  with unit is an extended derivation. A linear mapping  $T$  from a unital normed algebra  $\mathcal{A}$  into itself is

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said to be *spectrally bounded* if there exists a constant  $M \geq 0$  such that  $r(T(x)) \leq Mr(x)$  for all  $x \in \mathcal{A}$ , where  $r(\cdot)$  stands for the spectral radius.

The main aim of the present article is to take account of the stability problem for generalized derivations on Banach algebras corresponding to the following Jensen type functional equation

$$f\left(\frac{x+y}{k}\right) = \frac{f(x)}{k} + \frac{f(y)}{k},$$

where  $k$  is an integer greater than 1. This functional equation is introduced in [2].

## 2. Main Results

Throughout this article, the element  $e$  of an algebra  $\mathcal{A}$  will denote a unit. We first establish the generalized Hyers-Ulam stability of additive generalized derivations.

**Theorem 2.1.** *Let  $\mathcal{A}$  be a Banach algebra. Suppose that  $f : \mathcal{A} \rightarrow \mathcal{A}$  is a mapping with  $f(0) = 0$  for which there exists a function  $\varphi : \mathcal{A}^5 \rightarrow [0, \infty)$  such that*

$$\lim_{n \rightarrow \infty} \frac{\varphi(0, 0, k^n z, w, u)}{k^n} = 0, \quad (2.1)$$

$$\sum_{j=1}^{\infty} \frac{1}{k^{j-1}} \varphi(k^j x, k^j y, k^j z, k^j w, k^j u) < \infty, \quad (2.2)$$

$$\begin{aligned} & \left\| f\left(\frac{x+y}{k} + zwu\right) - \frac{f(x)}{k} - \frac{f(y)}{k} - f(zw)u + zf(w)u - zf(wu) \right\| \\ & \leq \varphi(x, y, z, w, u) \end{aligned} \quad (2.3)$$

for all  $x, y, z, w, u \in \mathcal{A}$ . Then there exists a unique additive generalized derivation  $d : \mathcal{A} \rightarrow \mathcal{A}$  satisfying

$$\|f(x) - d(x)\| \leq \sum_{j=1}^{\infty} \frac{1}{k^{j-1}} \varphi(k^j x, 0, 0, 0, 0) \quad (2.4)$$

for all  $x \in \mathcal{A}$ . Moreover,

$$x\{f(y) - d(y)\}z = 0 \quad (2.5)$$

for all  $x, y, z \in \mathcal{A}$ .

*Proof.* Substituting  $z = w = u = 0$  in (2.3), we get

$$\left\| f\left(\frac{x+y}{k}\right) - \frac{f(x)}{k} - \frac{f(y)}{k} \right\| \leq \varphi(x, y, 0, 0, 0) \quad (2.6)$$

for all  $x, y \in \mathcal{A}$ . Let us take  $y = 0$  and replace  $x$  by  $kx$  in the above relation. Then it follows that

$$\left\| f(x) - \frac{f(kx)}{k} \right\| \leq \varphi(kx, 0, 0, 0, 0) \quad (2.7)$$

for all  $x \in \mathcal{A}$ . An induction implies that

$$\left\| \frac{f(k^n x)}{k^n} - f(x) \right\| \leq \sum_{j=1}^n \frac{1}{k^{j-1}} \varphi(k^j x, 0, 0, 0, 0) \quad (2.8)$$

for all  $x \in \mathcal{A}$ . By virtue of (2.8), one can easily check that for all integers  $n > m \geq 0$ ,

$$\begin{aligned} \left\| \frac{f(k^n x)}{k^n} - \frac{f(k^m x)}{k^m} \right\| &= \frac{1}{k^m} \left\| \frac{f(k^{n-m} \cdot k^m x)}{k^{n-m}} - f(k^m x) \right\| \\ &\leq \sum_{j=m+1}^n \frac{1}{k^{j-1}} \varphi(k^j x, 0, 0, 0, 0) \end{aligned}$$

for all  $x \in \mathcal{A}$ . It follows that the sequence  $\{f(k^n x)/k^n\}$  is a Cauchy and so it is convergent, since  $\mathcal{A}$  is complete. Let  $d : \mathcal{A} \rightarrow \mathcal{A}$  be a mapping defined by  $d(x) := \lim_{n \rightarrow \infty} \frac{f(k^n x)}{k^n}$  for all  $x \in \mathcal{A}$ . Sending  $n \rightarrow \infty$  in (2.8), we arrive at (2.5).

Now, we assert that  $d$  is additive. Replacing  $x$  and  $y$  by  $k^n x$  and  $k^n y$  in (2.6), respectively, and then dividing both sides by  $k^n$ , we have

$$\left\| \frac{1}{k^n} f\left(\frac{k^n x + k^n y}{k}\right) - \frac{1}{k} \frac{f(k^n x)}{k^n} - \frac{1}{k} \frac{f(k^n y)}{k^n} \right\| \leq \frac{1}{k^n} \varphi(k^n x, k^n y, 0, 0, 0).$$

Taking the limit as  $n \rightarrow \infty$ , we obtain

$$d\left(\frac{x+y}{k}\right) = \frac{d(x)}{k} + \frac{d(y)}{k}. \quad (2.9)$$

Letting  $y = 0$  in (2.9) yields  $d(x/k) = d(x)/k$ . Thus we get  $d(x+y) = d(x) + d(y)$  for all  $x, y \in \mathcal{A}$ .

To verify the uniqueness of the additive mapping  $d$  subject to (2.4), assume that there exists another additive mapping  $D : \mathcal{A} \rightarrow \mathcal{A}$  satisfying the inequality (2.4). Since  $D(k^n x) = k^n D(x)$  and  $d(k^n x) = k^n d(x)$ , we see that

$$\begin{aligned} \|D(x) - d(x)\| &= \frac{1}{k^n} \|D(k^n x) - d(k^n x)\| \\ &\leq \frac{1}{k^n} [\|D(k^n x) - f(k^n x)\| + \|f(k^n x) - d(k^n x)\|] \\ &\leq \sum_{j=n+1}^{\infty} \frac{2}{k^{j-1}} \varphi(k^j x, 0, 0, 0, 0), \end{aligned}$$

which tends to zero as  $n \rightarrow \infty$  for all  $x \in \mathcal{A}$ . So that  $D = d$ .

Next, we are in the position to prove that

$$d(zwu) = d(zw)u - zd(w)u + zd(wu) \quad (2.10)$$

for all  $z, w, u \in \mathcal{A}$ . If we take  $x = y = 0$  in (2.3), we have

$$\|f(zwu) - f(zw)u + zf(w)u - zf(wu)\| \leq \varphi(0, 0, z, w, u) \quad (2.11)$$

for all  $z, w, u \in \mathcal{A}$ . If we replace  $z, w$  and  $u$  with  $k^n z, k^n w$  and  $k^n u$  in (2.11), respectively, and then divide both sides by  $k^{3n}$ , we get

$$\left\| \frac{f(k^{3n} zwu)}{k^{3n}} - \frac{f(k^{2n} zw)}{k^{2n}} u + z \frac{f(k^n w)}{k^n} u - z \frac{f(k^{2n} wu)}{k^{2n}} \right\| \leq \frac{1}{k^{3n}} \varphi(0, 0, k^n z, k^n w, k^n u).$$

Letting  $n \rightarrow \infty$ , we obtain the desired result (2.10).

We finally need to show that the formula (2.5) holds. Let  $\Delta : \mathcal{A}^3 \rightarrow \mathcal{A}$  be a mapping defined by

$$\Delta(z, w, u) = f(zwu) - f(zw)u + zf(w)u - zf(wu)$$

for all  $z, w, u \in \mathcal{A}$ . Since  $f$  satisfies the inequality given in (2.11), we have by (2.1)

$$\lim_{n \rightarrow \infty} \frac{\Delta(k^n z, w, u)}{k^n} = 0$$

for all  $z, w, u \in \mathcal{A}$ . We note that

$$\begin{aligned} d(zwu) &= \lim_{n \rightarrow \infty} \frac{f(k^n zwu)}{k^n} = \lim_{n \rightarrow \infty} \frac{f((k^n z)wu)}{k^n} \\ &= \lim_{n \rightarrow \infty} \frac{f(k^n zw)u - k^n z f(w)u + k^n z f(wu) + \Delta(k^n z, w, u)}{k^n} \\ &= \lim_{n \rightarrow \infty} \left\{ \frac{f(k^n zw)}{k^n} u - z f(w)u + z f(wu) \right\} + \lim_{n \rightarrow \infty} \frac{\Delta(k^n z, w, u)}{k^n} \\ &= d(zw)u - z f(w)u + z f(wu) \end{aligned} \quad (2.12)$$

for all  $z, w, u \in \mathcal{A}$ . Since  $d$  is additive, we can rewrite (2.12) as

$$\begin{aligned} &k^n d(zw)u - k^n z f(w)u + k^n z f(wu) \\ &= d((k^n z)wu) = d(z(k^n w)u) \\ &= k^n d(zw)u - z f(k^n w)u + z f(k^n wu), \end{aligned}$$

which implies that

$$-z f(w)u + z f(wu) = -z \frac{f(k^n w)}{k^n} u + z \frac{f(k^n wu)}{k^n}.$$

Letting  $n \rightarrow \infty$ , we obtain

$$-z f(w)u + z f(wu) = -z d(w)u + z d(wu)$$

for all  $z, w, u \in \mathcal{A}$ . Replace  $u$  by  $k^n u$  in the previous part and then divide both sides by  $k^n$  to find

$$-z f(w)u + z \frac{f(k^n wu)}{k^n} = -z d(w)u + z d(wu).$$

Passing the limit as  $n \rightarrow \infty$ , we get (2.5). This completes the proof of the theorem.  $\square$

Using the same method as in the proof of Theorem 2.1, we get the following.

**Theorem 2.2.** *Let  $\mathcal{A}$  be a Banach algebra and let  $\varphi : \mathcal{A}^5 \rightarrow [0, \infty)$  be a function such that*

$$\lim_{n \rightarrow \infty} k^n \varphi\left(0, 0, \frac{z}{k^n}, w, u\right) = 0, \quad (2.13)$$

$$\sum_{j=1}^{\infty} k^{3j} \varphi\left(\frac{x}{k^{j-1}}, \frac{y}{k^{j-1}}, \frac{z}{k^{j-1}}, \frac{w}{k^{j-1}}, \frac{u}{k^{j-1}}\right) < \infty \quad (2.14)$$

for all  $x, y, z, w, u \in \mathcal{A}$ . Assume that a mapping  $f : \mathcal{A} \rightarrow \mathcal{A}$  with  $f(0) = 0$  satisfies (2.3). Then there exists a unique additive generalized derivation  $d : \mathcal{A} \rightarrow \mathcal{A}$  satisfying

$$\|f(x) - d(x)\| \leq \sum_{j=1}^{\infty} k^j \varphi\left(\frac{x}{k^{j-1}}, 0, 0, 0, 0\right)$$

for all  $x \in \mathcal{A}$  and (2.5) holds.

From now on, we assume that  $\mathcal{A}$  is a unital Banach algebra. We also obtain the superstability of additive generalized derivations.



**Corollary 2.3.** Suppose that  $f : \mathcal{A} \rightarrow \mathcal{A}$  is a mapping with  $f(0) = 0$  for which there exists a function  $\varphi : \mathcal{A}^5 \rightarrow [0, \infty)$  satisfying (2.1), (2.2) and (2.3). Then  $f$  is an additive generalized derivation.

*Proof.* Letting  $x = z = e$  in (2.5) implies  $f = d$ , which completes the proof of the corollary.  $\square$

**Corollary 2.4.** Suppose that  $f : \mathcal{A} \rightarrow \mathcal{A}$  is a mapping with  $f(0) = 0$  for which there exists a function  $\varphi : \mathcal{A}^5 \rightarrow [0, \infty)$  satisfying (2.13), (2.14) and (2.3). Then  $f$  is an additive generalized derivation.

*Proof.* The proof is similar to the proof of the corollary 2.3.  $\square$

By the same way as in the proof of Theorem 2.1 and Corollary 2.3, we lead to the following.

**Corollary 2.5.** Suppose that  $f : \mathcal{A} \rightarrow \mathcal{A}$  is a mapping with  $f(0) = 0$  for which there exists a function  $\varphi : \mathcal{A}^4 \rightarrow [0, \infty)$  such that

$$\lim_{n \rightarrow \infty} \frac{\varphi(0, 0, k^n z, w)}{k^n} = 0, \quad (2.15)$$

$$\sum_{j=1}^{\infty} \frac{1}{k^{j-1}} \varphi(k^j x, k^j y, k^j z, k^j w) < \infty, \quad (2.16)$$

$$\left\| f\left(\frac{x+y}{k} + zw\right) - \frac{f(x)}{k} - \frac{f(y)}{k} - f(z)w + zf(e)w - zf(w) \right\| \leq \varphi(x, y, z, w) \quad (2.17)$$

for all  $x, y, z, w \in \mathcal{A}$ . Then  $f$  is an additive extended derivation.

Employing the same fashion as in the proof of Theorem 2.1 and Corollary 2.4, we obtain the following.

**Corollary 2.6.** Let  $\varphi : \mathcal{A}^4 \rightarrow [0, \infty)$  be a function such that

$$\lim_{n \rightarrow \infty} k^n \varphi\left(0, 0, \frac{z}{k^n}, w\right) = 0, \quad (2.18)$$

$$\sum_{j=1}^{\infty} k^{2j} \varphi\left(\frac{x}{k^{j-1}}, \frac{y}{k^{j-1}}, \frac{z}{k^{j-1}}, \frac{w}{k^{j-1}}\right) < \infty \quad (2.19)$$

for all  $x, y, z, w \in \mathcal{A}$ . Assume that a mapping  $f : \mathcal{A} \rightarrow \mathcal{A}$  with  $f(0) = 0$  satisfies (2.17). Then  $f$  is an additive extended derivation.

We now denote by  $\mathbb{U} := \{z \in \mathbb{C} : |z| = 1\}$ . The following theorem is a result for the superstability of functional equation stemming from spectrally bounded generalized derivations.

**Theorem 2.7.** Let  $f : \mathcal{A} \rightarrow \mathcal{A}$  be a mapping with  $f(0) = 0$  for which there exists a constant  $M \geq 0$  such that  $r(f(x)) \leq Mr(x)$  for all  $x \in \mathcal{A}$ . Suppose that  $\varphi : \mathcal{A}^5 \rightarrow [0, \infty)$  is a function satisfying (2.1), (2.2) and the inequality

$$\begin{aligned} & \left\| f\left(\frac{\alpha x + \beta y}{k} + zwu\right) - \alpha \frac{f(x)}{k} - \beta \frac{f(y)}{k} - f(zw)u + zf(w)u - zf(wu) \right\| \\ & \leq \varphi(x, y, z, w, u) \end{aligned} \quad (2.20)$$

for all  $x, y, z, w, u \in \mathcal{A}$  and all  $\alpha, \beta \in \mathbb{U}$ . Then  $f = L_{f(e)} + \delta$ , where  $L_{f(e)}$  is a left multiplication by  $f(e)$  and  $\delta$  is a derivation. In this case, both  $L_{f(e)}$  and  $\delta$  are spectrally bounded.

*Proof.* We consider  $\alpha = \beta = 1 \in \mathbb{U}$  in (2.20) and then  $f$  satisfies the inequality (2.3). It follows from the corollary 2.3 that  $f$  is an additive generalized derivation, where  $f(x) := \lim_{n \rightarrow \infty} \frac{f(k^n x)}{k^n}$  for all  $x \in \mathcal{A}$ .

Letting  $z = w = u = 0$  in (2.20), we have

$$\left\| f\left(\frac{\alpha x + \beta y}{k}\right) - \alpha \frac{f(x)}{k} - \beta \frac{f(y)}{k} \right\| \leq \varphi(x, y, 0, 0, 0) \quad (2.21)$$

for all  $x, y \in \mathcal{A}$  and all  $\alpha, \beta \in \mathbb{U}$ . If we replace  $x$  and  $y$  with  $k^n x$  and  $k^n y$  in (2.21), respectively, and then divide both sides by  $k^n$ , we see that

$$\left\| \frac{1}{k^n} f\left(\frac{\alpha k^n x + \beta k^n y}{k}\right) - \alpha \frac{1}{k} \frac{f(k^n x)}{k^n} - \beta \frac{1}{k} \frac{f(k^n y)}{k^n} \right\| \leq \frac{1}{k^n} \varphi(k^n x, k^n y, 0, 0, 0),$$

which tends to zero as  $n \rightarrow \infty$ . So we get

$$f\left(\frac{\alpha x + \beta y}{k}\right) = \alpha \frac{f(x)}{k} + \beta \frac{f(y)}{k}$$

for all  $x, y \in \mathcal{A}$  and all  $\alpha, \beta \in \mathbb{U}$ . From the additivity of  $f$ , we find that

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for all  $x, y \in \mathcal{A}$  and all  $\alpha, \beta \in \mathbb{U}$ . As we did in the proof of [10, Theorem 2.3] (or [11, Theorem 3.1]), we have that  $f(\lambda x) = \lambda f(x)$  for all  $x \in \mathcal{A}$ . Also, it is obvious that  $f(0x) = 0 = 0f(x)$  for all  $x \in \mathcal{A}$ , that is,  $f$  is linear. Therefore  $f$  is a generalized derivation. So  $f$  is an extended derivation and a spectrally bounded generalized derivation.

We now define a mapping  $\delta : \mathcal{A} \rightarrow \mathcal{A}$  by  $\delta(x) := f(x) - xf(e)$  for all  $x \in \mathcal{A}$ . So  $\delta$  is linear. Note that

$$\delta(xy) = f(xy) - xyf(e) = x(f(y) - yf(e)) + (f(x) - xf(e))y = x\delta(y) + \delta(x)y$$

for all  $x, y \in \mathcal{A}$ . Thus  $\delta$  is a derivation. In particular, setting  $L_{f(e)}(x) := xf(e)$  for all  $x \in \mathcal{A}$ , we obtain  $f = L_{f(e)} + \delta$ . According to the Brešar and and M. Mathieu's result [7], both  $L_{f(e)}$  and  $\delta$  are spectrally bounded. The proof of the theorem is ended.  $\square$

Now, we compare the following corollary with some results of [2].

**Corollary 2.8.** *Let  $f : \mathcal{A} \rightarrow \mathcal{A}$  be a mapping with  $f(0) = 0$  for which there exists a function  $\varphi : \mathcal{A}^4 \rightarrow [0, \infty)$  satisfying (2.15), (2.16) and the inequality*

$$\left\| f\left(\frac{\alpha x + \beta y}{k} + zw\right) - \alpha \frac{f(x)}{k} - \beta \frac{f(y)}{k} - f(z)w + zf(e)w - zf(w) \right\| \leq \varphi(x, y, z, w) \quad (2.22)$$

for all  $x, y, z, w \in \mathcal{A}$  and all  $\alpha, \beta \in \mathbb{U}$ . Then  $f$  is an extended derivation.

*Proof.* By the same reasoning described in the proof of Theorem 2.7, we find that  $f$  is an extended derivation. The proof of the corollary is complete.  $\square$

**Theorem 2.9.** Let  $f : \mathcal{A} \rightarrow \mathcal{A}$  be a mapping with  $f(0) = 0$  for which there exists a constant  $M \geq 0$  such that  $r(f(x)) \leq Mr(x)$  for all  $x \in \mathcal{A}$ . Suppose that  $\varphi : \mathcal{A}^5 \rightarrow [0, \infty)$  is a function satisfying (2.1), (2.2) and the inequality (2.20) for all  $x, y, z, w, u \in \mathcal{A}$  and all  $\alpha, \beta = 1, \mathbf{i}$ . If, in addition,  $f(tx)$  is continuous in  $t \in \mathbb{R}$  for each fixed  $x$  in  $\mathcal{A}$ , then  $f = L_{f(e)} + \delta$ , where  $L_{f(e)}$  is a left multiplication by  $f(e)$  and  $\delta$  is a derivation. In this case, both  $L_{f(e)}$  and  $\delta$  are spectrally bounded.

*Proof.* Let  $\alpha = \beta = 1$  in (2.1). Due to the corollary 2.3, we see that  $f$  is an additive generalized derivation  $f$ , where  $f(x) := \lim_{n \rightarrow \infty} \frac{f(k^n x)}{k^n}$  for all  $x \in \mathcal{A}$ . Employing the same method as in the proof of the main theorem of [15], we see that  $d$  is  $\mathbb{R}$ -linear.

Considering  $\alpha = \mathbf{i}$  and  $y = z = w = u = 0$  in (2.20), we get

$$\left\| f\left(\frac{\mathbf{i}x}{k}\right) - \mathbf{i}\frac{f(x)}{k} \right\| \leq \varphi(x, 0, 0, 0, 0)$$

for all  $x \in \mathcal{A}$ . Replacing  $x$  by  $k^{n+1}x$  in the previous part and then dividing  $k^n$  on both sides, we have

$$\left\| \frac{f(k^n \mathbf{i}x)}{k^n} - \mathbf{i}\frac{f(k^{n+1}x)}{k^{n+1}} \right\| \leq \frac{1}{k^n} \varphi(k^{n+1}x, 0, 0, 0, 0).$$

The right-hand side goes to zero as  $n \rightarrow \infty$ , so that

$$f(\mathbf{i}x) = \lim_{n \rightarrow \infty} \frac{f(k^n \mathbf{i}x)}{k^n} = \lim_{n \rightarrow \infty} \mathbf{i}\frac{f(k^{n+1}x)}{k^{n+1}} = \mathbf{i}f(x)$$

for all  $x \in \mathcal{A}$ . Then for all  $\lambda = a + \mathbf{i}b$  with  $a, b \in \mathbb{R}$ , one notes

$$\begin{aligned} f(\lambda x) &= f(ax + \mathbf{i}bx) = af(x) + bf(\mathbf{i}x) \\ &= af(x) + b\mathbf{i}f(x) = (a + b\mathbf{i})f(x) = \lambda f(x). \end{aligned}$$

Thus  $f$  is linear.

The remaining part of the theorem is similar to the proof of Theorem 2.7.  $\square$

**Corollary 2.10.** Let  $f : \mathcal{A} \rightarrow \mathcal{A}$  be a mapping with  $f(0) = 0$  for which there exists a function  $\varphi : \mathcal{A}^4 \rightarrow [0, \infty)$  satisfying (2.15), (2.16) and the inequality (2.22) for all  $x, y, z, w \in \mathcal{A}$  and all  $\alpha, \beta = 1, \mathbf{i}$ . If, in addition,  $f(tx)$  is continuous in  $t \in \mathbb{R}$  for each fixed  $x$  in  $\mathcal{A}$ , then  $f$  is an extended derivation.

*Proof.* The proof is similar to the proof of Theorem 2.9.  $\square$

*Remark.* Even though, in Theorem 2.7 and Theorem 2.9, we replace (2.1) and (2.2) by (2.13) and (2.14), the conclusions of Theorem 2.7 and Theorem 2.9 are still true. On the other hand, in Corollary 2.8 and Corollary 2.10, if we replace (2.15) and (2.16) with (2.18) and (2.19), then the results of Corollary 2.8 and Corollary 2.10 also hold. Furthermore, we can remove the assumption  $f(0) = 0$  in the previous facts as just mentioned. Indeed, we note that

$$\left\| \frac{(k - 2\mathbf{i})}{k} f(0) \right\| \leq \varphi(0, 0, 0, 0, 0), \quad \left( \left\| \frac{(k - 2\mathbf{i})}{k} f(0) \right\| \leq \varphi(0, 0, 0, 0), \text{ respectively} \right).$$

So, by the assumption of  $\varphi$ , we get  $f(0) = 0$ .

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# NEW SYSTEMS OF GENERALIZED MIXED VARIATIONAL INEQUALITIES WITH NONLINEAR MAPPINGS IN HILBERT SPACES

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**Abstract.** In this paper, we introduce and study a new system of generalized mixed variational inequality problems with nonlinear mappings in Hilbert spaces. We prove the existence of the solutions and the convergence of iterative sequences generated by the algorithm for the system. The results presented in this paper are the generalization and improvement of recent results.

## 1. INTRODUCTION

It is known variational inequality theory and complementarity problem are very powerful tools of the current mathematical technology. In recent years, classical variational inequalities and complementarity problems have been extended and generalized to study a large variety of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium and engineering sciences, etc. Standard variational inequality theory was introduced by Stampacchia [12] in 1964. In 1988, Noor [10] introduced and studied some new class of general variational inequality. Also, in 1998, he [11] introduced and studied some new class of variational inequality. The solvability of the variational inequalities based on some sort of iterative algorithm much depends on the suitable choice of the mappings and the proper space setting, as has been the case in most of the computational analysis.

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Recently, Chang *et al.* [2], Ding *et al.* [3], Kim-Kim [6], Kim-Kim ([7], [8]) and Verma ([13]-[17]) introduced some systems of nonlinear strongly monotone variational inequalities and studied the approximate solvability of these systems based on a system of projection methods.

The purpose of this paper is to consider, based on the resolvent method, the existence of solutions and approximation solvability of a class of new system of generalized nonlinear mixed variational inequalities with nonlinear mappings in Hilbert spaces. The results pretended in this paper generalize, improve and unify the corresponding results of Chang *et al.* [2], Ding *et al.* [3], Kim-Kim [6], Kim-Kim ([7], [8]), Noor ([10], [11]) and Verma ([13]-[17]).

## 2. PRELIMINARIES

Throughout this paper, let  $H$  be a real Hilbert space with the inner product  $\langle \cdot, \cdot \rangle$  and the norm  $\|\cdot\|$  and  $CB(H)$  be the family of all bounded closed convex subset of  $H$ . Let  $U, V : H \rightarrow CB(H)$ ,  $F, G : H \times H \times H \rightarrow H$ ,  $g_1, g_2 : H \rightarrow H$  be mappings,  $\phi_1, \phi_2 : H \rightarrow \mathbb{R} \cup \{+\infty\}$  be proper convex lower semicontinuous functions, and  $H(\cdot, \cdot)$  be the Hausdorff metric on  $CB(H)$  defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} D(x, B), \sup_{y \in B} D(A, y) \right\}.$$

We consider the following *new systems of generalized nonlinear mixed variational inequality problem*:

Find elements  $x, y \in H$ ,  $u \in U(x)$  and  $v \in V(y)$  such that  $g_1(x), g_2(y) \in H$  and

$$\begin{cases} \langle \rho F(x, u, y), z - g_1(x) \rangle \geq \rho \phi_1(g_1(x)) - \rho \phi_1(z), \\ \langle \gamma G(x, v, y), z - g_2(y) \rangle \geq \gamma \phi_2(g_2(y)) - \gamma \phi_2(z), \end{cases} \quad (2.1)$$

for all  $z \in H$  and  $\rho > 0, \gamma > 0$  are two constants.

### Special Cases

- (I) If  $\phi_1(x) = \delta_{K_1}(x)$ ,  $\phi_2(u) = \delta_{K_2}(u)$ , where  $\delta_{K_i}$  is the indicator function of a nonempty closed convex subset  $K_i$  for  $i = 1, 2$ , then the problem (2.1) reduce to finding  $x, y \in H$ ,  $u \in U(x)$  and  $v \in V(y)$  such that  $g_i(x) \in K_i$  for  $i = 1, 2$  and

$$\begin{cases} \langle \rho F(x, u, y), z - g_1(x) \rangle \geq 0, & \forall z \in K_1, \\ \langle \gamma G(x, v, y), z - g_2(y) \rangle \geq 0, & \forall z \in K_2, \end{cases} \quad (2.2)$$

where  $\rho > 0, \gamma > 0$  are two constants, which is called the *new systems of generalized nonlinear variational inequality problem* which was considered by Ding *et al.* [3].

- (II) If  $F(x, u, y) = T(y, x) + \frac{1}{\rho}g_1(x) - \frac{1}{\rho}g_2(y)$ ,  $G(x, v, y) = S(x, y) + \frac{1}{\gamma}g_2(y) - \frac{1}{\gamma}g_1(x)$ , for all  $x, y, u, v \in H$ , then the problem (2.1) reduced to finding  $x, y \in H$  such that  $g_1(x), g_2(y) \in H$  and

$$\begin{cases} \left\langle \rho T(y, x) + g_1(x) - g_2(y), z - g_1(x) \right\rangle \geq \rho(\phi_1(g_1(x)) - \phi_1(z)), \\ \left\langle \gamma S(x, y) + g_2(y) - g_1(x), z - g_2(y) \right\rangle \geq \gamma\phi_2(g_2(y)) - \phi_2(z), \end{cases} \quad (2.3)$$

for all  $z \in H$ , which was considered in Kim-Kim [7].

- (III) If  $\phi_1(x) = \delta_{K_1}(x)$ ,  $\phi_2(u) = \delta_{K_2}(u)$ ,  $K_1 = K_2 = K$ ,  $g_1 = g_2 = I$  (identity mapping),  $F(x, u, y) = T(y, x) + \frac{1}{\rho}x - \frac{1}{\rho}y$ ,  $G(x, v, y) = T(x, y) + \frac{1}{\gamma}y - \frac{1}{\gamma}x$  for all  $x, y, u, v \in H$ , then the problem (2.1) reduces to finding  $x, y \in K$  such that

$$\begin{cases} \left\langle \rho T(y, x) + x - y, z - x \right\rangle \geq 0, & \forall z \in K, \\ \left\langle \gamma T(x, y) + y - x, z - y \right\rangle \geq 0, & \forall z \in K, \end{cases} \quad (2.4)$$

where  $\rho > 0, \gamma > 0$  are two constants, which is called the *system of non-linear variational inequality* problem considered by Verma [17]. The special case of problem (2.4) was studied by Verma ([13], [16]).

- (IV) If  $\phi_1(x) = \delta_{K_1}(x)$ ,  $\phi_2(u) = \delta_{K_2}(u)$ ,  $g_1 = g_2 = I$  (identity mapping),  $F(x, u, y) = S(x, y)$ ,  $G(x, v, y) = T(x, y)$  for all  $x, y, u, v \in H$ , then the problem (2.1) reduce to finding  $x \in K_1, y \in K_2$  such that

$$\begin{cases} \left\langle \rho S(x, y), z - x \right\rangle \geq 0, & \forall z \in K_1, \\ \left\langle \gamma T(x, y), z - y \right\rangle \geq 0, & \forall z \in K_2, \end{cases} \quad (2.5)$$

which is just the problem considered in [5] with  $S, T$  being single-valued mappings.

In the sequel, we give some definitions and lemmas.

**Definition 2.1.** Let  $T : H \rightarrow H$  be mapping.

- (1) The mapping  $T$  is said to be *monotone* if

$$\langle T(x) - T(y), x - y \rangle \geq 0, \quad \forall x, y \in H.$$

- (2) The mapping  $T$  is said to be *r-strongly monotone* if there exists  $r > 0$  such that

$$\langle T(x) - T(y), x - y \rangle \geq r\|x - y\|^2, \quad \forall x, y \in H.$$

This implies that

$$\|T(x) - T(y)\| \geq r\|x - y\|,$$

that is,  $T$  is  $r$ -expansive and, when  $r = 1$ , it is expansive.

- (3) The mapping  $T$  is said to be *s-Lipschitz continuous* (or *Lipschitzian*) if there exists a constant  $s \geq 0$  such that

$$\|T(x) - T(y)\| \leq s\|x - y\|, \quad \forall x, y \in H.$$

- (4) The mapping  $T$  is said to be  $\mu$ -cocoercive ([4], [13], [16]) if there exists a constant  $\mu > 0$  such that

$$\langle T(x) - T(y), x - y \rangle \geq \mu \|T(x) - T(y)\|^2, \quad \forall x, y \in H.$$

Clearly, every  $\mu$ -cocoercive mapping  $T$  is  $\frac{1}{\mu}$ -Lipschitz continuous.

- (5) The mapping  $T$  is said to be *relaxed  $\gamma$ -cocoercive* if there exists a constant  $\gamma > 0$  such that

$$\langle T(x) - T(y), x - y \rangle \geq (-\gamma) \|T(x) - T(y)\|^2, \quad \forall x, y \in H.$$

- (6) The mapping  $T$  is said to be *relaxed  $(\gamma, r)$ -cocoercive* if there exist constants  $\gamma, r > 0$  such that

$$\langle T(x) - T(y), x - y \rangle \geq (-\gamma) \|T(x) - T(y)\|^2 + r\|x - y\|^2,$$

for all  $x, y \in H$ . For  $\gamma = 0$ ,  $T$  is  $r$ -strongly monotone, and for  $r = 0$ ,  $T$  is relaxed  $\gamma$ -cocoercive. This class of mappings is more general than the class of strongly monotone mappings.

**Remark 2.1.** We have the following implications:

$$\text{the strong monotonicity} \implies \text{the monotonicity},$$

$$\text{the strong monotonicity} \implies \text{the expansiveness}$$

and

$$\text{the } r\text{-strong monotonicity} \implies \text{the relaxed } (\gamma, r)\text{-cocoercivity}.$$

**Definition 2.2.** Let  $F : X \times X \times X \rightarrow X$  be a nonlinear mapping.  $F$  is said to be

- (i)  $\alpha$ -strongly monotone with respect to the first argument if there exists some  $\alpha > 0$  such that

$$\langle F(x, \cdot, \cdot) - F(y, \cdot, \cdot), x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in X,$$

- (ii)  $\xi$ -Lipschitz continuous with respect to the first argument if exists a constant  $\xi > 0$  such that

$$\|F(x, \cdot, \cdot) - F(y, \cdot, \cdot)\| \leq \xi \|x - y\|, \quad \forall x, y \in X.$$

Similarly, we can define the strong monotonicity and Lipschitzian continuity with respect to the second and the third argument of  $F(\cdot, \cdot, \cdot)$ .



**Definition 2.3.** ([9]) A mapping  $U : H \rightarrow CB(H)$  is said to be  $\psi$ -Lipschitz continuous if there exists a constant  $\psi > 0$  such that

$$H(U(v_1), U(v_2)) \leq \psi \|v_1 - v_2\|, \quad \forall v_i \in H, \quad i = 1, 2,$$

where  $H(\cdot, \cdot)$  is a Hausdorff metric on  $CB(H)$ .

**Lemma 2.1.** ([1], [18]) For a given  $u \in H$ , the point  $z \in H$  satisfies the following inequality

$$\langle u - z, v - u \rangle \geq \rho\phi(u) - \rho\phi(v) \quad \forall v \in H$$

if and only if

$$u = J_\phi^\rho(z),$$

where  $J_\phi^\rho = (I + \rho\partial\phi)^{-1}$  and  $\partial\phi$  denotes the subdifferential of a proper convex lower semicontinuous function  $\phi : H \rightarrow \mathbb{R} \cup \{+\infty\}$ .

**Remark 2.2.** It is well known that  $J_\phi^\rho$  is nonexpansive (see [1], [18]).

It is easy to prove that the following lemma is trivial from the Lemma 2.1.

**Lemma 2.2.** For given  $x, y \in H$ ,  $u \in U(x)$  and  $v \in V(y)$ ,  $(x, y, u, v)$  is a solution of the problem (2.1) if and only if

$$\begin{aligned} g_1(x) &= J_{\phi_1}^\rho \left( g_1(x) - \rho F(x, u, y) \right), \\ g_2(y) &= J_{\phi_2}^\gamma \left( g_2(y) - \gamma G(x, v, y) \right), \end{aligned}$$

where  $\rho, \gamma > 0$  are constants.

### 3. EXISTENCE AND CONVERGENCE

In this section, we construct some iterative algorithms for the problems (2.1). We also give the convergence analysis of the iterative sequences generated by the algorithm.

Now we give the algorithm for solving the problem (2.1) as follows.

**Algorithm 3.1.** For any given  $x_0, y_0 \in H$ , we choose  $u_0 \in U(x_0)$ ,  $v_0 \in V(y_0)$  and let

$$\begin{aligned} x_1 &= x_0 - g_1(x_0) + J_{\phi_1}^\rho \left( g_1(x_0) - \rho F(x_0, u_0, y_0) \right), \\ y_1 &= y_0 - g_2(y_0) + J_{\phi_2}^\gamma \left( g_2(y_0) - \gamma G(x_0, v_0, y_0) \right). \end{aligned}$$

By Nadler [9] there exists  $u_1 \in U(x_1)$  and  $v_1 \in V(y_1)$  such that

$$\begin{aligned} \|u_0 - u_1\| &\leq (1 + 1)H(U(x_0), U(x_1)), \\ \|v_0 - v_1\| &\leq (1 + 1)H(V(y_0), V(y_1)). \end{aligned}$$

Let

$$\begin{aligned}x_2 &= x_1 - g_1(x_1) + J_{\phi_1}^{\rho} \left( g_1(x_1) - \rho F(x_1, u_1, y_1) \right), \\y_2 &= y_1 - g_2(y_1) + J_{\phi_2}^{\gamma} \left( g_2(y_1) - \gamma G(x_1, v_1, y_1) \right).\end{aligned}$$

By induction, we obtain the sequence  $\{x_n\}$  and  $\{y_n\}$  such that

$$\begin{cases} u_n \in U(x_n), & \|u_n - u_{n+1}\| \leq \left(1 + \frac{1}{1+n}\right) H(U(x_n), U(x_{n+1})), \\ v_n \in V(y_n), & \|v_n - v_{n+1}\| \leq \left(1 + \frac{1}{1+n}\right) H(V(y_n), V(y_{n+1})), \\ x_{n+1} = x_n - g_1(x_n) + J_{\phi_1}^{\rho} \left( g_1(x_n) - \rho F(x_n, u_n, y_n) \right), \\ y_{n+1} = y_n - g_2(y_n) + J_{\phi_2}^{\gamma} \left( g_2(y_n) - \gamma G(x_n, v_n, y_n) \right). \end{cases} \quad (3.1)$$

for  $i = 0, 1, 2, \dots$ .

**Theorem 3.1.** Let  $g_i : H \rightarrow H$  be relaxed  $(\gamma_i, \eta_i)$ -cocoercive and  $\xi_i$ -Lipschitz continuous, for  $i = 1, 2$ . Let  $F : H \times H \times H \rightarrow H$  be  $l_1, l_2, l_3$ -Lipschitz continuous with respect to the first, second and third arguments, respectively, and relaxed  $(\delta, p)$ -cocoercive with respect to the first argument. Let  $G : H \times H \times H \rightarrow H$  be  $n_1, n_2, n_3$ -Lipschitz continuous with respect to the first, second and third arguments, respectively, and relaxed  $(\varepsilon, q)$ -cocoercive with respect to the third argument. Suppose  $U$  is  $\varphi$ -Lipschitz continuous and  $V$  is  $\beta$ -Lipschitz continuous. If

$$\begin{aligned}2\sqrt{1 - 2\eta_1 + (2\gamma_1 + 1)\xi_1^2} + \sqrt{1 - 2\rho p + (2\rho\delta + \rho^2)l_1^2} \\ + \rho l_2 \varphi + \gamma n_1 < 1\end{aligned} \quad (3.2)$$

and

$$\begin{aligned}2\sqrt{1 - 2\eta_2 + (2\gamma_2 + 1)\xi_2^2} + \sqrt{1 - 2\gamma q + (2\gamma\varepsilon + \gamma^2)n_3^2} \\ + \gamma n_2 \beta + \rho l_3 < 1,\end{aligned} \quad (3.3)$$

then there exist  $x, y \in H$ ,  $u \in U(x)$  and  $v \in V(y)$  which solve problem (2.1). Moreover, the iterative sequences  $\{x_n\}$  and  $\{y_n\}$  generated by Algorithm 3.1 converges to  $x$  and  $y$ , respectively.

*Proof.* From (3.1), we have

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
&= \left\| x_n - g_1(x_n) + J_{\phi_1}^\rho \left( g_1(x_n) - \rho F(x_n, u_n, y_n) \right) \right. \\
&\quad \left. - \left\{ x_{n-1} - g_1(x_{n-1}) + J_{\phi_1}^\rho \left( g_1(x_{n-1}) - \rho F(x_{n-1}, u_{n-1}, y_{n-1}) \right) \right\} \right\| \quad (3.4) \\
&\leq \left\| x_n - x_{n-1} - \left( g_1(x_n) - g_1(x_{n-1}) \right) \right\| \\
&\quad + \left\| g_1(x_n) - g_1(x_{n-1}) - \rho \left( F(x_n, u_n, y_n) - F(x_{n-1}, u_{n-1}, y_{n-1}) \right) \right\|.
\end{aligned}$$

We can know that

$$\begin{aligned}
& \left\| g_1(x_n) - g_1(x_{n-1}) - \rho \left( F(x_n, u_n, y_n) - F(x_{n-1}, u_{n-1}, y_{n-1}) \right) \right\| \\
&\leq \left\| x_n - x_{n-1} - \left( g_1(x_n) - g_1(x_{n-1}) \right) \right\| \\
&\quad + \left\| x_n - x_{n-1} - \rho \left( F(x_n, u_n, y_n) - F(x_{n-1}, u_n, y_n) \right) \right\| \quad (3.5) \\
&\quad + \rho \|F(x_{n-1}, u_n, y_n) - F(x_{n-1}, u_{n-1}, y_n)\| \\
&\quad + \rho \|F(x_{n-1}, u_{n-1}, y_n) - F(x_{n-1}, u_{n-1}, y_{n-1})\|.
\end{aligned}$$

Since  $g_1$  is  $\xi_1$ -Lipschitz continuous and relaxed  $(\gamma_1, \eta_1)$ -cocoercive, we have

$$\begin{aligned}
& \left\| x_n - x_{n-1} - \left( g_1(x_n) - g_1(x_{n-1}) \right) \right\|^2 \\
&= \|x_n - x_{n-1}\|^2 - 2 \left\langle x_n - x_{n-1}, g_1(x_n) - g_1(x_{n-1}) \right\rangle \\
&\quad + \|g_1(x_n) - g_1(x_{n-1})\|^2 \quad (3.6) \\
&\leq \|x_n - x_{n-1}\|^2 + 2\gamma_1 \|g_1(x_n) - g_1(x_{n-1})\|^2 \\
&\quad - 2\eta_1 \|x_n - x_{n-1}\|^2 + \|g_1(x_n) - g_1(x_{n-1})\|^2 \\
&\leq \left( 1 - 2\eta_1 + (2\gamma_1 + 1)\xi_1^2 \right) \|x_n - x_{n-1}\|^2.
\end{aligned}$$

And also,  $F$  is relaxed  $(\delta, p)$ -cocoercive and  $l_1$ -Lipschitz continuous with respect to the first argument, we have

$$\begin{aligned}
& \left\| x_n - x_{n-1} - \rho \left( F(x_n, u_n, y_n) - F(x_{n-1}, u_n, y_n) \right) \right\|^2 \\
&= \|x_n - x_{n-1}\|^2 - 2\rho \left\langle x_n - x_{n-1}, F(x_n, u_n, y_n) - F(x_{n-1}, u_n, y_n) \right\rangle \\
&\quad + \rho^2 \|F(x_n, u_n, y_n) - F(x_{n-1}, u_n, y_n)\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 + 2\rho\delta \|F(x_n, u_n, y_n) - F(x_{n-1}, u_n, y_n)\|^2 \\
&\quad - 2\rho p \|x_n - x_{n-1}\|^2 + \rho^2 \|F(x_n, u_n, y_n) - F(x_{n-1}, u_n, y_n)\|^2 \\
&\leq \left( 1 - 2\rho p + (2\rho\delta + \rho^2)l_1^2 \right) \|x_n - x_{n-1}\|^2.
\end{aligned} \tag{3.7}$$

Since  $F$  is  $l_2$ -Lipschitz continuous with respect to the second argument and  $U$  is  $\varphi$ -Lipschitz continuous, we obtain

$$\begin{aligned}
& \|F(x_{n-1}, u_n, y_n) - F(x_{n-1}, u_{n-1}, y_n)\| \\
&\leq l_2 \|u_n - u_{n-1}\| \\
&\leq l_2 \left( 1 + \frac{1}{n} \right) H(U(x_n), U(x_{n-1})) \\
&\leq \left( 1 + \frac{1}{n} \right) l_2 \varphi \|x_n - x_{n-1}\|.
\end{aligned} \tag{3.8}$$

Since  $F$  is  $l_3$ -Lipschitz continuous with respect to the third argument, we obtain

$$\|F(x_{n-1}, u_{n-1}, y_n) - F(x_{n-1}, u_{n-1}, y_{n-1})\| \leq l_3 \|y_n - y_{n-1}\|. \tag{3.9}$$

Substituting (3.6)-(3.9) into (3.5), we have

$$\begin{aligned}
& \left\| g_1(x_n) - g_1(x_{n-1}) - \rho \left( F(x_n, u_n, y_n) - F(x_{n-1}, u_{n-1}, y_{n-1}) \right) \right\| \\
&\leq \sqrt{1 - 2\eta_1 + (2\gamma_1 + 1)\xi_1^2} \|x_n - x_{n-1}\| \\
&\quad + \sqrt{1 - 2\rho p + (2\rho\delta + \rho^2)l_1^2} \|x_n - x_{n-1}\| \\
&\quad + \rho l_2 \varphi \left( 1 + \frac{1}{n} \right) \|x_n - x_{n-1}\| + \rho l_3 \|y_n - y_{n-1}\|.
\end{aligned} \tag{3.10}$$

Substituting (3.6) and (3.10) into (3.4), we have

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
&\leq \left( 2\sqrt{1 - 2\eta_1 + (2\gamma_1 + 1)\xi_1^2} + \sqrt{1 - 2\rho p + (2\rho\delta + \rho^2)l_1^2} \right) \|x_n - x_{n-1}\| \\
&\quad + \rho l_2 \varphi \left( 1 + \frac{1}{n} \right) \|x_n - x_{n-1}\| + \rho l_3 \|y_n - y_{n-1}\|.
\end{aligned} \tag{3.11}$$

Similarly, we have

$$\begin{aligned} & \|y_{n+1} - y_n\| \\ & \leq \left(2\sqrt{1 - 2\eta_2 + (2\gamma_2 + 1)\xi_2^2} + \sqrt{1 - 2\gamma q + (2\gamma\varepsilon + \gamma^2)n_3^2}\right. \\ & \quad \left.+ \gamma n_2\beta\left(1 + \frac{1}{n}\right)\right)\|y_n - y_{n-1}\| + \gamma n_1\|x_n - x_{n-1}\|. \end{aligned} \quad (3.12)$$

Now, (3.11) and (3.12) imply

$$\begin{aligned} & \|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \\ & \leq \left(2\sqrt{1 - 2\eta_1 + (2\gamma_1 + 1)\xi_1^2} + \sqrt{1 - 2\rho p + (2\rho\delta + \rho^2)l_1^2}\right. \\ & \quad \left.+ \rho l_2\varphi\left(1 + \frac{1}{n}\right) + \gamma n_1\right)\|x_n - x_{n-1}\| \\ & \quad + \left(2\sqrt{1 - 2\eta_2 + (2\gamma_2 + 1)\xi_2^2} + \sqrt{1 - 2\gamma q + (2\gamma\varepsilon + \gamma^2)n_3^2}\right. \\ & \quad \left.+ \gamma n_2\beta\left(1 + \frac{1}{n}\right) + \rho l_3\right)\|y_n - y_{n-1}\| \\ & \leq \theta_n(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|), \end{aligned} \quad (3.13)$$

where

$$\begin{aligned} \theta_n = \max \Big\{ & 2\sqrt{1 - 2\eta_1 + (2\gamma_1 + 1)\xi_1^2} + \sqrt{1 - 2\rho p + (2\rho\delta + \rho^2)l_1^2} \\ & + \rho l_2\varphi\left(1 + \frac{1}{n}\right) + \gamma n_1, \\ & 2\sqrt{1 - 2\eta_2 + (2\gamma_2 + 1)\xi_2^2} + \sqrt{1 - 2\gamma q + (2\gamma\varepsilon + \gamma^2)n_3^2} \\ & + \gamma n_2\beta\left(1 + \frac{1}{n}\right) + \rho l_3 \Big\}. \end{aligned}$$

We know that  $\theta_n \rightarrow \theta$  as  $n \rightarrow \infty$ , where

$$\begin{aligned} \theta = \max \Big\{ & 2\sqrt{1 - 2\eta_1 + (2\gamma_1 + 1)\xi_1^2} + \sqrt{1 - 2\rho p + (2\rho\delta + \rho^2)l_1^2} \\ & + \rho l_2\varphi + \gamma n_1, \\ & 2\sqrt{1 - 2\eta_2 + (2\gamma_2 + 1)\xi_2^2} + \sqrt{1 - 2\gamma q + (2\gamma\varepsilon + \gamma^2)n_3^2} \\ & + \gamma n_2\beta + \rho l_3 \Big\}. \end{aligned}$$

It follows from (3.2) and (3.3) that  $0 \leq \theta < 1$ . Hence  $\theta_n < 1$  for sufficiently large  $n$ . Thus (3.13) implies that  $\{x_n\}$  and  $\{y_n\}$  are both Cauchy sequences in  $H$ , and so  $\{x_n\}$  converges to  $x \in H$ ,  $\{y_n\}$  converges to  $y \in H$ . From (3.1),

we have

$$\begin{aligned}\|u_{n+1} - u_n\| &\leq \left(1 + \frac{1}{1+n}\right) H(U(x_{n+1}), U(x_n)) \\ &\leq \left(1 + \frac{1}{1+n}\right) \varphi \|x_{n+1} - x_n\|,\end{aligned}$$

and

$$\begin{aligned}\|v_{n+1} - v_n\| &\leq \left(1 + \frac{1}{1+n}\right) H(V(y_{n+1}), V(y_n)) \\ &\leq \left(1 + \frac{1}{1+n}\right) \beta \|y_{n+1} - y_n\|.\end{aligned}$$

Thus,  $\{u_n\}$  and  $\{v_n\}$  are also Cauchy sequences in  $H$ . Let  $\{u_n\}$  converges to  $u$  and  $\{v_n\}$  converges to  $v$ . Now, we prove that  $u \in U(x)$  and  $v \in V(y)$ . In fact,

$$\begin{aligned}D(u, U(x)) &= \inf_{z \in U(x)} \|u - z\| \\ &\leq \|u - u_n\| + D(u_n, U(x)) \\ &\leq \|u - u_n\| + H(U(x_n), U(x)) \\ &\leq \|u - u_n\| + \varphi \|x_n - x\|.\end{aligned}$$

Letting  $n \rightarrow \infty$ , we know that  $D(u, U(x)) = 0$  and so  $u \in U(x)$ . Similarly, we have  $v \in V(y)$ . Since  $g_1, g_2, J_{\phi_1}^\rho, J_{\phi_2}^\gamma, F$  and  $G$  are all continuous, we have

$$\begin{aligned}x &= x - g_1(x) + J_{\phi_1}^\rho \left( g_1(x) - \rho F(x, u, y) \right), \\ y &= y - g_2(y) + J_{\phi_2}^\gamma \left( g_2(y) - \gamma G(x, v, y) \right).\end{aligned}$$

The result follows then from Lemma 2.2. This completes the proof.  $\square$

**Remark 3.1.** Let  $\rho > 0, \gamma > 0$  be numbers satisfying the conditions:

$$l_2\varphi < l_1, \quad \rho l_2\varphi < 1 - e_1 - \gamma n_1,$$

$$\left| \rho + \frac{\delta l_1^2 - p + l_2\varphi}{l_1^2 - (l_2\varphi)^2} \right| < \sqrt{\frac{(e_1 + \gamma n_1)^2 + \frac{(\delta l_1^2 - p + l_2\varphi)^2}{l_1^2 - (l_2\varphi)^2}}{l_1^2 - (l_2\varphi)^2}}$$

and

$$n_2\beta < n_3, \quad \gamma n_2\beta < 1 - e_2 - \rho l_3,$$

$$\left| \gamma + \frac{\varepsilon n_3^2 - q + n_2\beta}{n_3^2 - (n_2\beta)^2} \right| < \sqrt{\frac{(e_2 + \rho l_3)^2 + \frac{(\varepsilon n_3^2 - q + n_2\beta)^2}{n_3^2 - (n_2\beta)^2}}{n_3^2 - (n_2\beta)^2}},$$

where

$$e_1 = 2\sqrt{1 - 2\eta_1 + (2\gamma_1 + 1)\xi_1^2}, \quad e_2 = 2\sqrt{1 - 2\eta_2 + (2\gamma_2 + 1)\xi_2^2}.$$

Then (3.2) and (3.3) holds.

Taking  $\gamma_i = 0$  ( $i = 1, 2$ ),  $\delta = 0$  and  $\varepsilon = 0$ , by Definition 2.1–(6), in Theorem 3.1, we can obtain the following theorem.

**Theorem 3.2.** *Let  $g_i : H \rightarrow H$  be  $\eta_i$ -strongly monotone and  $\xi_i$ -Lipschitz continuous, for  $i = 1, 2$ . Let  $F : H \times H \times H \rightarrow H$  be  $l_1, l_2, l_3$ -Lipschitz continuous with respect to the first, second and third arguments, respectively, and  $p$ -strongly monotone with respect to the first argument. Let  $G : H \times H \times H \rightarrow H$  be  $n_1, n_2, n_3$ -Lipschitz continuous with respect to the first, second and third arguments, respectively, and  $q$ -strongly monotone with respect to the third argument. Suppose  $U$  is  $\varphi$ -Lipschitz continuous and  $V$  is  $\beta$ -Lipschitz continuous. If*

$$2\sqrt{1 - 2\eta_1 + \xi_1^2} + \sqrt{1 - 2\rho p + \rho^2 l_1^2} + \rho l_2 \varphi + \gamma n_1 < 1$$

and

$$2\sqrt{1 - 2\eta_2 + \xi_2^2} + \sqrt{1 - 2\gamma q + \gamma^2 n_3^2} + \gamma n_2 \beta + \rho l_3 < 1,$$

then there exist  $x, y \in H$ ,  $u \in U(x)$  and  $v \in V(y)$  which solve problem (2.1). Moreover, the iterative sequences  $\{x_n\}$  and  $\{y_n\}$  generated by Algorithm 3.1 converges to  $x$  and  $y$ , respectively.

**Remark 3.2.**

- (i) Theorem 3.1 and Theorem 3.2 are generalization of the results in Chang *et al.* [2], Ding *et al.* [3], Kim-Kim [6], Kim-Kim ([7], [8]), Noor ([10], [11]) and Verma ([15], [16]).
- (ii) If  $\phi_i = \delta_{K_i}$ , where  $\delta_{K_i}$  is the indicator function of a nonempty closed convex subset  $K_i$  for  $i = 1, 2$ , in Theorem 3.2, then we can obtain the result of Ding *et al.* [3].

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# STRONG CONVERGENCE OF ITERATIVE SCHEMES FOR NONEXPANSIVE SEMIGROUPS IN BANACH SPACES

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**ABSTRACT.** Strong convergences of the implicit iterative scheme and the explicit iterative scheme for nonexpansive semigroup are established in a reflexive and strictly convex Banach space having a uniformly Gâteaux differentiable norm. Certain different control conditions of the explicit iterative scheme are given.

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## 1. Introduction

Let  $E$  be a real Banach space and  $C$  be a nonempty closed convex subset of  $E$ . Recall that a mapping  $f : C \rightarrow C$  is a *contraction* on  $C$  if there exists a constant  $k \in (0, 1)$  such that  $\|f(x) - f(y)\| \leq k\|x - y\|$ ,  $x, y \in C$ . We use  $\Sigma_C$  to denote the collection of mappings  $f$  verifying the above inequality. That is,  $\Sigma_C = \{f : C \rightarrow C \mid f \text{ is a contraction with constant } k\}$ . Note that each  $f \in \Sigma_C$  has a unique fixed point in  $C$ .

Now let  $T : C \rightarrow C$  be a nonexpansive mapping (recall that a mapping  $T : C \rightarrow C$  is *nonexpansive* if  $\|Tx - Ty\| \leq \|x - y\|$ ,  $x, y \in C$ ) and  $F(T)$  denote the set of fixed points of  $T$ ; that is,  $F(T) = \{x \in C : x = Tx\}$ .

Recall that a family  $\{T(t) : t \geq 0\}$  of mappings from  $C$  into itself is called a *nonexpansive semigroup* on  $C$  if it satisfies the following conditions:

- (1)  $T(t_1 + t_2)x = T(t_1)T(t_2)x$  for any  $t_1, t_2 \in \mathbb{R}^+$  and  $x \in C$ ;
- (2)  $T(0)x = x$  for each  $x \in C$ ;
- (3) for each  $x \in C$ ,  $t \mapsto T(t)x$  is continuous;
- (4)  $\|T(t)x - T(t)y\| \leq \|x - y\|$  for each  $t \in \mathbb{R}^+$  and  $x, y \in C$ .

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Given a real number  $t \in (0, 1)$ , a contraction  $f \in \Sigma_C$  and a nonexpansive mapping  $T$ , let a contraction  $T_t := T_t^f : C \rightarrow C$  be defined by

$$T_t z = t f(z) + (1 - t) T z, \quad z \in C.$$

and let  $z_t := z_t^f \in C$  be the unique fixed point of  $T_t$ . Then  $z_t$  is the unique solution of the fixed point equation

$$z_t = t f(z_t) + (1 - t) T z_t. \quad (1.1)$$

A special case of (1.1) has been considered by Browder [3] in a Hilbert space as follows. Fix  $u \in C$  and define a contraction  $G_t$  on  $C$  by

$$G_t x = t u + (1 - t) T x, \quad x \in C.$$

Let  $x_t \in C$  be the unique fixed point of  $G_t$ . Thus

$$x_t = t u + (1 - t) T x_t.$$

(Such a sequence  $\{x_t\}$  is said to be an approximating fixed point of  $T$  since it possesses the property that if  $\{x_t\}$  is bounded, then  $\lim_{t \rightarrow 0} \|T x_t - x_t\| = 0$ .) In 1967, the strong convergence of  $\{x_t\}$  as  $t \rightarrow 0$  for a self-mapping  $T$  of a bounded  $C$  was proved in a Hilbert space independently by Browder [3] and Halpern [9]. In 1980, Reich [14] extended the result of Browder [3] to a uniformly smooth Banach space and showed that the limit defines the (unique) sunny nonexpansive retraction from  $C$  onto  $F(T)$ . Takahashi and Ueda [18] improved results of Reich [14] to a reflexive Banach space with a uniformly Gâteaux differentiable norm (see also Ha and Jung [8]).

On the other hand, to order to extend Browder's and Reich's results to the nonexpansive semigroup  $\{T(t) : t \geq 0\}$  case, Shioji-Takahashi [15] introduced in a Hilbert space the implicit iterative scheme

$$x_n = \alpha_n u + (1 - \alpha_n) \sigma_{t_n}(x_n), \quad n \geq 1, \quad (1.2)$$

where  $\{\alpha_n\}$  is a sequence in  $(0, 1)$ ,  $\{t_n\}$  is a sequence of positive real number divergent to  $\infty$  and for each  $t > 0$  and  $x \in C$ ,  $\sigma_t(x)$  is the average given by

$$\sigma_t(x) = \frac{1}{t} \int_0^t T(s)x ds.$$

Under suitable conditions on the sequence  $\{\alpha_n\}$ , they proved the strong convergence of  $\{x_n\}$  defined by (1.2) to a point in  $F := \bigcap_{t \geq 0} F(T(t))$ . In 2003, Suzuki [16] introduced firstly in Hilbert space the following implicit iterative scheme:

$$x_n = \alpha_n u + (1 - \alpha_n) T(t_n)x_n, \quad n \geq 1 \quad (1.3)$$

for the nonexpansive semigroup case and proved strong convergence of the iterative scheme (1.3) with appropriate conditions imposed upon sequences  $\{\alpha_n\}$  and  $\{t_n\}$ . In 2005, Xu [20] proved that Suzuki's result holds in a uniformly convex Banach space with a weakly continuous duality mapping.

In 2005, Aleyner and Reich [1] first introduced the following explicit iterative scheme

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T(t_n)x_n, \quad n \geq 0 \quad (1.4)$$

in a reflexive Banach space having a uniformly Gâteaux differentiable norm such that each nonempty, bounded, closed and convex subset of  $E$  has the fixed point property for nonexpansive mappings (Note that all these assumptions are fulfilled whenever  $E$  is uniformly smooth). Under the following conditions on  $\{\alpha_n\}$  and  $\{t_n\}$ ;

- (i)  $\alpha_n \rightarrow 0$  ( $n \rightarrow \infty$ );  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;
- (ii)  $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ;
- (iii)  $t_0 < t_1 < t_2 < \dots < t_n < \dots$ ,  $\lim_{n \rightarrow \infty} t_n = \infty$ ,

and the uniformly asymptotic regularity on  $\{T(t) : t \geq 0\}$ , they showed that the sequence  $\{x_n\}$  defined by (1.4) converges strongly to  $Qu$ , where  $Q$  is the unique sunny nonexpansive retraction from  $C$  onto  $F := \bigcap_{t \geq 0} F(T(t))$ ,  $Qu = s\text{-}\lim_{t \rightarrow \infty} x_t$  and  $x_t$  is the unique solution of the following equation:

$$x_t = \alpha_t u + (1 - \alpha_t)T(t)x_t, \quad t \in (0, \infty),$$

where  $\{\alpha_t\}_{t \in (0, \infty)}$  is a net in  $(0, 1)$  such that  $\lim_{t \rightarrow \infty} \alpha_t = 0$ .

Recently, Chang and Yang [4] considered the the following composite iterative scheme

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)T(t_n)x_n \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \quad n \geq 0 \end{cases}$$

in either a reflexive Banach space having a uniformly Gâteaux differentiable norm or a uniformly convex Banach space having a weakly sequentially continuous duality mapping, where  $\{\alpha_n\}$  and  $\{t_n\} \subset \mathbb{R}^+$  satisfy the conditions (i)–(iii) and  $\{\beta_n\} \subset [0, a]$ , for some constant  $a \in (0, 1)$ , satisfies the following condition

- (iv)  $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ .

On the another hand, the viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping was proposed by Moudafi [12]. In 2004, in order to extend Theorem 2.2 of Moudafi [12] to a Banach space setting, Xu [19] consider the the following explicit iterative scheme: for  $T : C \rightarrow C$  nonexpansive mapping,  $f \in \Sigma_C$  and  $\alpha_n \in (0, 1)$ ,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0.$$

Moreover, in [19], he also studied the strong convergence of  $x_t$  defined by (1.1) as  $t \rightarrow 0$  in either a Hilbert space or a uniformly smooth Banach space and showed

that the strong  $\lim_{t \rightarrow 0} z_t$  is the unique solution of certain variational inequality. This result of Xu [19] also improved Theorem 2.1 of Moudafi [12] as the continuous version.

In this paper, motivated by above-mentioned results, we consider two iterative schemes as the viscosity approximation method for nonexpansive semigroup  $\{T(t) : t \geq 0\}$  on  $C$ ; for  $f \in \Sigma_C$ ,

$$z_t = \lambda_t f(z_t) + (1 - \lambda_t)T(t)z_t, \quad t \in (0, \infty) \quad (1.5)$$

where  $\{\lambda_t\}_{t \in (0, \infty)}$  is a net in  $(0, 1)$  such that  $\lim_{t \rightarrow \infty} \lambda_t = 0$ , and

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)T(t_n)x_n \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)y_n, \quad n \geq 0. \end{cases} \quad (1.6)$$

where  $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$  and  $\{t_n\} \subset \mathbb{R}^+$ . First, by using the uniform asymptotic regularity on  $\{T(t) : t \geq 0\}$ , we establish a strong convergence theorem for the sequence  $\{z_t\}$  defined by (1.5) in a reflexive and strictly convex Banach space having a uniformly Gâteaux differentiable norm. Then, under certain different control conditions on  $\{\alpha_n\}$  and  $\{\beta_n\}$  and the uniform asymptotic regularity on  $\{T(t) : t \geq 0\}$ , we prove in the same Banach space that the sequence  $\{x_n\}$  generated by (1.6) converges strongly to a common fixed point of  $\{T(t) : t \geq 0\}$  which is a solution of a certain variational inequality. The main results improve and develop the corresponding results of Aleyner and Reich [1], Chang and Yang [4], Shioji-Takahashi [15], Suzuki [16] and Xu [21].

## 2. Preliminaries and Lemmas

Let  $E$  be a real Banach space with norm  $\|\cdot\|$  and let  $E^*$  be its dual. The value of  $f \in E^*$  at  $x \in E$  will be denoted by  $\langle x, f \rangle$ . When  $\{x_n\}$  is a sequence in  $E$ , then  $x_n \rightarrow x$  will denote strong convergence of the sequence  $\{x_n\}$ .

A Banach space  $E$  is called *strictly convex* if its unit sphere  $U = \{x \in E : \|x\| = 1\}$  does not contain any linear segment ([6,7]). This condition is equivalent to the following:

$$\|x\| = \|y\| = 1, \quad x \neq y \Rightarrow \left\| \frac{x+y}{2} \right\| < 1.$$

The (*normalized*) *duality mapping*  $J$  from  $E$  into the family of nonempty (by Hahn-Banach theorem) weak-star compact subsets of its dual  $E^*$  is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$$

for each  $x \in E$ .

The norm of  $E$  is said to be *Gâteaux differentiable* (and  $E$  is said to be *smooth*) if

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.1)$$

exists for each  $x, y$  in its unit sphere  $U = \{x \in E : \|x\| = 1\}$ . It is said to be *uniformly Gâteaux differentiable* if for  $y \in U$ , the limit is attained uniformly for  $x \in U$ . The space  $E$  is said to have a *uniformly Fréchet differentiable norm* (and  $E$  is said to be *uniformly smooth*) if the limit in (2.1) is attained uniformly for  $(x, y) \in U \times U$ . It is known that  $E$  is smooth if and only if each duality mapping  $J$  is single-valued. It is also well-known that if  $E$  has a uniformly Gâteaux differentiable norm,  $J$  is uniformly norm to weak\* continuous on each bounded subsets of  $E$  ([5]).

Let  $C$  be a nonempty closed convex subset of  $E$ .  $C$  is said to have the *fixed point property* for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset  $D$  of  $C$  has a fixed point in  $D$  ([6]). Let  $D$  be a subset of  $C$ . Then a mapping  $Q : C \rightarrow D$  is said to be a retraction from  $C$  onto  $D$  if  $Qx = x$  for all  $x \in D$ . A retraction  $Q : C \rightarrow D$  is said to be *sunny* if  $Q(Qx + t(x - Qx)) = Qx$  for all  $x \in C$  and  $t \geq 0$  with  $Qx + t(x - Qx) \in C$ . A subset  $D$  of  $C$  is said to be a *sunny nonexpansive retract* of  $C$  if there exists a sunny nonexpansive retraction of  $C$  onto  $D$ . In a smooth Banach space  $E$ , it is well-known ([7, p. 48]) that  $Q$  is a sunny nonexpansive retraction from  $C$  onto  $D$  if and only if the following condition holds:

$$\langle x - Qx, J(z - Qx) \rangle \leq 0, \quad x \in C, \quad z \in D. \quad (2.2)$$

Let LIM be a continuous linear functional on  $l^\infty$ . According to time and circumstances, we LIM( $a_n$ ) instead of LIM( $a$ ) for  $a = \{a_n\} \in l^\infty$ . LIM is said to be *Banach limit* if

$$\text{LIM}(a_n) = \text{LIM}(a_{n+1})$$

for every  $a = \{a_n\} \in l^\infty$ . Using the Hahn-Banach theorem, or the Tychonoff fixed point theorem, we can prove the existence of a Banach limit. We know that if LIM is a Banach limit, then

$$\liminf_{n \rightarrow \infty} a_n \leq \text{LIM}(a_n) \leq \limsup_{n \rightarrow \infty} a_n$$

for all  $a = \{a_n\} \in l^\infty$ .

We need the following lemmas for the proof of our main results. (Lemma 2.1 was also given in [13]. Lemma 2.2 is Lemma 2 of [17] and Lemma 2.3 is essentially Lemma 2 in [11] (also see [19]). Lemma 2.4 was given in [8, 18], which is essentially a variant of Lemma 1.2 in [13]). We refer also [5, 6, 7] for Lemmas 2.5 and 2.6.

**Lemma 2.1.** *Let  $E$  be a real Banach space and  $J$  be the duality mapping. Then, for any given  $x, y \in E$ , one has*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle$$

for all  $j(x + y) \in J(x + y)$ .

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**Lemma 2.2.** *Let  $\{x_n\}$  and  $\{z_n\}$  be bounded sequences in a Banach space  $E$  and let  $\{\gamma_n\}$  be a sequence in  $[0, 1]$  which satisfies the following condition:*

$$0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1.$$

*Suppose that*

$$x_{n+1} = \gamma_n x_n + (1 - \gamma_n) z_n, \quad n \geq 0,$$

*and*

$$\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

*Then  $\|z_n - x_n\| = 0$ .*

**Lemma 2.3.** *Let  $\{s_n\}$  be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n) s_n + \lambda_n \beta_n + \gamma_n, \quad n \geq 0,$$

*where  $\{\lambda_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  satisfy the following conditions:*

- (i)  $\{\lambda_n\} \subset [0, 1]$  and  $\sum_{n=0}^{\infty} \lambda_n = \infty$  or, equivalently,  $\prod_{n=0}^{\infty} (1 - \lambda_n) = 0$ ;
- (ii)  $\limsup_{n \rightarrow \infty} \beta_n \leq 0$  or  $\sum_{n=1}^{\infty} \lambda_n \beta_n < \infty$ ;
- (iii)  $\gamma_n \geq 0$  ( $n \geq 0$ ),  $\sum_{n=0}^{\infty} \gamma_n < \infty$ .

*Then  $\lim_{n \rightarrow \infty} s_n = 0$ .*

**Lemma 2.4.** *Let  $C$  be a nonempty closed convex subset of a Banach space  $E$  having a uniformly Gâteaux differentiable norm and  $\{x_n\}$  be a bounded sequence in  $E$ . Let LIM be a Banach limit on  $l^\infty$  and  $q \in C$ . Then*

$$\text{LIM} \|x_n - q\|^2 = \min_{y \in C} \text{LIM} \|x_n - y\|^2$$

*if and only if*

$$\text{LIM} \langle x - q, J(x_n - q) \rangle \leq 0$$

*for all  $x \in C$ , where  $J$  is the duality mapping of  $E$ .*

**Lemma 2.5.** *Let  $C$  be a closed convex of a reflexive and strictly convex Banach space  $E$ . Then  $C^o = \{x \in E : \|x\| = \inf\{\|y\| : y \in C\}\}$  is a singleton.*

**Lemma 2.6.** *Let  $E$  be a smooth Banach space,  $C$  a nonempty closed convex subset of  $E$  and  $T : C \rightarrow C$  a nonexpansive mapping. If  $J$  is the duality mapping on  $E$ , then*

$$\langle (I - T)(x) - (I - T)(y), J(x - y) \rangle \geq 0, \quad \text{for all } x, y \in C.$$

Finally, recall that a nonexpansive semigroup  $\{T(t) : t \geq 0\}$  on  $C$  is said to be *uniformly asymptotically regular* (shortly, u.a.r) on bounded subsets of  $C$  if

$$T(s + t)x = T(s)T(t)x$$

for all  $s, t \geq 0$  and  $x \in C$  and for all bounded subset  $K$  of  $C$  there holds

$$\lim_{t \rightarrow \infty} \sup_{x \in K} \|T(s)T(t)x - T(t)x\| = 0, \quad (2.3)$$

uniformly for all  $s \geq 0$ . Note that both these assumptions hold when the trajectories of the nonexpansive semigroup  $\{T(t) : t \geq 0\}$  converge uniformly on bounded subsets of  $E$ .

### 3. Main results

First, we study the existence of solutions of certain variational inequality.

For any  $t \geq 0$ ,  $T(t) : C \rightarrow C$  is nonexpansive and so, for any  $\lambda_t \in (0, 1)$  and  $f \in \Sigma_C$ ,  $\lambda_t f + (1 - \lambda_t)T(t) : C \rightarrow C$  defines a strict contraction mapping. Thus, by the Banach contraction mapping principle, there exists a unique fixed point  $x_t^f$  satisfying

$$z_t^f = \lambda_t f(z_t^f) + (1 - \lambda_t)T(t)z_t^f. \quad (A)$$

For simplicity we will write  $z_t$  for  $z_t^f$  provided no confusion occurs.

Now we show that the sequence  $\{z_t\}$  defined by (A) converges strongly some common fixed point of  $\{T(t) : t \geq 0\}$ .

**Theorem 3.1.** *Let  $E$  be a reflexive and strictly convex Banach space having a uniformly Gâteaux differentiable norm. Let  $C$  be a nonempty closed convex subset of  $E$  and  $\{T(t) : t \geq 0\}$  a u.a.r. nonexpansive semigroup from  $C$  into itself with  $F := \bigcap_{t \geq 0} F(T(t)) \neq \emptyset$ . Let  $\{z_t\}$  be defined by (A) and  $\lambda_t \in (0, 1)$  such that  $\lim_{t \rightarrow \infty} \lambda_t = 0$ . Then as  $t \rightarrow \infty$ ,  $\{z_t\}$  converges strongly to a point in  $F$ . If we define  $Q : \Sigma_C \rightarrow F$  by*

$$Q(f) := \lim_{t \rightarrow \infty} z_t, \quad f \in \Sigma_C, \quad (3.1)$$

*then  $Q(f)$  is the unique solution in  $F$  of the variational inequality*

$$\langle (I - f)(Q(f)), J(Q(f) - p) \rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F.$$

*Proof.* Let  $\{z_{t_n}\}$  be a subsequence of  $\{z_t\}$  such that  $\lim_{n \rightarrow \infty} t_n = \infty$ . Let  $p \in F$ . Then

$$\begin{aligned} \|z_{t_n} - p\| &\leq \lambda_{t_n} \|f(z_{t_n}) - p\| + (1 - \lambda_{t_n}) \|T(t_n)z_{t_n} - T(t_n)p\| \\ &\leq \lambda_{t_n} \|f(z_{t_n}) - p\| + (1 - \lambda_{t_n}) \|z_{t_n} - p\|. \end{aligned}$$

This gives that

$$\begin{aligned} \|z_{t_n} - p\| &\leq \|f(z_{t_n}) - p\| \leq \|f(z_{t_n}) - f(p)\| + \|f(p) - p\| \\ &\leq k \|z_{t_n} - p\| + \|f(p) - p\|, \end{aligned}$$

and so  $\|z_{t_n} - p\| \leq \frac{1}{1-k} \|f(p) - p\|$ . In particular,  $\{z_{t_n}\}$  is bounded, so are  $\{f(z_{t_n})\}$  and  $\{T(t_n)z_{t_n}\}$ .

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Now define a functional  $\phi$  on  $C$  by

$$\phi(x) = \text{LIM} \|z_{t_n} - x\|^2,$$

where LIM is a Banach limit on  $l^\infty$ . Since  $\phi$  is continuous and convex,  $\phi(z) \rightarrow \infty$  as  $\|z\| \rightarrow \infty$ , and  $E$  is reflexive,  $\phi$  attains its infimum over  $C$  [2, p. 79]. Let

$$K = \{x \in C : \phi(x) = \min_{y \in C} \phi(y)\}.$$

It is easily seen that  $K$  is a nonempty closed convex bounded subset of  $E$ . Moreover,  $K$  is invariant under  $T(r)$  for any  $r \geq 0$ . In fact, since

$$\|z_{t_n} - T(t_n)z_{t_n}\| = \lambda_{t_n} \|f(z_{t_n}) - T(t_n)z_{t_n}\| \rightarrow 0 \quad (\text{as } n \rightarrow \infty)$$

and for any  $r > 0$ ,

$$\begin{aligned} & \|z_{t_n} - T(r)x\| \\ & \leq \|z_{t_n} - T(t_n)z_{t_n}\| + \|T(t_n)z_{t_n} - T(r)T(t_n)z_{t_n}\| + \|T(r)T(t_n)z_{t_n} - T(r)x\| \\ & \leq \|z_{t_n} - T(t_n)z_{t_n}\| + \sup_{y \in B} \|T(t_n)y - T(r)T(t_n)y\| + \|T(r)T(t_n)z_{t_n} - T(r)x\|, \end{aligned}$$

where  $B$  is a bounded subset of  $E$  containing  $\{z_{t_n}\}$ , it follows from (2.3) that for each  $x \in K$

$$\begin{aligned} \phi(T(r)x) &= \text{LIM} \|z_{t_n} - T(r)x\|^2 \\ &\leq \text{LIM} \|T(r)T(t_n)z_{t_n} - T(r)x\|^2 \\ &\leq \text{LIM} \|T(t_n)z_{t_n} - x\|^2 = \text{LIM} \|z_{t_n} - x\|^2 = \phi(x). \end{aligned}$$

So,  $K$  contains a fixed point of  $\{T(r)\}$  for any  $r > 0$ . Indeed, define

$$K^o = \{v \in K : \|v - p\| = \min_{y \in K} \|y - p\|\}.$$

Then, by Lemma 2.5,  $K^o$  is a singleton. Denote such a singleton by  $z$ . Then we have

$$\|T(r)z - p\| = \|T(r)z - T(r)p\| \leq \|z - p\|$$

and hence  $T(r)z = z$  for any  $r > 0$ . That is,  $K$  contains a common fixed point of  $\{T(t) : t \geq 0\}$ . Let  $q \in K \cap F$  be a such a common fixed point. Since  $q$  is a minimizer of  $\phi$  over  $C$ , it follows from Lemma 2.4 that for any  $x \in C$ ,

$$\text{LIM} \langle x - q, J(z_{t_n} - q) \rangle \leq 0. \quad (3.2)$$

On the other hand, for any  $p \in F$ ,

$$z_{t_n} - p = (1 - \lambda_{t_n})(T(t_n)z_{t_n} - p) + \lambda_{t_n}(f(z_{t_n}) - p).$$



It follows that

$$\begin{aligned}\|z_{t_n} - p\|^2 &= (1 - \lambda_{t_n})\langle T(t_n)z_{t_n} - p, J(z_{t_n} - p) \rangle + \lambda_{t_n}\langle f(z_{t_n}) - p, J(z_{t_n} - p) \rangle \\ &\leq (1 - \lambda_{t_n})\|z_{t_n} - p\|^2 + \lambda_{t_n}(k\|z_{t_n} - p\|^2 + \langle f(p) - p, J(z_{t_n} - p) \rangle),\end{aligned}$$

and so

$$\|z_{t_n} - p\|^2 \leq \frac{1}{1 - k}\langle f(p) - p, J(z_{t_n} - p) \rangle. \quad (3.3)$$

Combining (3.2) and (3.3), we obtain

$$\text{LIM}\|z_{t_n} - q\|^2 \leq 0.$$

Hence there is a subsequence  $\{z_{t_{n_j}}\}$  of  $\{z_{t_n}\}$  such that  $\lim_{j \rightarrow \infty} \|z_{t_{n_j}} - q\| = 0$ . Assume that there exists another subsequence  $\{z_{t_{n_k}}\}$  of  $\{z_{t_n}\}$  such that  $\lim_{k \rightarrow \infty} \|z_{t_{n_k}} - \tilde{q}\| = 0$ , where  $\tilde{q} \in K \cap F$ . Then (3.3) implies that

$$\|q - \tilde{q}\|^2 \leq \frac{1}{1 - k}\langle f(q) - \tilde{q}, J(q - \tilde{q}) \rangle. \quad (3.4)$$

Similarly we have

$$\|\tilde{q} - q\|^2 \leq \frac{1}{1 - k}\langle f(\tilde{q}) - q, J(\tilde{q} - q) \rangle. \quad (3.5)$$

Adding (3.4) and (3.5), we get  $\|q - \tilde{q}\|^2 \leq 0$ , that is,  $q = \tilde{q}$ .

The same argument shows that if  $t_l \rightarrow \infty$ , then the subsequence  $\{z_{t_l}\}$  of  $\{z_t\}$  converges strongly to the same limit. Thus, as  $t \rightarrow \infty$ ,  $\{z_t\}$  converges strongly to a point in  $F$ .

If we define  $Q : \sum_C \rightarrow F$  by  $Q(f) = \lim_{t \rightarrow \infty} z_t$ ,  $f \in \sum_C$ , then  $Q(f)$  solves the variational inequality

$$\langle (I - f)(Q(f)), J(Q(f) - p) \rangle \leq 0, \quad f \in \sum_C, \quad p \in F.$$

In fact, since

$$(I - f)(z_t) = -\frac{1 - \lambda_t}{\lambda_t}(I - T(t))(z_t),$$

by Lemma 2.6, we have for  $p \in F$ ,

$$\langle (I - f)(z_t), J(z_t - p) \rangle = -\frac{1 - \lambda_t}{\lambda_t} \langle (I - T(t))(z_t) - (I - T(t))(p), J(z_t - p) \rangle \leq 0.$$

Noting the fact that  $J$  is uniformly continuous on bounded subsets of  $E$  from the strong topology of  $E$  to the weak\* topology of  $E^*$  and taking the limit as  $t \rightarrow \infty$ , we obtain

$$\langle (I - f)(Q(f)), J(Q(f) - p) \rangle \leq 0, \quad f \in \sum_C, \quad p \in F.$$

□

**Remark 3.1.** If  $f(x) = u$ ,  $x \in C$ , is a constant in Theorem 3.1, then

$$\langle Qu - u, J(Qu - p) \rangle \leq 0, \quad u \in C, \quad p \in F.$$

Hence by (2.2),  $Q$  reduces to the sunny nonexpansive retraction from  $C$  to  $F$ .

**Remark 3.2.** (1) Theorem 3.1 improves the corresponding result in Aleyner and Reich [1] to the viscosity method in the different Banach space.

(2) Theorem 3.1 appears to be independent of the result in Aleyner and Reich [1]. On the one hand, it is easy to find examples of spaces which satisfy the fixed point property for nonexpansive mapping, which are not strictly convex. However, it appears to be unknown whether a reflexive and strictly convex space satisfy the fixed point property for nonexpansive mappings.

(3) Theorem 3.1 also develops the corresponding result of Shioji and Takahashi [15], Suzuki [16] and Xu [21] to the viscosity method in more general Banach space.

By using Theorem 3.1, we have the following result.

**Theorem 3.2.** *Let  $E$  be a reflexive and strictly convex Banach space having a uniformly Gâteaux differentiable norm. Let  $C$  be a nonempty closed convex subset of  $E$  and  $\{T(t) : t \geq 0\}$  a u.a.r. nonexpansive semigroup from  $C$  into itself with  $F := \bigcap_{t \geq 0} F(T(t)) \neq \emptyset$ . Let  $\{\alpha_n\} \subset (0, 1)$ ,  $\{\beta_n\} \subset [0, 1]$  and  $\{t_n\} \subset \mathbb{R}^+$  be sequences satisfying the following conditions:*

(C1)  $\alpha_n \rightarrow 0$  ( $n \rightarrow \infty$ );  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;

(C2)  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n \leq a < 1$  for all  $n \geq 0$  and for some constant  $a \in (0, 1)$ ;

(C3)  $0 \leq t_0 < t_1 < t_2 < \cdots < t_n < \cdots$ ,  $\lim_{n \rightarrow \infty} t_n = \infty$ .

Let  $f \in \sum_C$  and  $x_0 \in C$  be chosen arbitrarily. Let  $\{x_n\}$  be defined by

$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1 - \beta_n)T(t_n)x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)y_n \end{cases} \quad (\text{IS})$$

Then  $\{x_n\}$  converges strongly to  $Q(f) \in F$ , where  $Q(f)$  solves a variational inequality

$$\langle (I - f)(Q(f)), J(Q(f) - p) \rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F.$$

*Proof.* First we note that by Theorem 3.1, there exists a solution  $Q(f) \in F$  of a variational inequality

$$\langle (I - f)(Q(f)), J(Q(f) - p) \rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F,$$

where  $Q(f) = \lim_{t \rightarrow \infty} z_t$  and  $z_t$  is defined by (A).

We proceed with following steps:

**Step 1.** We show that  $\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{1}{1-k}\|f(p) - p\|\}$  for  $p \in F$ .  
Indeed, let  $p \in F$  and  $d = \max\{\|x_0 - p\|, \frac{1}{1-k}\|f(p) - p\|\}$ . Noting that

$$\|y_n - p\| \leq \beta_n \|x_n - p\| + (1 - \beta_n) \|T(t_n)x_n - p\| \leq \|x_n - p\|,$$

we have

$$\begin{aligned} \|x_1 - p\| &\leq (1 - \alpha_0) \|y_0 - p\| + \alpha_0 \|f(x_0) - p\| \\ &\leq (1 - \alpha_0) \|x_0 - p\| + \alpha_0 (\|f(x_0) - f(p)\| + \|f(p) - p\|) \\ &\leq (1 - \alpha_0) \|x_0 - p\| + \alpha_0 (k \|x_0 - p\| + \|f(p) - p\|) \\ &\leq (1 - (1 - k)\alpha_0) \|x_0 - p\| + \alpha_0 \|f(p) - p\| \\ &\leq (1 - (1 - k)\alpha_0) d + \alpha_0 (1 - k) d = d. \end{aligned}$$

Using an induction, we obtain  $\|x_{n+1} - p\| \leq d$ . Hence  $\{x_n\}$  is bounded, and so are  $\{T(t_n)x_n\}$ ,  $\{f(x_n)\}$  and  $\{y_n\}$ .

**Step 2.** We show that  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ . To this end, set  $\gamma_n = (1 - \alpha_n)\beta_n$ ,  $n \geq 0$ . Then it follow from (C1) and (C2) that

$$0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1. \quad (3.1)$$

Define

$$x_{n+1} = \gamma_n x_n + (1 - \gamma_n) z_n. \quad (3.2)$$

Observe that

$$\begin{aligned} z_{n+1} - z_n &= \frac{x_{n+2} - \gamma_{n+1}x_{n+1}}{1 - \gamma_{n+1}} - \frac{x_{n+1} - \gamma_n x_n}{1 - \gamma_n} \\ &= \frac{\alpha_{n+1}f(x_{n+1}) + (1 - \alpha_{n+1})y_{n+1} - \gamma_{n+1}x_{n+1}}{1 - \gamma_{n+1}} \\ &\quad - \frac{\alpha_n f(x_n) + (1 - \alpha_n)y_n - \gamma_n x_n}{1 - \gamma_n} \\ &= \left( \frac{\alpha_{n+1}f(x_{n+1})}{1 - \gamma_{n+1}} - \frac{\alpha_n f(x_n)}{1 - \gamma_n} \right) \\ &\quad - \frac{(1 - \alpha_n)[\beta_n x_n + (1 - \beta_n)T(t_n)x_n] - \gamma_n x_n}{1 - \gamma_n} \\ &\quad + \frac{(1 - \alpha_{n+1})[\beta_{n+1}x_{n+1} + (1 - \beta_{n+1})T(t_{n+1})x_{n+1}] - \gamma_{n+1}x_{n+1}}{1 - \gamma_{n+1}} \\ &= \left( \frac{\alpha_{n+1}f(x_{n+1})}{1 - \gamma_{n+1}} - \frac{\alpha_n f(x_n)}{1 - \gamma_n} \right) + \frac{(1 - \alpha_{n+1})(1 - \beta_{n+1})T(t_{n+1})x_{n+1}}{1 - \gamma_{n+1}} \\ &\quad - \frac{(1 - \alpha_n)(1 - \beta_n)T(t_n)x_n}{1 - \gamma_n} \end{aligned} \quad (3.3)$$

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$$\begin{aligned}
&= \left( \frac{\alpha_{n+1}f(x_{n+1})}{1-\gamma_{n+1}} - \frac{\alpha_n f(x_n)}{1-\gamma_n} \right) + (T(t_{n+1})x_{n+1} - T(t_n)x_n) \\
&\quad - \frac{\alpha_{n+1}}{1-\gamma_{n+1}}T(t_{n+1})x_{n+1} + \frac{\alpha_n}{1-\gamma_n}T(t_n)x_n \\
&= \left( \frac{\alpha_{n+1}f(x_{n+1})}{1-\gamma_{n+1}} - \frac{\alpha_n f(x_n)}{1-\gamma_n} \right) + (T(t_{n+1})x_{n+1} - T(t_{n+1})x_n) \\
&\quad + (T(t_{n+1} - t_n)T(t_n)x_n - T(t_n)x_n) \\
&\quad - \frac{\alpha_{n+1}}{1-\gamma_{n+1}}T(t_{n+1})x_{n+1} + \frac{\alpha_n}{1-\gamma_n}T(t_n)x_n.
\end{aligned}$$

It follows from (3.3) and (C3) that

$$\begin{aligned}
&\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\gamma_{n+1}}(\|f(x_{n+1})\| + \|T(t_{n+1})x_{n+1}\|) + \frac{\alpha_n}{1-\gamma_n}(\|f(x_n)\| + \|T(t_n)x_n\|) \quad (3.4) \\
&\quad + \sup_{x \in D} \|T(t_{n+1} - t_n)T(t_n)x - T(t_n)x\|,
\end{aligned}$$

where  $D = \{x \in C : \|x - p\| \leq \max\{\|x_0 - p\|, \frac{1}{1-k}\|f(p) - p\|\}\}$ . Since  $\{f(x_n)\}$  and  $\{T(t_n)x_n\}$  are bounded, by (2.3), (C1), (3.1) and (3.4) we obtain that

$$\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence by Lemma 2.2, we have

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0. \quad (3.5)$$

It then follows from (3.1) and (3.2) that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.$$

**Step 3.** We show that  $\lim_{n \rightarrow \infty} \|x_n - T(t_n)x_n\| = 0$ . Indeed, as a consequence with control condition (C1), we know

$$\|x_{n+1} - y_n\| \leq \alpha_n \|f(x_n) - y_n\| \leq \alpha_n (\|f(x_n)\| + \|y_n\|) \rightarrow 0 \text{ (as } n \rightarrow \infty). \quad (3.6)$$

By (IS) we have  $\|y_n - T(t_n)x_n\| = \beta_n \|x_n - T(t_n)x_n\|$  and so

$$\begin{aligned}
\|x_n - T(t_n)x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - T(t_n)x_n\| \\
&= \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n \|x_n - T(t_n)x_n\|.
\end{aligned}$$

Simplifying it and using Step 2 and (3.6), we obtain

$$\begin{aligned}
(1-a)\|x_n - T(t_n)x_n\| &\leq (1-\beta_n)\|x_n - T(t_n)x_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \rightarrow 0 \text{ (as } n \rightarrow \infty)
\end{aligned}$$

This implies that

$$\|x_n - T(t_n)x_n\| \rightarrow 0 \text{ (as } n \rightarrow \infty\text{)}.$$

**Step 4.** We show that  $\lim_{n \rightarrow \infty} \|T(r)x_n - x_n\| = 0$  uniformly in  $r \in \mathbb{R}^+$ . In fact, it follows from Step 3 and (2.3) that

$$\begin{aligned} \|T(r)x_n - x_n\| &\leq \|T(r)x_n - T(r)T(t_n)x_n\| + \|T(r)T(t_n)x_n - T(t_n)x_n\| \\ &\quad + \|T(t_n)x_n - x_n\| \\ &\leq 2\|x_n - T(t_n)x_n\| + \sup_{x \in D} \|T(r)T(t_n)x - T(t_n)x\| \\ &\rightarrow 0 \text{ (as } n \rightarrow \infty\text{)} \end{aligned}$$

uniformly  $r \in \mathbb{R}^+$ , where  $D = \{x \in C : \|x - p\| \leq \max\{\|x_0 - p\|, \frac{1}{1-k}\|f(p) - p\|\}\}$ .

**Step 5.** We show that  $\limsup_{n \rightarrow \infty} \langle (I - f)(Q(f)), J(Q(f) - x_n) \rangle \leq 0$ . To prove this, let a subsequence  $\{x_{n_j}\}$  of  $\{x_n\}$  be such that

$$\limsup_{n \rightarrow \infty} \langle (I - f)(Q(f)), J(Q(f) - x_n) \rangle = \lim_{j \rightarrow \infty} \langle (I - f)(Q(f)), J(Q(f) - x_{n_j}) \rangle$$

and  $x_{n_j} \rightarrow p$  for some  $p \in C$ . Now let  $z_t$  be defined by  $z_t = \lambda_t f(z_t) + (1 - \lambda_t)T(t)z_t$  for each  $t \in \mathbb{R}^+$  and  $0 < \lambda_t < 1$  with  $\lim_{t \rightarrow \infty} \lambda_t = 0$ . Then we can write

$$z_t - x_n = (1 - \lambda_t)(T(t)z_t - x_n) + \lambda_t(f(z_t) - x_n).$$

Applying Lemma 2.1, we have

$$\|z_t - x_n\|^2 \leq (1 - \lambda_t)^2 \|T(t)z_t - x_n\|^2 + 2\lambda_t \langle f(z_t) - x_n, J(z_t - x_n) \rangle.$$

Putting

$$\begin{aligned} a_j(t) &= (1 - \lambda_t)^2 \|T(t)x_{n_j} - x_{n_j}\| \\ &\quad \times (2\|z_t - x_{n_j}\| + \|T(t)x_{n_j} - x_{n_j}\|) \rightarrow 0 \text{ (as } j \rightarrow \infty\text{)} \end{aligned}$$

by Step 4 and using Lemma 2.1, we obtain

$$\begin{aligned} \|z_t - x_{n_j}\|^2 &\leq (1 - \lambda_t)^2 \|T(t)z_t - x_{n_j}\|^2 + 2\lambda_t \langle f(z_t) - x_{n_j}, J(z_t - x_{n_j}) \rangle \\ &\leq (1 - \lambda_t)^2 (\|T(t)z_t - T(t)x_{n_j}\| + \|T(t)x_{n_j} - x_{n_j}\|)^2 \\ &\quad + 2\lambda_t \langle f(z_t) - z_t, J(z_t - x_{n_j}) \rangle + 2\lambda_t \|z_t - x_{n_j}\|^2 \\ &\leq (1 - \lambda_t)^2 \|z_t - x_{n_j}\|^2 + a_j(t) \\ &\quad + 2\lambda_t \langle f(z_t) - z_t, J(z_t - x_{n_j}) \rangle + 2\lambda_t \|z_t - x_{n_j}\|^2. \end{aligned}$$

The last inequality implies

$$\langle z_t - f(z_t), J(z_t - x_{n_j}) \rangle \leq \frac{\lambda_t}{2} \|z_t - x_{n_j}\|^2 + \frac{1}{2\lambda_t} a_j(t).$$

It follows that

$$\lim_{j \rightarrow \infty} \langle z_t - f(z_t), J(z_t - x_{n_j}) \rangle \leq \frac{\lambda_t}{2} M, \quad (3.7)$$

where  $M > 0$  is a constant such that  $M \geq \|z_t - x_n\|^2$  for all  $n \geq 0$  and  $\lambda_t \in (0, 1)$ . Taking the lim sup as  $t \rightarrow \infty$  in (3.7) and noticing the fact that the two limits are interchangeable due to the fact that  $J$  is uniformly continuous on bounded subsets of  $E$  from the strong topology of  $E$  to the weak\* topology of  $E^*$ , we have

$$\limsup_{j \rightarrow \infty} \langle Q(f) - f(Q(f)), J(Q(f) - x_{n_j}) \rangle \leq 0.$$

Indeed, letting  $t \rightarrow \infty$ , from (3.7) we have

$$\limsup_{t \rightarrow \infty} \limsup_{j \rightarrow \infty} \langle z_t - f(z_t), J(z_t - x_{n_j}) \rangle \leq 0.$$

So, for any  $\varepsilon > 0$ , there exists a positive number  $\delta_1$  such that for any  $t > \delta_1$ ,

$$\limsup_{j \rightarrow \infty} \langle z_t - f(z_t), J(z_t - x_{n_j}) \rangle \leq \frac{\varepsilon}{2}.$$

Moreover, since  $z_t \rightarrow Q(f)$  as  $t \rightarrow \infty$ , the set  $\{z_t - x_{n_j}\}$  is bounded and the duality mapping  $J$  is norm-to-weak\* uniformly continuous on bounded subset of  $E$ , there exists  $\delta_2 > 0$  such that, for any  $t > \delta_2$ ,

$$\begin{aligned} & |\langle Q(f) - f(Q(f)), J(Q(f) - x_{n_j}) \rangle - \langle z_t - f(z_t), J(z_t - x_{n_j}) \rangle| \\ &= |\langle Q(f) - f(Q(f)), J(Q(f) - x_{n_j}) - J(z_t - x_{n_j}) \rangle \\ &\quad + \langle Q(f) - f(Q(f)) - (z_t - f(z_t)), J(z_t - x_{n_j}) \rangle| \\ &\leq |\langle Q(f) - f(Q(f)), J(z_t - x_{n_j}) - J(Q(f) - x_{n_j}) \rangle| \\ &\quad + \|Q(f) - f(Q(f)) - (z_t - f(z_t))\| \|z_t - x_{n_j}\| < \frac{\varepsilon}{2}. \end{aligned}$$

Choose  $\delta = \max\{\delta_1, \delta_2\}$ , we have for all  $t > \delta$  and  $j \in \mathbb{N}$ ,

$$\langle Q(f) - f(Q(f)), J(Q(f) - x_{n_j}) \rangle < \langle z_t - f(z_t), J(z_t - x_{n_j}) \rangle + \frac{\varepsilon}{2},$$

which implies that

$$\limsup_{j \rightarrow \infty} \langle Q(f) - f(Q(f)), J(Q(f) - x_{n_j}) \rangle \leq \limsup_{j \rightarrow \infty} \langle z_t - f(z_t), J(z_t - x_{n_j}) \rangle + \frac{\varepsilon}{2}.$$

Since  $\limsup_{j \rightarrow \infty} \langle z_t - f(z_t), J(z_t - x_{n_j}) \rangle \leq \frac{\varepsilon}{2}$ , we have

$$\limsup_{j \rightarrow \infty} \langle Q(f) - f(Q(f)), J(Q(f) - x_{n_j}) \rangle \leq \varepsilon.$$

Since  $\varepsilon$  is arbitrary, we obtain that

$$\limsup_{j \rightarrow \infty} \langle (I - f)(Q(f)), J(Q(f) - x_{n_j}) \rangle \leq 0.$$

**Step 6.** We show that  $\lim_{n \rightarrow \infty} \|x_n - Q(f)\| = 0$ . By using (IS), we have

$$x_{n+1} - Q(f) = \alpha_n(f(x_n) - Q(f)) + (1 - \alpha_n)(y_n - Q(f)).$$

Applying Lemma 2.1, we obtain

$$\begin{aligned} & \|x_{n+1} - Q(f)\|^2 \\ & \leq (1 - \alpha_n)^2 \|y_n - Q(f)\|^2 + 2\alpha_n \langle f(x_n) - Q(f), J(x_{n+1} - Q(f)) \rangle \\ & \leq (1 - \alpha_n)^2 \|x_n - Q(f)\|^2 + 2\alpha_n \langle f(x_n) - f(Q(f)), J(x_{n+1} - Q(f)) \rangle \\ & \quad + 2\alpha_n \langle f(Q(f)) - Q(f), J(x_{n+1} - Q(f)) \rangle \\ & \leq (1 - \alpha_n)^2 \|x_n - Q(f)\|^2 + 2k\alpha_n \|x_n - Q(f)\| \|x_{n+1} - Q(f)\| \\ & \quad + 2\alpha_n \langle f(Q(f)) - Q(f), J(x_{n+1} - Q(f)) \rangle \\ & \leq (1 - \alpha_n)^2 \|x_n - Q(f)\|^2 + k\alpha_n (\|x_n - Q(f)\|^2 + \|x_{n+1} - Q(f)\|^2) \\ & \quad + 2\alpha_n \langle f(Q(f)) - Q(f), J(x_{n+1} - Q(f)) \rangle. \end{aligned}$$

It then follows that

$$\begin{aligned} \|x_{n+1} - Q(f)\|^2 & \leq \frac{1 - (2 - k)\alpha_n + \alpha_n^2}{1 - k\alpha_n} \|x_n - Q(f)\|^2 \\ & \quad + \frac{2\alpha_n}{1 - k\alpha_n} \langle f(Q(f)) - Q(f), J(x_{n+1} - Q(f)) \rangle \\ & \leq \frac{1 - (2 - k)\alpha_n}{1 - k\alpha_n} \|x_n - Q(f)\|^2 + \frac{\alpha_n^2}{1 - k\alpha_n} M \\ & \quad + \frac{2\alpha_n}{1 - k\alpha_n} \langle (I - f)(Q(f)), J(Q(f) - x_{n+1}) \rangle \end{aligned} \tag{3.7}$$

where  $M = \sup_{n \geq 0} \|x_n - Q(f)\|^2$ . Put

$$\begin{aligned} \lambda_n &= \frac{2(1 - k)\alpha_n}{1 - k\alpha_n} \quad \text{and} \\ \delta_n &= \frac{M\alpha_n}{2(1 - k)} + \frac{1}{1 - k} \langle (I - f)(Q(f)), J(Q(f) - x_{n+1}) \rangle. \end{aligned}$$

From (C1) and Step 5, it follows that  $\lambda_n \rightarrow 0$ ,  $\sum_{n=0}^{\infty} \lambda_n = \infty$  and  $\limsup_{n \rightarrow \infty} \delta_n \leq 0$ . Since (3.7) reduces to

$$\|x_{n+1} - Q(f)\|^2 \leq (1 - \lambda_n) \|x_n - Q(f)\|^2 + \lambda_n \delta_n,$$

from Lemma 2.3, we conclude that  $\lim_{n \rightarrow \infty} \|x_n - Q(f)\| = 0$ . This completes the proof.  $\square$

**Corollary 3.1.** *Let  $E$  be a uniformly convex and uniformly smooth Banach space and  $C$  a nonempty closed convex subset of  $E$ . Let  $\{T(t) : t \geq 0\}$  be a u.a.r. nonexpansive semigroup from  $C$  into itself with  $F := \bigcap_{t \geq 0} F(T(t)) \neq \emptyset$ . Let  $\{\alpha_n\} \subset (0, 1)$ ,  $\{\beta_n\} \subset [0, 1]$  and  $\{t_n\} \subset \mathbb{R}^+$  be sequences satisfying the conditions (C1)–(C3) in Theorem 3.2. Let  $f \in \Sigma_C$  and  $x_0 \in C$  be chosen arbitrarily. Let  $\{x_n\}$  be defined by*

$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1 - \beta_n)T(t_n)x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)y_n. \end{cases}$$

*Then  $\{x_n\}$  converges strongly to  $Q(f) \in F$ , where  $Q(f)$  solves a variational inequality*

$$\langle (I - f)(Q(f)), J(Q(f) - p) \rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F.$$

**Remark 3.3.** (1) Theorem 3.2 improves Theorem 2.1 and Theorem 3.1 of Chang and Yang [4] to the viscosity method under certain different control conditions in the different Banach space. In particular, Theorem 3.2 removes the conditions  $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$  and  $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$  imposed on control sequences  $\{\alpha_n\}$  and  $\{\beta_n\}$  in Theorem 2.1 and Theorem 3.1 of [4].

(2) Theorem 3.2 also appears to be independent of Theorem 3.1 in Chang and Yang [4].

(3) In general, the condition (C2) in Theorem 3.2 and the condition  $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$  are not comparable; neither of them implies other.

(4) Theorem 3.2 also develops Theorem 3.1 of Aleyner and Reich [1] to the viscosity method. Moreover, by using iterative scheme (IS), the condition  $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$  imposed on control sequence  $\{\alpha_n\}$  was removed.

(5) We point out that our results are applicable to, in particular, in all  $L^p$  spaces,  $1 < p < \infty$ .

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# A note on the generalized $q$ -Euler numbers (2)

By

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**Abstract.** Recently, the new  $q$ -Euler numbers and polynomials related to Frobenius-Euler numbers and polynomials are constructed by Kim (see[3]). In this paper, we study the generalized  $q$ -Euler numbers and polynomials attached to Dirichlet's character  $\chi$  related to the new  $q$ -Euler numbers and polynomials which is constructed in [3]. Finally, we will derive some interesting congruence on the generalized  $q$ -Euler numbers and polynomials attached to  $\chi$ .

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**Key Words and Phrases:**  $q$ -Euler numbers,  $q$ -Euler polynomials.

## §1. Introduction

Let  $\mathbb{Z}, \mathbb{R}$  and  $\mathbb{C}$  denote the ring of integers, the field of real numbers and the complex number field. and let  $p$  be a fixed an odd prime number. Assume that  $q$  is an indeterminate in  $\mathbb{C}$  with  $q \in \mathbb{C}$  with  $|q| < 1$ . As the  $q$ -symbol  $[x]_q$ , we denote  $[x]_q = \frac{1-q^x}{1-q}$ . Recently,  $q$ -Euler polynomials are defined as

$$\frac{[2]_q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}, \text{ for } |t + \log q| < \pi, \quad (\text{see [3]}).$$

In the special case  $x = 0$ ,  $E_{n,q} = E_{n,q}(0)$  are call the  $n$ -th  $q$ -Euler numbers (see [3]). These  $q$ -Euler numbers and polynomials are closely related to Frobenius-Euler numbers and polynomials and these numbers are studied by Simsek-Cangul-Ozden, Cenkci-Kurt and Can and several authors (see [1-2, 18-26]). In this paper, we study the generalized  $q$ -Euler numbers and polynomials attached to  $\chi$  related to the  $q$ -Euler numbers and polynomials,  $E_{n,q}(x)$ , which is constructed in [3]. Finally, we will derive some interesting congruence on the generalized  $q$ -Euler numbers and polynomials attached to  $\chi$ .

## §2. Congruence for $q$ -Euler numbers and polynomials

The ordinary Euler polynomials are defined as

$$e^{xt} \frac{2}{e^t + 1} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (\text{see [1-5]}),$$

where we use the technical method notation by replacing  $E^n(x)$  by  $E_n(x)$  ( $n \geq 0$ ), symbolically (see [1-2]). Let us consider the generating function of  $q$ -Euler polynomials  $E_{n,q}(x)$  as follows:

$$F_q(x, t) = \frac{[2]_q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}, \quad (1)$$

and we also note that

$$\sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = \frac{[2]_q}{qe^t + 1} e^{xt} = \frac{1 - (-q^{-1})}{e^t - (-q^{-1})} e^{xt} = \sum_{n=0}^{\infty} H_n(-q^{-1}, x) \frac{t^n}{n!},$$

where  $H_n(-q^{-1}, x)$  are called the  $n$ -th Frobenius-Euler polynomials (see [3]). From (1), we note that

$$\lim_{q \rightarrow 1} F_q(x, t) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \quad (2)$$

By (1) and (2), we see that

$$\lim_{q \rightarrow 1} E_{n,q}(x) = E_n(x).$$

In (1), it is easy to show that

$$\sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = F_q(x, t) = \frac{[2]_q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \binom{n}{l} E_{l,q} x^{n-l} \right) \frac{t^n}{n!}.$$

By comparing the coefficients on the both sides, we have

$$E_{n,q}(x) = \sum_{l=0}^n \binom{n}{l} E_{l,q} x^{n-l}, \quad \text{where } E_{l,q} \text{ are the } l\text{-th } q\text{-Euler numbers.} \quad (3)$$

Let  $\chi$  be the Dirichlet's character with conductor  $d \equiv 1 \pmod{2}$ . Then we define generating function of the generalized  $q$ -Euler numbers attached to  $\chi$ ,  $E_{n,\chi,q}$  as follows:

$$F_{q,\chi}(t) = \frac{[2]_q \sum_{l=0}^{d-1} \chi(l) q^l (-1)^l e^{lt}}{q^d e^{dt} + 1} = \sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!}. \quad (4)$$

From (4), we note that

$$\lim_{q \rightarrow 1} F_{q,\chi}(t) = \frac{2 \sum_{a=0}^{d-1} \chi(a) (-1)^a e^{at}}{e^{dt} + 1} = \sum_{n=0}^{\infty} E_{n,\chi} \frac{t^n}{n!}, \quad (5)$$

where  $E_{n,\chi}$  are the  $n$ -th ordinary Euler numbers attached to  $\chi$ . By (4) and (5), we see that

$$\lim_{q \rightarrow 1} E_{n,\chi,q} = E_{n,\chi}.$$

From (4), we can also derive

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!} &= F_{q,\chi}(t) = [2]_q \sum_{k=0}^{\infty} \chi(k) (-q)^k e^{kt} \\ &= \sum_{n=0}^{\infty} \left( [2]_q \sum_{k=0}^{\infty} \chi(k) (-q)^k k^n \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left( d^n \sum_{a=0}^{d-1} (-q)^a \chi(a) E_{n,q^d} \left( \frac{a}{d} \right) \right) \frac{t^n}{n!}. \end{aligned} \quad (6)$$

By comparing the coefficients on the both sides of (6), we have

$$E_{n,\chi,q} = [2]_q \sum_{k=0}^{\infty} \chi(k) (-q)^k k^n = d^n \sum_{a=0}^{d-1} (-q)^a \chi(a) E_{n,q^d} \left( \frac{a}{d} \right). \quad (7)$$

Finally, we define the generating function of the generalized  $q$ -Euler polynomials attached to  $\chi$ ,  $E_{n,\chi,q}(x)$  as follows:

$$F_{q,\chi}(x, t) = \sum_{n=0}^{\infty} E_{n,\chi,q}(x) \frac{t^n}{n!} = [2]_q \sum_{k=0}^{\infty} \chi(k) (-q)^k e^{(x+k)t}. \quad (8)$$

By (8), we easily see that

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,\chi,q}(x) \frac{t^n}{n!} &= F_{q,\chi}(x, t) = [2]_q \sum_{k=0}^{\infty} \chi(k) (-q)^k e^{(x+k)t} \\ &= \sum_{n=0}^{\infty} \left( [2]_q \sum_{k=0}^{\infty} \chi(k) (-q)^k (x+k)^n \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left( d^n \sum_{a=0}^{d-1} (-q)^a \chi(a) E_{n,q^d} \left( \frac{a+x}{d} \right) \right) \frac{t^n}{n!}. \end{aligned} \quad (9)$$

Thus, we have

$$E_{n,\chi,q}(x) = d^n \sum_{a=0}^{d-1} (-q)^a \chi(a) E_{n,q^d} \left( \frac{a+x}{d} \right) = \sum_{\ell=0}^n \binom{n}{\ell} x^{n-\ell} E_{\ell,\chi,q} = [2]_q \sum_{k=0}^{\infty} \chi(k) (-q)^k (x+k)^n. \quad (10)$$

Let  $d \in \mathbb{N}$  with  $d \equiv 1 \pmod{2}$ . Then, we see that

$$\begin{aligned} q^d F_{q,\chi}(d, t) + F_{q,\chi}(t) &= [2]_q \sum_{k=0}^{\infty} \chi(k) (-q)^k e^{(d+k)t} + [2]_q \sum_{k=0}^{\infty} \chi(k) (-q)^k e^{kt} \\ &= [2]_q \sum_{k=0}^{d-1} \chi(k) (-q)^k e^{kt}. \end{aligned} \quad (11)$$

A note on the generalized  $q$ -Euler numbers (2)

From (11), we have

$$\sum_{n=0}^{\infty} \left( q^d E_{n,\chi,q}(d) + E_{n,\chi,q} \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left\{ [2]_q \sum_{k=0}^{d-1} \chi(k) (-q)^k k^n \right\} \frac{t^n}{n!}.$$

Therefore, we obtain the following theorem.

**THEOREM 1.** For  $q \in \mathbb{C}$  with  $|q| < 1$ ,  $n \in \mathbb{Z}_+$  and  $d \in \mathbb{N}$  with  $d \equiv 1 \pmod{2}$ , we have

$$q^d E_{n,\chi,q}(d) + E_{n,\chi,q} = [2]_q \sum_{k=0}^{d-1} \chi(k) (-q)^k k^n.$$

Let  $p$  be a positive odd integer and let  $N \in \mathbb{N}$ . Then we have

$$\begin{aligned} [2]_q \sum_{a=0}^{dp^N-1} \chi(a) (-q)^a a^n &= q^{dp^N} E_{n,\chi,q}(dp^N) + E_{n,\chi,q} \\ &= q^{dp^N} \sum_{j=0}^n \binom{n}{j} (dp^N)^j E_{n-j,\chi,q} + E_{n,\chi,q} \\ &= q^{dp^N} \sum_{j=1}^n \binom{n}{j} (dp^N)^j E_{n-j,\chi,q} + (q^{dp^N} + 1) E_{n,\chi,q} \\ &\equiv 2E_{n,\chi,q} \pmod{dp^N}, \end{aligned}$$

because  $q^{ndp^N} \equiv 1 \pmod{dp^N}$ . Therefore, we obtain the following theorem.

**THEOREM 2.** Let  $p$  be a positive odd integer and  $q \in \mathbb{C}$  with  $|q| < 1$  and  $(q-1, dp) = 1$ . For  $d \in \mathbb{N}$  with  $d \equiv 1 \pmod{2}$ , we have

$$[2]_q \sum_{a=0}^{dp^N-1} \chi(a) (-q)^a a^n \equiv 2E_{n,\chi,q} \pmod{dp^N}.$$

**REMARK.** Define

$$L_{E,q}(s, \chi|x) = [2]_q \sum_{n=0}^{\infty} \frac{(-q)^n \chi(n)}{(n+x)^s},$$

where  $s \in \mathbb{C}$ , and  $x \neq 0, -1, -2, \dots$ . For  $k \in \mathbb{Z}_+$ , we have  $L_{E,q}(-k, \chi|x) = E_{k,\chi,q}(x)$ .

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# AN ESTIMATION TO THE SOLUTION OF AN INITIAL VALUE PROBLEM VIA $q$ -BERNSTEIN POLYNOMIALS

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ABSTRACT. In the present paper, we give an estimation to the difference  $|B_{n-1}(f; q; x) - B_n(f; q; x)|$ , where  $B_n(f; q; x)$  is the  $q$ -Bernstein Polynomials (see [15]) and then we construct an approximating sequence for the solution of the initial value problem.

## 1. INTRODUCTION

Let  $C[0, 1]$  denotes the set of continuous functions on  $[0, 1]$ . In [2], S. Bernstein introduced the following well-known linear, positive operators

$$(1.1) \quad B_n(f; x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

and he showed that if  $f \in C[0, 1]$ , then  $B_n(f; x) \Rightarrow f(x)$  where " $\Rightarrow$ " represents the uniform convergence. One can find a detailed monograph about the Bernstein Polynomials in [5].

The  $q$ -generalization of the Bernstein Polynomials was introduced by G. M. Phillips, in [14], by the following way:

$$(1.2) \quad B_n(f; q; x) = \sum_{r=0}^n f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1 - q^s x),$$

where an empty product is 1 and  $f_r = f([r]/[n])$ . In (1.2), the value  $[r]$  denotes the  $q$ -integer of  $r$ , which is given by

$$[r] = \begin{cases} \frac{1 - q^r}{1 - q}, & q \in \mathbb{R}^+ - \{1\} \\ r, & q = 1 \end{cases}.$$

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Also the  $q$ -factorial of the number  $r$  has the definition

$$[r]! = \begin{cases} 1[2]\dots[r], & r = 1, 2, \dots \\ 1, & r = 0 \end{cases},$$

and the  $q$ -binomial coefficient is defined by

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]![n-r]!}.$$

It is clear that the operators  $B_n(f; q; x)$  defined by (1.2) are the generalization of the Bernstein polynomials defined by (1.1), since

$$\lim_{q \rightarrow 1} B_n(f; q; x) = B_n(f; x).$$

For the  $q$ -Bernstein polynomials  $B_n(f; q; x)$  defined by (1.2), Phillips [14] obtained the moments as

$$B_n(1; q; x) = 1, \quad B_n(t; q; x) = x, \quad B_n(t^2; q; x) = x^2 + \frac{x(1-x)}{[n]}.$$

On the other hand, since the operators  $B_n(f; q; x)$  are monotone, using the well known Bohman-Korovkin theorem, Phillips gave the following convergence theorem:

**Theorem 1.1.** (Phillips [14]) *Let  $0 < q = q_n < 1$  and  $q_n \rightarrow 1$  as  $n \rightarrow \infty$ . Then for any  $f \in C[0, 1]$ , the operators  $B_n(f; q; x)$  converges uniformly to  $f(x)$  on  $[0, 1]$ .*

In order to give the properties of the derivative, it is convenient to rewrite the operators  $B_n(f; q; x)$  in the following form (see [14] and [15])

$$B_n(f; q; x) = \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} \Delta^r f_0 x^r$$

where the  $q$ -differences are defined through the following recurrence formula:

$$\Delta^0 f_i = f_i$$

for  $i = 0, 1, \dots, n$  and

$$\Delta^{r+1} f_i = \Delta^r f_{i+1} - q^r \Delta^r f_i$$

for  $k = 0, 1, \dots, n - i - 1$ . One can easily show by induction on  $r$  that  $q$ -differences satisfy the relation

$$\Delta^r f_i = \sum_{j=0}^r (-1)^j q^{j(j-1)/2} \begin{bmatrix} r \\ j \end{bmatrix} f_{i+r-j}.$$

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In [15], in the examination of the properties of the derivative, Phillips introduced the following linear positive operators:

$$\begin{aligned} B_{n-1}^*(f; q; x) &= \sum_{r=0}^{n-1} f\left(\frac{[r]}{[n]}\right) \begin{bmatrix} n-1 \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-2} (1 - q^s x) \\ &= \sum_{r=0}^{n-1} \begin{bmatrix} n-1 \\ r \end{bmatrix} \Delta^r f_0 x^r. \end{aligned} \quad (1.3)$$

Also it was shown in [15] that,  $B_{n-1}^*(f; q; x)$  converges uniformly to  $f$  on  $[0, 1]$  as  $q_n \rightarrow 1$ , since

$$\begin{aligned} B_{n-1}^*(1; q; x) &= 1 \quad (n \geq 1), \quad B_{n-1}^*(t; q; x) = x - \frac{q^{n-1}}{[n]} x \quad (n \geq 2), \\ B_{n-1}^*(t^2; q; x) &= x^2 + \frac{1}{q[n]} \left(1 - \frac{1}{[n]}\right) \\ &\quad + \frac{1}{q^2} \left(1 - q^2 - \frac{2+q}{[n]} + \frac{1+q}{[n]^2}\right) x^2 \quad (n \geq 3). \end{aligned} \quad (1.4)$$

Investigation the properties of  $q$ -based operators has been an active research field during the last decade ( see [4], [20], [7], [8], [9], [10], [16], [17], [18], [12], [19], [13]). Detailed review of the results obtained until 2007 and a number of open problems can be found in [11].

In this paper, we obtain an estimation to the difference

$$|B_{n-1}(f; q; x) - B_n(f; q; x)|,$$

which is needed in the approximation to a solution of the initial value problem. Finally, using the  $q$ -Bernstein polynomials, we introduce a sequence which converges to a solution of the initial value problem.

2. AN ESTIMATION TO THE DIFFERENCE  $|B_{n-1}(f; q; x) - B_n(f; q; x)|$ 

The aim of this section is to find an estimation to the difference  $|B_{n-1}(f; q; x) - B_n(f; q; x)|$  where  $B_n(f; q; x)$  is the Generalized Bernstein polynomials.

In [[6], p. 10, eq. (3.2) and (3.3)], we have the following equality

$$B_{n-1}(f; q; x) - B_n(f; q; x) = \sum_{r=1}^{n-1} \begin{bmatrix} n \\ r \end{bmatrix} a_r x^r \prod_{s=0}^{n-r-1} (1 - q^s x)$$

where

$$a_r = \frac{[n-r]}{[n]} f\left(\frac{[r]}{[n-1]}\right) + q^{n-r} \frac{[r]}{[n]} f\left(\frac{[r-1]}{[n-1]}\right) - f\left(\frac{[r]}{[n]}\right).$$

The following lemma is needed for obtaining an estimation to the difference  $|B_{n-1}(f; q; x) - B_n(f; q; x)|$ .

**Lemma 2.1.** (*Cheney and Sharma [3]*) Let  $|f''| \leq k, 0 \leq \alpha, \beta$  and  $\alpha + \beta = 1$ , then

$$|\alpha f(x) + \beta f(y) - f(\alpha x + \beta y)| \leq \frac{k}{4} (x - y)^2.$$

**Theorem 2.2.** Let  $f : [0, 1] \rightarrow \mathbb{R}$  is a function such that  $|f''| \leq k$ , then we have

$$|B_{n-1}(f; q; x) - B_n(f; q; x)| \leq \frac{k}{4q^2[n-1]^2}.$$

*Proof.* Choosing

$$\alpha = \frac{[n-r]}{[n]}, \beta = q^{n-r} \frac{[r]}{[n]}, x = \frac{[r]}{[n-1]}, y = \frac{[r-1]}{[n-1]}$$

one can easily see that  $\alpha + \beta = 1$  and  $\alpha x + \beta y = \frac{[r]}{[n]}$ . Therefore using Lemma 2.1, we obtain that

$$\begin{aligned} |B_{n-1}(f; q; x) - B_n(f; q; x)| &= \sum_{r=1}^{n-1} \begin{bmatrix} n \\ r \end{bmatrix} a_r x^r \prod_{s=0}^{n-r-1} (1 - q^s x) \\ &\leq \frac{k}{4} \sum_{r=1}^{n-1} \left( \frac{q^{2r-2}}{[n-1]^2} \right) \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1 - q^s x) \\ &\leq \frac{k}{4q^2[n-1]^2}. \end{aligned}$$

Whence the result.  $\square$

**Theorem 2.3.** Let  $y_n(x)$  be defined through the following recurrence relation:

$$(3.1) \quad y_0(x) = y_0, \quad y_n(x) = y_0 + \int_0^x B_n[f(t, y_{n-1}(t)); q_n; s] ds.$$

where  $f$  has continuous partial derivatives of first order and  $0 < q_n \leq 1$  with  $q_n \rightarrow 1$ . Also assume that  $f$  and its first derivatives have the same common bound  $M$  in the strip  $0 \leq x \leq 1, -\infty < y < \infty$ . Then we have

$$|y'_n(x)| \leq M, \quad |y''_n(x)| < 2(M + \epsilon)$$

where  $\epsilon$  is a positive real number.

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*Proof.* Since  $B_n[1; q_n; x] = 1$ , it is obvious that

$$|y'_n(x)| = B_n[f(t, y_{n-1}(t)); q_n; x] \leq \sup_{0 \leq x \leq 1} f(x, y_{n-1}(x)) \leq M.$$

To prove the second inequality, let

$$\begin{aligned} B_{n-1}^*(f; q_n; x) &= \sum_{r=0}^{n-1} f_r \left[ \begin{matrix} n-1 \\ r \end{matrix} \right] x^r \prod_{s=0}^{n-r-2} (1 - q_n^s x) \\ &= \sum_{r=0}^{n-1} \left[ \begin{matrix} n-1 \\ r \end{matrix} \right] \Delta^r f_0 x^r. \end{aligned}$$

Now, letting  $F(x) = f(x, y_{n-1}(x))$ , it can be computed that

$$\begin{aligned} y''_n(x) &= B'_n(F; q_n; x) = B_{n-1}^*([n]\Delta F_r; q_n; x) \\ (3.2) \quad &+ \sum_{r=0}^{n-1} \left( \frac{r+1}{[r+1]} - 1 \right) \left[ \begin{matrix} n-1 \\ r \end{matrix} \right] \Delta^r([n]\Delta F_r) x^r. \end{aligned}$$

Recalling the inequality

$$\left| \frac{r+1}{[r+1]} - 1 \right| \leq \frac{1}{3^n}$$

given in [15], we get, using (3.2)

$$\begin{aligned} y''_n(x) &\leq \left( 1 + \frac{1}{3^n} \right) B_{n-1}^*([n]\Delta F_r; q_n; x) \\ &\leq 2B_{n-1}^*([n]\Delta F_r; q_n; x) \\ &= 2 \left[ \sum_{r=0}^{n-1} F'_r \left[ \begin{matrix} n-1 \\ r \end{matrix} \right] x^r \prod_{s=0}^{n-r-2} (1 - q_n^s x) \right. \\ &\quad \left. + \sum_{r=0}^{n-1} ([n]\Delta F_r - F'_r) \left[ \begin{matrix} n-1 \\ r \end{matrix} \right] x^r \prod_{s=0}^{n-r-2} (1 - q_n^s x) \right]. \end{aligned}$$

Since  $f$  has continuous partial derivatives of first order, by the mean value theorem, there exists a real number  $\theta$  ( $0 < \theta < 1$ ) such that

$$F'_r - [n]\Delta F_r = F' \left( \frac{[r]}{[n]} \right) - q_n^r F' \left( \frac{[r] + \theta q_n^r}{[n]} \right).$$

Since  $f$  has continuous partial derivatives and  $q_n \rightarrow 1$ , given any  $\epsilon > 0$ , there exists a positive integer  $N = N(\epsilon)$  such that  $|[n]\Delta F - F'_r| < \epsilon$  for  $0 \leq r \leq n-1$ .

Therefore we get

$$|y''_n(x)| < 2(M + \epsilon)$$

since

$$B_{n-1}^*(F'_r; q; x) \leq \sup_{0 \leq x \leq 1} |F'_r| \leq M.$$

□

### 3. APPROXIMATION TO A SOLUTION OF THE INITIAL VALUE PROBLEM

In this section, inspiring from Arama's theorem [1], it is aimed to show that the sequence  $y_n(x)$  given by (3.1) approximates to a solution of the initial value problem.

**Theorem 3.1.** *The sequence  $y_n(x)$  defined by (3.1) converges uniformly to a solution of the initial value problem*

$$y' = f(x, y), \quad y(0) = y_0$$

for  $x \in [0, 1]$ , provided that  $f$  and its first two derivatives are bounded in the strip  $0 \leq x \leq 1, -\infty < y < \infty$ , that  $f$  has continuous partial derivatives of first order and that  $f$  satisfies

$$|f(x, y_1) - f(x, y_2)| \leq \xi |y_1 - y_2|$$

where  $\xi < 1$ .

*Proof.* First of all, it will be shown that the series

$$y_0 + \sum_{n=1}^{\infty} [y_n(x) - y_{n-1}(x)]$$

converges uniformly in  $[0, 1]$ . Let  $\epsilon_n(x) = y_n(x) - y_{n-1}(x)$ . Then

$$\begin{aligned} |\epsilon_n(x)| &\leq \int_0^x |B_n[f(t, y_{n-1}(t)); q; s] - B_{n-1}[f(t, y_{n-2}(t)); q; s]| ds \\ &\leq \int_0^x |B_n[f(t, y_{n-1}(t)); q; s] - B_{n-1}[f(t, y_{n-1}(t)); q; s]| ds \\ &\quad + \int_0^x |B_{n-1}[f(t, y_{n-1}(t)); q; s] - B_{n-1}[f(t, y_{n-2}(t)); q; s]| ds \\ &= I_1 + I_2. \end{aligned}$$

Using Theorem 3.2, we can write that

$$(4.1) \quad I_1 \leq \int_0^x \frac{k}{4q^2[n-1]^2} ds \leq \frac{k}{4q^2[n-1]^2}$$

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since  $x \in [0, 1]$ .

Now, let us verify that

$$k = \sup_{0 \leq x \leq 1} \left| \frac{d^2}{dx^2} f(x, y_n(x)) \right|$$

is finite. Since

$$\frac{d^2}{dx^2} f(x, y_n(x)) = f_{11} + (2f_{12} + f_{22}y'_n)y'_n + f_{22}y''_n$$

then using Theorem 3.3 and the hypotheses of the theorem, we obtain

$$\left| \frac{d^2}{dx^2} f(x, y_n(x)) \right| < M + (2M + M^2)M + 2M(M + \epsilon).$$

On the other hand, using monotonicity, we can write

$$I_2 \leq \int_0^x B_{n-1}(|f(t, y_{n-1}(t)) - f(t, y_{n-2}(t))|; q; s) ds.$$

We get, from the hypothesis of the theorem that,

$$|f(t, y_{n-1}(t)) - f(t, y_{n-2}(t))| \leq \xi |y_{n-1}(t) - y_{n-2}(t)| = \xi |\epsilon_{n-1}(t)|.$$

Thus we have

$$(4.2) \quad I_2 \leq \xi \sup_{0 \leq t \leq 1} |\epsilon_{n-1}(t)|.$$

Since

$$\begin{aligned} |y_n(x)| &\leq |y_0| + \int_0^x |B_n[f(t, y_{n-1}(t)); q; s]| ds \\ &\leq |y_0| + M \end{aligned}$$

then

$$\sup_{0 \leq x \leq 1} |\epsilon_n(x)| \leq 2(|y_0| + M).$$

From (4.1) and (4.2), it is obtained that

$$|\epsilon_n(x)| \leq \xi \sup_{0 \leq t \leq 1} |\epsilon_{n-1}(t)| + O(q_n^{-2}[n-1]^{-2}).$$

This shows that  $y_n(x)$  converges to a function  $y(x)$  uniformly in  $[0, 1]$ .

Now, let us show that  $y(x)$  is a solution of the initial value problem.

Differentiating term by term, we get

$$\begin{aligned} |\epsilon'_n(x)| &\leq |B_n[f(t, y_{n-1}(t)); q; x] - B_{n-1}[f(t, y_{n-1}(t)); q; x]| \\ &\quad + |B_{n-1}[f(t, y_{n-1}(t)); q; x] - B_{n-1}[f(t, y_{n-2}(t)); q; x]| \\ &\leq \frac{k}{4q^2[n-1]^2} + \xi \sup_{0 \leq t \leq 1} |\epsilon_{n-1}(t)|. \end{aligned}$$

Considering that  $y_n(x) \rightarrow y(x)$ , we conclude that

$$y'(x) = \lim_{n \rightarrow \infty} y'_n(x) = \lim_{n \rightarrow \infty} B_n[f(t, y_{n-1}(t)); q; x].$$

Finally, taking

$$A_n(x) = B_n[f(t, y_{n-1}(t)); q; x] - f(x, y(x))$$

and considering the following inequality

$$\begin{aligned} |A_n(x)| \leq & |B_n[f(t, y_{n-1}(t)); q; x] - B_n[f(t, y(t)); q; x]| \\ & + |B_n[f(t, y(t)); q; x] - f(x, y(x))| \end{aligned}$$

completes the proof since, the first term at the right hand side does not exceed

$$\sup_{0 \leq t \leq 1} |f(t, y_{n-1}(t)) - f(t, y(t))| \leq \xi \sup_{0 \leq t \leq 1} |y_{n-1}(t) - y(t)|$$

and the second term converges to zero.  $\square$

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## Direct Integration Preconditioning For Solving Optimal Control Problems

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**Abstract:** In this paper, direct integration preconditioning is proposed to solve optimal control problems governed by ordinary differential equations. Legendre approximations are used to reduce the problem to a constrained optimization problem. Error estimation for the Legendre approximations is derived and a technique that gives an optimal approximation of the problems is introduced. Numerical results are included to confirm the efficiency and accuracy of the method.

**Keywords:** Spectral methods; approximation by Legendre polynomials; optimal control problems.

### 1. Introduction

Spectral methods using expansion in orthogonal polynomials such as Chebyshev or Ultraspherical polynomials have proven successful in the numerical approximation of various boundary value problems; see for instance, Canuto et al [1], Gottlib and Orszag [10] and Szegö [12]. If these polynomials are used as basis functions, then the rate of decay of the expansion coefficients is determined by the smoothness properties of the function being expanded. This choice of trial functions is responsible for the superior approximation properties of spectral methods when compared with finite difference and finite element methods. For spectral and pseudospectral methods, explicit expressions for the expansion coefficients of the derivatives in terms of the expansion coefficients of the solution are needed. Doha [2] obtained a general formula when the basis functions are the Ultraspherical polynomials. In [6], the author introduced a Chebyshev spectral procedure for solving ordinary and partial differential equations by transforming them into integral formulae. He used El-gendi [3] to obtain an approximation for the finite integrals.

Optimal control problems governed by ordinary differential equations are discussed by many authors, among them Martin [5], [8], [11] and [13]. A variety of numerical methods for solving this optimal control problem exists. The most common approach is to replace the unknowns of the problem by some approximation function and to determine the unknowns by minimizing the resulting constrained optimization problem. Martin [11] consider the problem of time-optimal boundary control of a one-dimensional vibrating system subject to a control constraint that prescribes an upper bound for the  $L^2$ -norm of the image of the control function under a Volterra operator. He uses Newton's method to compute the zero of the optimal value function of certain parametric auxiliary problems, where the steering time is the parameter.

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The proposed algorithm describes an alternative technique. The system dynamics, the state variables can be obtained by transforming the boundary value problem for ordinary differential equations into integral formulas. Start with a Legendre spectral approximation for the highest-order derivative and generate approximations to the lowest-order derivatives through successive integrations. Therefore, the differential and integral expressions that arise in the system dynamics, the performance index, the initial (or boundary) conditions (and even for general multipoint boundary conditions) are converted into algebraic equations with unknown coefficients. This algorithm is of the finite element type and results in static optimization problems with a relatively small number of variables. This approach yields a static optimization problem. This means that the optimal control problem is reduced to a parameter static optimization problem, which consists of the minimization of an objective function, subject to a system of algebraic constraints that are linear in the state variables, irrespective of whether the dynamic system itself is linear or nonlinear. In such cases, the static optimization problem can be efficiently performed using the penalty partial quadratic interpolation technique [4]. They derived error estimation for this approximation, and introduced an algorithm that gives an optimal approximation of the integrals.

The paper is organized as follows: In section 2, optimal control problem formula and some useful notations are presented. In section 3, the Legendre pseudospectral integration is presented to define an optimal preconditioner. The direct integration preconditioning technique is presented in section 4. In section 5, error estimation of the preconditioning Legendre approximations is presented. Two numerical examples of optimal control problems are solved in section 6.

## 2. Optimal control problems with linear terminal constraints

We consider the problem of finding the control  $u(t)$  which minimizes the cost functional.

$$J = h(x, x^{(1)}, \dots, x^{(n-1)}, T) + \int_0^T g(x, x^{(1)}, \dots, x^{(n-1)}, u, \tau) d\tau, \quad (2.1)$$

Subject to

$$F(x, x^{(1)}, \dots, x^{(n)}, u, \tau) = 0, \quad 0 \leq \tau \leq T, \quad (2.2)$$

where  $x^{(r)} = \frac{d^r x}{dx^r}$ ,  $r = 1, 2, \dots, n$ .

and linear initial constraints,

$$L(x(0), x^{(1)}(0), x^{(2)}(0), \dots, x^{(n-1)}(0)) = 0, \quad (2.3)$$

and terminal constraints,

$$M(x(T), x^{(1)}(T), x^{(2)}(T), \dots, x^{(n-1)}(T)) = 0, \quad (2.4)$$

Where, the time  $T$  is assumed to be fixed,  $L$  and  $M$  are vector functions of dimension  $l$  and  $m$ , respectively, with  $n \leq l + m \leq 2n$ . The state variable  $x(\tau) \in R^N$ , the control variable  $u(\tau) \in R^M$ ,  $g(\tau) \in R^N$  are real valued continuous functions on  $[0, T]$ .

Before starting to reformulate the optimal control problem, we need to change the time interval  $\tau \in [0, T]$  into  $t \in [-1, 1]$  by substituting

$$t = \frac{2\tau}{T} - 1$$

in order to use Legendre polynomials, defined on the interval  $[-1, 1]$ . Hence the optimal control problem becomes: Minimize

$$J = h(x, x^{(1)}, \dots, x^{(n-1)}, 1) + \frac{T}{2} \int_{-1}^1 g(x, x^{(1)}, \dots, x^{(n-1)}, u, t) dt \quad (2.5)$$

Subject to

$$F\left(x, \left(\frac{2}{T}\right)x^{(1)}, \dots, \left(\frac{2}{T}\right)^n x^{(n)}, u, t\right) = 0, \quad -1 \leq t \leq 1 \quad (2.6)$$

and linear initial and terminal constraints,

$$L\left(x(-1), \left(\frac{2}{T}\right)x^{(1)}(-1), \dots, \left(\frac{2}{T}\right)^n x^{(n)}(-1)\right) = 0, \quad (2.7)$$

$$M\left(x(1), \left(\frac{2}{T}\right)x^{(1)}(1), \dots, \left(\frac{2}{T}\right)^n x^{(n)}(1)\right) = 0. \quad (2.8)$$

### 3. Pseudospectral Legendre integration approximations [7]

We present here the Legendre approximations of any function  $f(t) \in C^\infty[-1, 1]$ , using  $(N + 1)$  Legendre-Gauss-Lobatto (LGL) points as:  $\{t_i : (1 - t_i^2)P'_N(t_i) = 0, i = 0, 1, \dots, N\}$ .

$$f(x) = \sum_{j=0}^N a_j P_j(t), \quad (3.1)$$

where

$$a_j \cong \frac{2j+1}{N(N-1)} \sum_{k=0}^N \frac{P_j(t_k) f(t_k)}{[P_{N-1}(t_k)]^2}, \quad j = 0, 1, \dots, N. \quad (3.2)$$

Approximate the integrals of a function  $f(x)$  by interpolating the function with a polynomial  $P_N f$  at Legendre-Gauss-Lobatto (LGL) points.

The values of the integrals  $\int_{-1}^t \int_{-1}^t \dots \int_{-1}^t (P_N f)(t) dt dt \dots dt$  at the same  $(N + 1)$  points, in fact, be

expressed as a fixed linear combination of the given function values and the whole relationship may be written in matrix form

$$\left[ \int_{-1}^t \int_{-1}^t \dots \int_{-1}^t (P_N f)(t) dt dt \dots dt \right]_{n\text{-times}} = B^{(n)} [P_N f]$$

Setting  $[P_N f] = [f(t_0), f(t_1), \dots, f(t_N)]^T$  as the vector consisting of values of  $f(x)$  at  $(N + 1)$

collocation points,  $\left[ \int_{-1}^t (P_N f)(t) dt \right] = \left[ \int_{-1}^{t_0} (P_N f)(t) dt, \dots, \int_{-1}^{t_N} (P_N f)(t) dt \right]^T$  as the values of the

integrals at the collocation points and  $B^{(n)} = b_{ij}^{(n)}$ ,  $i, j = 0, 1, \dots, N$  as the collocation integral matrix;

$$b_{ik}^{(n)} = \sum_{j=0}^N \sum_{m=0}^{[j/2]} \frac{2j+1}{N(N-1)[P_{N-1}(t_k)]^2} c_m^j P_j(t_k) \left\{ \frac{(j-2m)!}{(j-2m+n)!} [t_i^{j-2m+n} - (-1)^{j-2m+n}] \right\}$$

$$\left. - \sum_{r=1}^{n-1} \sum_{s=0}^r \frac{(-1)^{j-2m+n-r} (j-2n)! t_i^s}{(r-s)! s! (j-2m+n-r)!} \right\}, \quad 0 \leq i, k \leq N. \quad (3.3)$$

and

$$c_k^n = \frac{(-1)^k (2n-2k)!}{2^n (n-k)! (n-2k)! k!}.$$

#### 4. Direct integration preconditioning technique

Legendre spectral approximation is adopted here to approximate the solution of the problem.

We start with Legendre approximation for the highest-order derivative,  $x^{(n)}$ , and generate approximations to the lowest-order derivatives  $x^{(n-1)}$ ;  $x^{(n-2)}$ ; ... and  $x^{(0)}$ , through successive integrations of the approximation of the highest-order derivative, as follows:

Suppose that

$$x^{(n)}(t) = \Psi(t) \quad (4.1)$$

where  $\Psi(t_i), i = 0, 1, \dots, N$  are some unknowns. By integration, and making use of the given conditions, we get

$$\begin{aligned} x^{(n-1)}(t) &= \int_{-1}^t \Psi(t) dt + c_0, \\ x^{(n-2)}(t) &= \int_{-1}^t \int_{-1}^t \Psi(t) dt dt + c_0 t + c_1 \\ &\vdots \\ x(t) &= \int_{-1}^t \int_{-1}^t \dots \int_{-1}^t \Psi(t) dt dt \dots dt + \sum_{r=0}^{n-1} c_{n-r} t^r. \end{aligned} \quad (4.2)$$

Now we apply present Legendre integral approximation, then we have

$$\begin{aligned} x^{(n-1)}(t_i) &= \sum_{j=0}^N b_{ij} \Psi(t_j) + c_0, \quad i = 0, 1, \dots, N \\ x^{(n-2)}(t_i) &= \sum_{j=0}^N b_{ij}^{(2)} \Psi(t_j) + c_0 t_i + c_1 \\ &\vdots \\ x(t_i) &= \sum_{j=0}^N b_{ij}^{(n)} \Psi(t_j) + \sum_{r=0}^{n-1} c_r t_i^r. \end{aligned} \quad (4.3)$$

where the constants  $c_r, r = 0, 1, \dots, n-1$  may be defined from the given conditions. Making use of the approximation for the control variable as  $u(t_i) = u(t_i)$ , the optimal control problem (2.5)-(2.8) are replaced by the constrained optimization problems

Minimize

$$J = h(x, x^{(1)}, \dots, x^{(n-1)}, 1) + \frac{T}{2} \sum_{i=0}^N b_{Ni} g \left( \sum_{j=0}^N b_{ij}^{(n)} \Psi(t_j) + \sum_{r=0}^{n-1} C_r t_i^r, \dots, \sum_{j=0}^N b_{ij} \Psi(t_j) + C_0, u(t_i) \right) \quad (4.4)$$

Subject to

$$F \left( \left\{ \sum_{j=0}^N b_{ij}^{(n)} \Psi(t_j) + \sum_{r=0}^{n-1} C_r t_i^r \right\}, \dots, \left( \frac{2}{T} \right)^{n-1} \left\{ \sum_{j=0}^N b_{ij} \Psi(t_j) + C_0 \right\}, \left( \frac{2}{T} \right)^n \Psi(t_i), u(t_i) \right) = 0. \quad (4.5)$$

The constrained optimization problem is then takes the form:

$$\text{Minimize} \quad J = J[\alpha_i], i = 0, \dots, N, \quad (4.6)$$

$$\text{Subject to} \quad F[\alpha_i] = 0, i = 0, 1, \dots, N, \quad (4.7)$$

where  $\alpha_i = [\Psi(t_0), \Psi(t_1), \dots, \Psi(t_N)]$ .

Equations (4.6) and (4.7) are solved by using penalty partial quadratic interpolation technique [4]. We therefore use either

$$|J(\gamma_{N+1}, \lambda_{N+1}) - J(\gamma_N, \lambda_N)| < \varepsilon_1 \quad \text{Or} \quad \left( \sum_{i=0}^N F_i^2 \right)^{1/2} < \varepsilon_2$$

or both to decide whether the computed solution is close enough to the optimal solution.

## 5. Error Estimation of the Preconditioning Legendre Approximations

### Theorem 5.1:

Let  $f(t)$  be approximated by Legendre polynomials, then there exists a number  $\xi = \xi(t) \in [-1, 1]$  such that:

$$f(t) = \sum_{k=0}^N a_k P_k(t),$$

$$\int_{-1}^{t_i} \int_{-1}^t \dots \int_{-1}^t f(t) dt \dots dt = \sum_{k=0}^N b_{ik}^{(n)} f(t_k) + E_n(t_i, \xi), \quad (5.1)$$

n-times

where

$$E_n(t_i, \xi) = \frac{f^{(N+1)}(\xi)}{(N+1)! K_{N+1}} \int_{-1}^{t_i} \int_{-1}^t \dots \int_{-1}^t P_{N+1}(t) dt \dots dt \quad (5.2)$$

n-times

and

$$K_{N+1} = \frac{(2N+1)!}{2^N (N+1)! N!}.$$

**Proof:** See Ref. [7].

### Theorem 5.2

Assume that the optimal control problem (2.5)-(2.8) is approximated by Legendre integral method and assuming that  $x^{(n+N+1)}(t)$  is bounded i.e.

$$\|x^{(n+N+1)}(t)\| \leq D, \quad (5.3)$$

then there exists a number  $\xi(t) \in [-1, 1]$  such that

$$E(t_i, \xi) \leq \frac{D}{(N+1)! K_{N+1}} \int_{-1}^{t_i} \int_{-1}^t \dots \int_{-1}^t P_{N+1}(t) dt \dots dt, \quad (5.4)$$

n-times

$$E_F(t_i) =$$

$$F \left( \sum_{j=0}^N b_{ij}^{(n)} \Psi(t_j) + E_n(t_i, \xi) + \sum_{r=0}^{n-1} C_r t_i^r, \sum_{j=0}^N b_{ij}^{(n-1)} \Psi(t_j) + E_{n-1}(t_i, \xi) + \sum_{r=0}^{n-2} C_r t_i^r, \dots, \sum_{j=0}^N b_{ij} \Psi(t_j) + C_0 + E_1(t_i, \xi), u(t_i) \right)$$

$$-F\left(\sum_{j=0}^N b_{ij}^{(n)}\Psi(t_j) + \sum_{r=0}^{n-1} C_r t_i^r, \sum_{j=0}^N b_{ij}^{(n-1)}\Psi(t_j) + \sum_{r=0}^{n-2} C_r t_i^r, \dots, \sum_{j=0}^N b_{ij}\Psi(t_j) + C_0, u(t_i)\right). \quad (5.5)$$

**Proof:**

Firstly, let  $E(t_i, \xi)$  denote the error in approximation  $x(t_i)$  with (4.3), namely

$$E(t_i) = \int_{-1}^{t_i} \int_{-1}^t \dots \int_{-1}^t \Psi(t) dt dt \dots dt - \sum_{j=0}^N b_{ij}^{(n)} \Psi(t_j), \quad (5.6)$$

$n$ -times

then, making use of (5.1) and (5.2), the error in the approximation (4.3) can be written as:

$$\begin{aligned} E(t_i, \xi) &= \frac{\Psi^{(N+1)}(\xi)}{(N+1)! K_{N+1}} \int_{-1}^{t_i} \int_{-1}^t \dots \int_{-1}^t P_{N+1}(t) dt dt \dots dt \\ &= \frac{\mathcal{X}^{(n+N+1)}(\xi)}{(N+1)! K_{N+1}} \int_{-1}^{t_i} \int_{-1}^t \dots \int_{-1}^t P_{N+1}(t) dt dt \dots dt \end{aligned}$$

$n$ -times

Thus, making use of (5.3),

$$E(t_i, \xi) \leq \frac{D}{(N+1)! K_{N+1}} \int_0^{t_i} \int_0^t \dots \int_0^t P_{N+1}(t) dt dt \dots dt.$$

$n$ -times

Secondly, the original constraint (2.6) in view of (4.2) becomes

$$F\left(\left\{\int_0^{t_i} \int_0^t \dots \int_0^t \Psi(t) dt + \sum_{r=0}^{n-1} C_r t_i^r\right\}, \left(\frac{2}{T}\right)\left\{\int_0^{t_i} \int_0^t \dots \int_0^t \Psi(t) dt + \sum_{r=0}^{n-2} C_r t_i^r\right\}, \dots, \left(\frac{2}{T}\right)^n \Psi(t_i), u(t_i)\right) = 0$$

$n$ -times       $(n-1)$ -times

Making use of (5.1) then,

$$\begin{aligned} &F\left(\sum_{j=0}^N b_{ij}^{(n)}\Psi(t_j) + E_n(t_i, \xi) + \sum_{r=0}^{n-1} C_r t_i^r, \right. \\ &\quad \left. \dots, \left(\frac{2}{T}\right)^{n-1} \sum_{j=0}^N b_{ij}\Psi(t_j) + E_1(t_i, \xi) + C_0, \left(\frac{2}{T}\right)^n \Psi(t_i), u(t_i)\right) = 0. \end{aligned} \quad (5.7)$$

Subtracting (4.5) from (5.7), we obtain

$$\begin{aligned} E_F(t_i) &= F\left(\sum_{j=0}^N b_{ij}^{(n)}\Psi(t_j) + E_n(t_i, \xi) + \sum_{r=0}^{n-1} C_r t_i^r, \right. \\ &\quad \left. \dots, \left(\frac{2}{T}\right)^{n-1} \sum_{j=0}^N b_{ij}\Psi(t_j) + E_1(t_i, \xi) + C_0, \left(\frac{2}{T}\right)^n \Psi(t_i), u(t_i)\right) - \\ &\quad F\left(\left\{\sum_{j=0}^N b_{ij}^{(n)}\Psi(t_j) + \sum_{r=0}^{n-1} C_r t_i^r\right\}, \dots, \left(\frac{2}{T}\right)^{n-1} \left\{\sum_{j=0}^N b_{ij}\Psi(t_j) + C_0\right\}, \left(\frac{2}{T}\right)^n \Psi(t_i), u(t_i)\right). \end{aligned}$$

with  $E_n(t_i, \xi)$  is defined in (5.2).

## 6. Numerical Examples

Now, we consider the following problems to show the effectiveness of our technique.

### Example 1:

Among all piecewise differentiable control variables, find the optimal control  $u(t)$  which minimizes

$$J = \int_0^1 [x^2(\tau) + x'^2(\tau) + 0.005u^2(\tau)] d\tau, \quad (6.1)$$

$$\text{subject to} \quad x''(\tau) + x'(\tau) - u(\tau) = 0, \quad (6.2)$$

$$x(\tau) - 8(\tau - 0.5)^2 + 0.5 \leq 0. \quad (6.3)$$

And

$$x(0) = 0, \quad x'(0) = -1. \quad (6.4)$$

The first step in solving this problem by the proposed method is to transform the time interval into  $t \in [-1, 1]$ . This will lead to the following problem.

Minimizes

$$J = \frac{1}{2} \int_{-1}^1 [x^2(t) + 4x'^2(t) + 0.005u^2(t)] dt, \quad (6.5)$$

$$\text{Subject to} \quad 4x''(t) + 2x'(t) - u(t) = 0, \quad (6.6)$$

$$x(t) - 2t^2 + 0.5 \leq 0. \quad (6.7)$$

And

$$x(0) = 0, \quad x'(0) = -1. \quad (6.8)$$

We give the approximation (4.1)-(4.3) for the state variable  $x(t)$  so

Let  $x''(t_i) = \Psi(t_i) = \Psi_i$

$$x'(t) = \int_{-1}^t \Psi(s) ds + c_1$$

$$x(t) = \int_{-1}^t \int_{-1}^s \Psi(s) ds ds + c_1 t + c_2.$$

By using Legendre method and condition (6.8), we have  $c_1 = c_2 = -1$  then

$$x'(t_i) = \sum_{k=0}^N b_{ik} \Psi(t_k) - 1,$$

$$x(t_i) = \sum_{k=0}^N b_{ik}^{(2)} \Psi(t_k) - t_i - 1.$$

We also use one of the approximations,  $u(t_i)$ ,  $i = 0, 1, \dots, N$  for the control variable, and then the problem can be converted to the following constrained optimization problem:

Minimize

$$J = \frac{1}{2} \sum_{i=0}^N b_{Ni} \left( \left[ \sum_{k=0}^N b_{ik}^{(2)} \Psi_k - t_i - 1 \right]^2 + 4 \left[ \sum_{k=0}^N b_{ik} \Psi_k - 1 \right]^2 + 0.005u^2(t_i) \right), \quad (6.9)$$

$$\text{subject to} \quad 4\Psi(t_i) - \left[ \sum_{k=0}^N b_{ik} \Psi_k - 1 \right] - u(t_i) = 0. \quad (6.10)$$

We approximate the inequality constraint by adding a slack variable as we show previously,



$$\left( \sum_{k=0}^N b_{ik}^{(2)} \Psi_k - t_i - 1 \right) - 2t_i^2 + 0.5 + A_{N+i}^2 = 0. \quad (6.11)$$

Solving this problem (6.9)-(6.11) using the proposed method by 9<sup>th</sup> order Legendre, we find the optimal value to be  $J^* = 0.71426412$ . The optimal state and the optimal control are shown in Figures (1) and (2), respectively. Elnagar [8] used of Cell Averaging Chebyshev method by 9<sup>th</sup> order Chebyshev for solve this example and have  $J^* = 0.74096103$ .

Fig. (1) state variable  $x(t)$  of example (1)

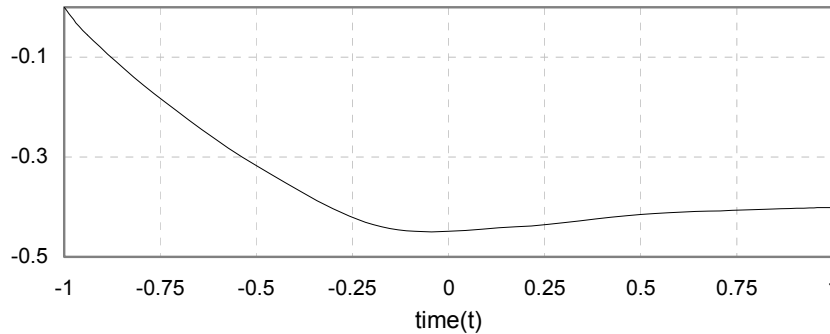
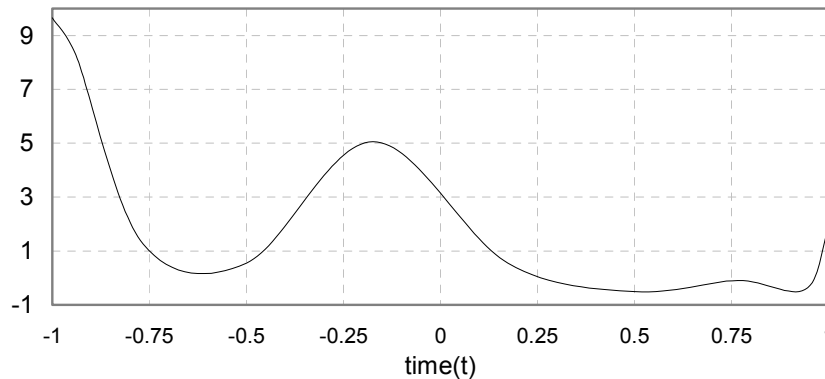


Fig. (2) control variable  $u(t)$  of example (1)



### Example 2: The Controlled Linear Oscillator

Consider the optimal control problem of a liner oscillator the performance index

$$J = \frac{1}{2} \int_{-T}^0 u^2(\tau) d\tau \quad (6.12)$$

is minimized over all admissible control functions  $u(\tau)$ .

Subject to the differential equation

$$\ddot{x}(\tau) + \omega^2 x(\tau) = u(\tau), \quad -T \leq \tau \leq 0 \quad (6.13)$$

with the boundary conditions

$$x(-T) = x_0, \quad x(0) = 0$$

The first step in solving this problem by the proposed method is to transform the time interval into  $t \in [-1, 1]$  by  $\tau = \frac{T}{2}(t-1)$ . This will lead to the following problem:

Minimize

$$J = \frac{T}{4} \int_{-1}^1 u^2(t) dt \quad (6.14)$$

Subject to

$$x''(t) = \frac{T^2}{4} (-\omega^2 x + u(t)), \quad (6.15)$$

With the boundary conditions

$$x(-1) = x_0 \text{ and } x(1) = 0. \quad (6.16)$$

For solve the problem let

$$x''(t_i) = \Psi(t_i) = \Psi_i$$

and use the approximation Eqs. (4.1)-(4.3) for the state variable as:

$$x'(t) = \int_{-1}^t \Psi(s) ds + c_1$$

$$x(t) = \int_{-1}^t \int_{-1}^s \Psi(s) ds ds + c_1 t + c_2.$$

using Legendre method, we get

$$x'(t_i) = \sum_{k=0}^N b_{ik} \Psi(t_k) + c_1,$$

$$x(t_i) = \sum_{k=0}^N b_{ik}^{(2)} \Psi(t_k) + c_1 t_i + c_2$$

From the boundary condition Eq. (6.16),

$$x(-1) = -c_1 + c_2 = x_0,$$

$$x(1) = \sum_{k=0}^N b_{Nk}^{(2)} \Psi(t_k) + c_1 + c_2 = 0$$

then,

$$c_2 = \frac{1}{2} \left( x_0 - \sum_{k=0}^N b_{Nk}^{(2)} \Psi(t_k) \right) \text{ and } c_1 = \frac{-1}{2} \left( x_0 + \sum_{k=0}^N b_{Nk}^{(2)} \Psi(t_k) \right),$$

hence

$$x(t_i) = \sum_{k=0}^N b_{ik}^{(2)} \Psi(t_k) + \frac{1}{2} x_0 (1 - t_i) - \frac{1}{2} x_0 (1 + t_i) \sum_{k=0}^N b_{Nk}^{(2)} \Psi(t_k)$$

then the problem can be converted to the following constrained optimization problem:

Minimize

$$J = \frac{T}{4} \sum_{i=0}^N b_{Ni} u^2(t_i) \quad (6.17)$$

Subject to

$$\Psi(t_i) - \frac{T^2}{4} \left( -\omega^2 \left\{ \sum_{k=0}^N b_{ik}^{(2)} \Psi(t_k) + \frac{1}{2} x_0 (1 - t_i) - \frac{1}{2} x_0 (1 + t_i) \sum_{k=0}^N b_{Nk}^{(2)} \Psi(t_k) \right\} + u(t_i) \right) = 0. \quad (6.18)$$

At  $\omega=1$ ,  $T=2$  and  $x_0=0.5$ , we get the optimal results of the cost functional  $J^* = 0.18485854$  with  $N=18$ . Table (1) has optimal value of the cost functional  $J^*$  for

different values of  $N$ . The optimal state and the optimal control are shown in Figs. (3) and (4), respectively.

Table (1):  $J^*$  of present method with other methods

Methods	$N, M$	$J^*$
Van Dooren [13]	M=4	0.18491700
	M=7	0.18485854
	M=10	0.18485854
Elnagar [8]	M=5	0.18485790
	M=6	0.18485854
Present method	N=8,10,12	0.18485851
		0.18485851
		0.18485851

FIG.(3) State variable  $x(t)$  of example (2)

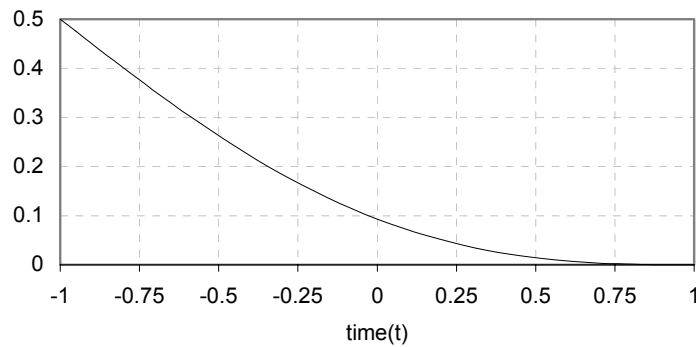
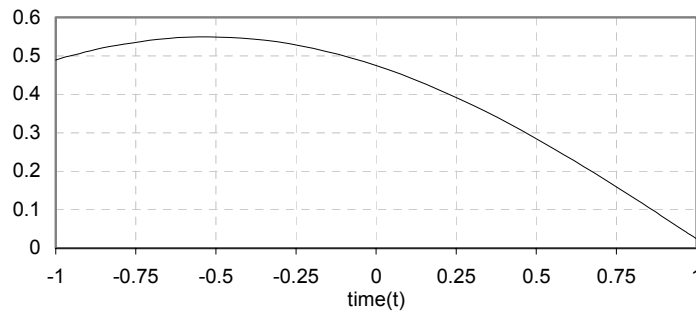


fig. (4) control variable  $u(t)$  of example (2)



## 7. Conclusion

The basic idea of our present method is to transform the optimal control problems governed by ordinary differential equations to a constrained optimization problem, by using Legendre approximations. We solve the resulting constrained optimization problem since it is easier than solving the original problem. Here we use (PQI) method, which may be more suitable in such case, where the number of constraints is increases.

The major advantages of this method is that, we can deal directly with the highest-order derivatives in the differential equation without transforming it to a system of first order, and that will reduce the number of the unknowns. In this way, the optimal control problem is replaced by a parameter optimization problem which consists of the minimization of the performance index subject to algebraic constraints. Finally, the method has been extended to the linear and nonlinear optimal control problems.

The tables given previously show that the suggested technique is quite reliable. It can be successfully applied to both linear and nonlinear ordinary differential problems and related optimal control problems. The methods produce an accurate solution at small number of nodes. The comparison of the maximum absolute error resulting from the proposed method and those obtained by Elnagar [8] and Van Dooren [13] show favorable agreement and always it is more accurate than these treatments.

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## B-spline solution of the Black-Scholes partial differential equation

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### Abstract

The numerical solutions of several mathematical models in the financial economics are arising. Most of the models are based on the Black-Scholes partial differential equations. In this paper, the Black-Scholes option pricing model which has been used frequently is solved by using the B-spline functions. The numerical experiments showed that the present method is an applicable technique and gives an exciting results for European option pricing.

*Keywords:* Black-Scholes equation; B-spline method.

### 1. Introduction

Mathematical modeling and simulation have become essential tools in the financial industry. Related people spend a lot of time to simulate and predict the price movements for financial assets like stocks, options and bonds. Much of the mathematics employed in this area especially in academic research, is highly sophisticated and spans the fields of analysis, probability, statistics, differential equations and numerical analysis. During the past years financial securities have become necessary tools for corporations and investors. The

movement in stock prices creates a risk, and options can be used to hedge assets and portfolios to control this risk. In finance and economics in general, option pricing theory is one of the major theories. A call (put) option gives the holder the right to buy (sell) the underlying asset by a certain date  $T$  (expiration date or maturity date) at a certain price (strike price). Options that can be exercised only on the expiration date known as European options whereas American options can be exercised at any time up to the expiration date [1]. Related parties could plan for the financial future with some certainty regardless of the price of the commodity on the open market at the time the option expired. However, the seller of the option incurs the risk of having to buy or sell an asset at a loss and must be compensated for this risk through the sales price of the option. Determination of the option value is a major concern of financial engineering. The numerical solutions of several mathematical models are arising in financial economics for the valuation of both European and American call options on different types assets are considered in several researches. All the models are based on the Black- Scholes partial differential equation. The newly developed Least-Squares Monte Carlo method offers a simple and well-organized technique for valuing American-type options. Compared to the other valuation techniques, the Least-squares Monte Carlo method does not necessitate advanced mathematical techniques and has the supplementary advantage of being able to easily handle multiple and complex stochastic processes concurrently [2]. The methods for valuing options can be divided into analytical and numerical techniques. The first analytic model for valuing simple financial options was created by Black and Scholes [3]. After Black and Scholes, the research has aimed at developing numerical methods capable of dealing with valuing complex options, such as the American-type options with multiple uncertain state variables. The most popular numerical techniques for valuing options are: finite difference method [4], binomial lattice method [5], and Monte Carlo Simulation by Boyle [6]. Finite difference method for option valuation has been utilized by Courtadon [7] as well. The most significant drawback of both finite difference and lattice method is that they are impractical for valuing complex options with multiple uncertain state variables. On the contrary to both the finite difference and lattice methods, Monte Carlo Simulation can handle competently with the situations where there are multiple and complex stochastic variables. Tilley [8] was the primary researcher to suggest a modification to Monte Carlo Simulation in order to make it applicable for valuing American options. Another research establishes a dual way

to price American options, formed on simulating the paths of the option payoff [9]. The method introduced, leads to candidate hedging policies for the option, and estimates of the risk involved in using them. Monte Carlo Simulation can be applied to asset pricing problems with multiple state variables and possible path dependencies because convergence of Monte Carlo method is independent of the number of state variables. Another study relates to Monte Carlo Simulation to the problem of shaping free exercise boundaries for pricing American type options [10]. Along with numerical methods for valuing derivatives, lattice based models like binomial are useful for pricing American options, but have difficulty with path dependent contracts. Monte Carlo Simulation is a good way for path dependent problems but computation time increases harshly when there is more than one stochastic variable. This problem is handled by a technique introduced by Raymar and Zwecher [11]. Their method is fast and accurate in basic cases, can be used easily on much more complex options as well. The biggest problem in assessing its performance on the most difficult cases is that there are no benchmarks available for accuracy. Their techniques solve valuation problems that no other approach can touch and the technique is also applicable to many complex equity and fixed-income derivatives. In another work a mathematical model is presented which give the value of the call option in any moment prior to the expiry date. The main interest of options on assets comes from limiting the risk due to unexpected fallings of the asset price [12]. There are also many other methods for American option pricing problem like the method of lines by Meyer and Van der Hoek [13]. Most of the latest literature dealing with the modeling of financial assets assumes that the essential dynamics of equity prices follow a jump procedure. Some financial models, capture a number of main characteristics of the dynamics of stock prices [14]. In this specific research, the article named Compact Finite Difference Method for American Option Pricing is the starting point. In that research, three ways of combining compact finite difference methods for American option price on a single asset has been developed. All the three methods work for both short term and long term options [15].

In this paper, we deal with the European put options on shares which may pay continuous dividends. Consider European put option pricing problem in the following simple form:

$$u_\tau = u_{xx} + g(x, \tau), \quad (1)$$

where  $x \in (-\infty, +\infty)$ ,  $\tau \in (0, (\sigma^2/2)T)$ ,

$$k_1 = 2r/\sigma^2,$$

$$k_2 = 2(r - D)/\sigma^2 \text{ and}$$

$$g(x, \tau) = e^{k_1\tau}((k_1 - k_2)e^{x-(k_2-1)\tau} - k_1).$$

This problem has the following the initial and the boundary conditions:

$$u(x, 0) = \max(e^x - 1, 0), x \in (-\infty, +\infty), \quad (2)$$

$$\lim_{x \rightarrow \infty} u(x, \tau) = e^{k_1\tau}(e^{x-(k_2-1)\tau} - 1), \quad (3)$$

$$\lim_{x \rightarrow -\infty} u(x, \tau) = 1 + e^{k_1\tau}(e^{x-(k_2-1)\tau} - 1), \quad (4)$$

where  $T$  is the duration (in years) of the option contract,  $\sigma$  is the stands for the volatility in return,  $D$  is the dividend yield of the asset, and  $r$  is the risk free interest rate.

## 2. The third-degree B-splines

A detailed description of B-spline functions generated by subdivision can be found in [16]. Consider equally-spaced knots of a partition  $\pi : a = x_0 < x_1 < \dots < x_n = b$  on  $[a, b]$ . Let  $S_3[\pi]$  be the space of continuously-differentiable, piecewise, third-degree polynomials on  $\pi$ . That is,  $S_3[\pi]$  is the space of third-degree splines on  $\pi$ . Consider the B-splines basis in  $S_3[\pi]$ . The third-degree B-splines are defined as

$$B_0(x) = \frac{1}{6h^3} \begin{cases} x^3 & 0 \leq x < h \\ -3x^3 + 12hx^2 - 12h^2x + 4h^3 & h \leq x < 2h \\ 3x^3 - 24hx^2 + 60h^2x - 44h^3 & 2h \leq x < 3h \\ -x^3 + 12hx^2 - 48h^2x + 64h^3 & 3h \leq x < 4h \end{cases} \quad (5)$$

$$B_{i-1}(x) = B_0(x - (i-1)h), \quad i = 2, 3, \dots$$



Table 1: Values of  $B_i$ ,  $B'_i$  and  $B''_i$ 

	$\mathbf{x}_i$	$\mathbf{x}_{i+1}$	$\mathbf{x}_{i+2}$	$\mathbf{x}_{i+3}$	$\mathbf{x}_{i+4}$
$B_i$	0	1	4	1	0
$B'_i$	0	-3/h	0/h	3/h	0
$B''_i$	0	6/h <sup>2</sup>	-12/h <sup>2</sup>	6/h <sup>2</sup>	0

To solve Eq.(1),  $B_i$ ,  $B'_i$  and  $B''_i$  evaluated at the nodal points are needed. Their coefficients are summarized in Table 1.

### 3. B-spline solution for the Black-Scholes option pricing model

In this section a spline method for solving option pricing model is outlined, which is based on the collocation approach [17]. Let

$$S(x) = \sum_{j=-3}^{n-1} C_j B_j(x) \quad (6)$$

be an approximate solution of Eq.(1), where  $C_i$  are unknown real coefficients and  $B_j(x)$  are third-degree B-spline functions. Let  $x_0, x_1, \dots, x_n$  be  $n+1$  grid points in the interval  $[a, b]$ , so that

$$x_i = a + ih, \quad i = 0, 1, \dots, n; \quad x_0 = a, \quad x_n = b, \quad h = (b-a)/n.$$

Difference schemes for this problem considered as following:

$$\frac{u_{i+1} - u_i}{\Delta\tau} = u_{xx} + g(x, \tau) \quad (7)$$

$$-(\Delta\tau)u''_{i+1} + u_{i+1} = u_i + (\Delta\tau)g(x, \tau) \quad (8)$$

and the initial conditions are given in (2):

$$u(x, 0) = \max(e^x - 1, 0), \quad x \in (-\infty, +\infty), \quad (9)$$

$$u(x, 0) = u_0, \quad (10)$$

Substituting (9-10) in (8) then is obtained as follows

$$t = 0 + \Delta\tau \quad -(\Delta\tau)u_1'' + u_1 = u_0 + (\Delta\tau)g(x, \Delta\tau) \quad (11)$$

$$t = 0 + 2\Delta\tau \quad -(\Delta\tau)u_2'' + u_2 = u_1 + (\Delta\tau)g(x, 2\Delta\tau) \quad (12)$$

$$\begin{array}{ccc} \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \end{array}$$

$$t = 0 + n\Delta\tau \quad -(\Delta\tau)u_n'' + u_n = u_{n-1} + (\Delta\tau)g(x, n\Delta\tau) \quad (13)$$

The approximate solution of the equation (11)-(13) are sought in the form of the B-spline functions  $S(x)$ , it follows that

$$t = 0 + \Delta\tau \quad -(\Delta\tau)S_1'' + S_1 = u_0 + (\Delta\tau)g(x, \Delta\tau) \quad (14)$$

$$t = 0 + 2\Delta\tau \quad -(\Delta\tau)S_2'' + S_2 = u_1 + (\Delta\tau)g(x, 2\Delta\tau) \quad (15)$$

$$\begin{array}{ccc} \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \end{array}$$

$$t = 0 + n\Delta\tau \quad -(\Delta\tau)S_n'' + S_n = u_{n-1} + (\Delta\tau)g(x, n\Delta\tau) \quad (16)$$

and boundary conditions (3)-(4) can be rewritten as follows

$$\sum_{j=-3}^{n-1} C_j B_j(x) = 1 + e^{k_1\tau}(e^{x-(k_2-1)\tau} - 1) \text{ for } \lim_{x \rightarrow -\infty}, \quad (17)$$

$$\sum_{j=-3}^{n-1} C_j B_j(x) = e^{k_1\tau}(e^{x-(k_2-1)\tau} - 1) \text{ for } \lim_{x \rightarrow \infty}. \quad (18)$$

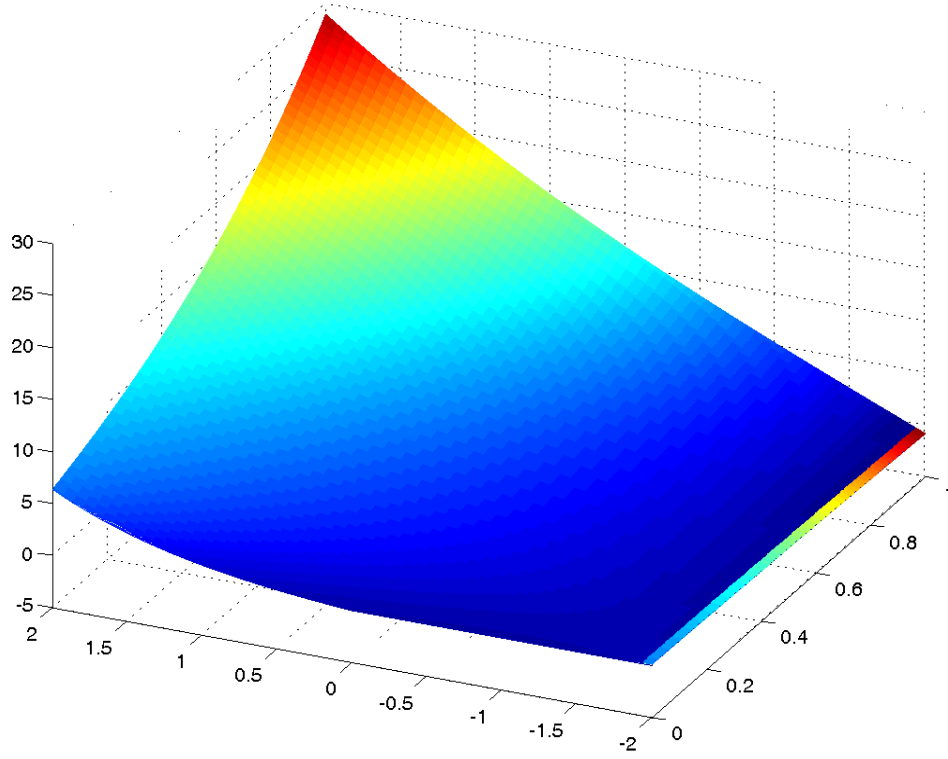


Figure 1: Numerical results for the data set:  $\sigma^2 = 0.3$ ,  $D = 0.02$ ,  $r = 0.04$  and  $n = 111$ ,  $k = 0.001$ .

Solving the spline Eq.(14) and using boundary conditions (17)-(18), we have obtained approximate solution. It is easy to see that, the same approximation is applied the other equations (15)-(16). As an illustration of the good performance of the above algorithms. We shall present the numerical results obtained for data set:

$$x \in (-2, 2) , \tau \in (0, 1),$$

$$\sigma^2 = 0.3 ,$$

$$D = 0.02 ,$$

$$r = 0.04 .$$

The time and share value steps have been taken to  $k=0.001$ ,  $h=1/111$ , in Fig.1 the computed solutions are shown. All computations were carried out using MATLAB 6.5.

#### 4. Conclusions

B-spline method has been considered for the numerical solution of European Black-Scholes option pricing model. As well-known that the Black-Scholes partial differential equation can be transformed into the heat equation [18]. In a previous work, we have showed that the proposed method gives good results for the heat equation [19]. In this paper, we have shown that the present method is an alternative technique for the solution of the Black-Scholes model too. The results of numerical testing show that the numerical method is very efficient and accurate for the Black-Scholes model.

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# MULTILATERAL GENERATING FUNCTIONS FOR THE MULTIPLE LAGUERRE AND MULTIPLE HERMITE POLYNOMIALS

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**ABSTRACT.** The main object of this paper is to derive several substantially more general families of bilinear, bilateral, and mixed multilateral finite-series relationships and generating functions for the multiple Laguerre and multiple Hermite polynomials. Some applications of the above statements are also given.

## 1. INTRODUCTION

Multiple orthogonal polynomials, which are the extension of orthogonal polynomials has been an active research field during the last few decades. Their roots come from Hermite Pade approximation of a system of  $r$  functions.

There are two subjects in the theory of multiple orthogonal polynomials: classical multiple orthogonal polynomials and general multiple orthogonal polynomials. The classical multiple orthogonal polynomials are those polynomials which have a Rodrigues formula and there exists a first order differential operator such that, when applied to these multiple orthogonal polynomials, gives another set of multiple orthogonal polynomials. For the classical multiple orthogonal polynomials, we refer [3],[15],[10] and for the general multiple orthogonal polynomials, we refer [7],[8],[16]. Surveys of results on multiple orthogonal polynomials together with an extensive bibliography on the subject were given in [2].

In this paper we consider the multiple Laguerre polynomials of the first and second kind and multiple Hermite polynomials which are defined as follows:

The multiple Laguerre polynomials of the first kind are orthogonal on  $[0, \infty)$  with respect to the  $r$  weights  $w_j(x) = x^{\alpha_j} e^{\beta x}$  ( $j = 1, 2, \dots, r$ ), where  $\beta < 0$ ;  $\alpha_1, \dots, \alpha_r > -1$  and  $\alpha_i - \alpha_j \notin \mathbb{Z}$  whenever  $i \neq j$ . The multiple Laguerre  $I$  polynomials  $L_{\vec{n}}^{(\vec{\alpha}, \beta)}(x)$ , for the multi-index  $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$  and  $\vec{\alpha} = (\alpha_1, \dots, \alpha_r)$ , are of degree  $|\vec{n}| = n_1 + \dots + n_r$  and satisfy the orthogonality conditions

$$\int_0^\infty L_{\vec{n}}^{(\vec{\alpha}, \beta)}(x) x^{k+\alpha_i} e^{\beta x} dx = 0, \quad k = 0, 1, \dots, n_i - 1; \quad i = 1, 2, \dots, r.$$

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*Key words and phrases.* Rodrigues formula, Generating function, Multiple Laguerre polynomials, Multiple Hermite polynomials, Finite-series relationship.

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The Rodrigues formula for the multiple Laguerre polynomials of the first kind is given by ( see [10])

$$L_{\vec{n}}^{(\vec{\alpha}, \beta)}(x) = w_r^{-1}(w_r w_{r-1}^{-1} x^{n_r} (\dots (w_2 w_1^{-1} x^{n_2} (w_1 x^{n_1})^{(n_1)})^{(n_2)} \dots)^{(n_{r-1})})^{(n_r)}.$$

The multiple Laguerre *II* polynomials  $L_{\vec{n}}^{(\alpha, \vec{\beta})}(x)$  are defined by ( see [10])

$$L_{\vec{n}}^{(\alpha, \vec{\beta})}(x) = w_r^{-1}(w_r w_{r-1}^{-1} (\dots (w_2 w_1^{-1} (w_1 x^{\lfloor \vec{n} \rfloor})^{(n_1)})^{(n_2)} \dots)^{(n_{r-1})})^{(n_r)},$$

where  $\alpha > -1$  and  $\beta_j < 0$  ( $j = 1, 2, \dots, r$ ). They are orthogonal on  $(0, \infty)$  with respect to the  $r$  weights  $w_j(x) = x^\alpha e^{\beta_j x}$  ( $j = 1, 2, \dots, r$ ),  $\beta_i - \beta_j \notin \mathbb{Z}$  whenever  $i \neq j$ .

The multiple Hermite polynomials  $H_{\vec{n}}^{(\vec{\alpha}, \delta)}(x)$  are defined by

$$H_{\vec{n}}^{(\delta; \vec{\alpha})}(x) = w_r^{-1}(w_r w_{r-1}^{-1} (\dots (w_2 w_1^{-1} (w_1)^{(n_1)})^{(n_2)} \dots)^{(n_{r-1})})^{(n_r)},$$

where  $\delta < 0$  and  $\vec{\alpha} = (\alpha_1, \dots, \alpha_r)$ . They are orthogonal on  $(-\infty, \infty)$  with respect to the  $r$  weights  $w_j(x) = e^{\frac{\delta}{2}x^2 + \alpha_j x}$  ( $j = 1, 2, \dots, r$ ),  $\alpha_i - \alpha_j \notin \mathbb{Z}$  whenever  $i \neq j$ .

The Rodrigues formula allows to obtain the differential equations and explicit expressions of the multiple orthogonal polynomials (see [3],[15]). Recently, using Rodrigues formulas [10], some generating functions have been obtained for the multiple Laguerre *I* polynomials:

$$\begin{aligned} (1) \quad G_1(x; t_1, \dots, t_r) &= \sum_{n_1, \dots, n_r=0}^{\infty} L_{n_1, \dots, n_r}^{(\alpha_1, \dots, \alpha_r; \beta)}(x) \frac{t_1^{n_1}}{n_1!} \dots \frac{t_r^{n_r}}{n_r!} \\ &= \left[ \prod_{i=1}^r \frac{1}{(1-t_i)^{\alpha_i+1}} \right] \exp \left( \beta x \left( \frac{1}{(1-t_1) \dots (1-t_r)} - 1 \right) \right) \\ &\quad (|t_i| < 1; i = 1, 2, \dots, r), \end{aligned}$$

for the multiple Laguerre *II* polynomials:

$$\begin{aligned} (2) \quad G_2(x; t_1, \dots, t_r) &= \sum_{n_1, \dots, n_r=0}^{\infty} L_{n_1, \dots, n_r}^{(\alpha; \beta_1, \dots, \beta_r)}(x) \frac{t_1^{n_1}}{n_1!} \dots \frac{t_r^{n_r}}{n_r!} \\ &= \frac{1}{(1-t_1 - \dots - t_r)^{\alpha+1}} \exp \left( \frac{\beta_1 t_1 + \dots + \beta_r t_r}{1-t_1 - \dots - t_r} x \right) \\ &\quad |t_1 + t_2 + \dots + t_r| < 1, \end{aligned}$$

and for the multiple Hermite polynomials:

$$\begin{aligned} (3) \quad G_3(x; t_1, \dots, t_r) &= \sum_{n_1, \dots, n_r=0}^{\infty} H_{n_1, \dots, n_r}^{(\delta; \alpha_1, \dots, \alpha_r)}(x) \frac{t_1^{n_1}}{n_1!} \dots \frac{t_r^{n_r}}{n_r!} \\ &= \exp \left( \delta x \sum_{i=1}^r t_i + \frac{\delta}{2} \left( \sum_{i=1}^r t_i \right)^2 + \sum_{i=1}^r \alpha_i t_i \right). \end{aligned}$$

The main object of this paper is to derive several substantially more general families of bilinear, bilateral, and mixed multilateral finite-series relationships and generating functions for the multiple Laguerre and multiple Hermite polynomials. Some applications of the above statements are also given.



# MULTILATERAL GENERATING FUNCTIONS

## 2. MAIN RESULTS

In recent years by making use of the familiar group-theoretic (Lie algebraic) method some certain mixed trilateral finite-series relationships have been proved for orthogonal polynomials (see, for instance, [13]). In this section, we obtain families of bilinear, bilateral, mixed multilateral finite-series relationship and generating functions for the multiple Laguerre and multiple Hermite polynomials, by applying the similar method as considered in [13],[6] (and recently in [11],[1],[12],[14]), instead of using group theoretic method.

In the first three theorem of this section, we consider the case  $r = 2$ . In fact for  $r = 2$ , generating relation (1) turns out to be

$$(4) \quad G_1^*(x; t_1, t_2) = \sum_{n,k=0}^{\infty} L_{n,k}^{(\alpha_1, \alpha_2; \beta)}(x) \frac{t_1^n}{n!} \frac{t_2^k}{k!} \\ = \frac{1}{(1-t_1)^{\alpha_1+1} (1-t_2)^{\alpha_2+1}} \exp \left( \frac{\beta(t_1+t_2-t_1t_2)}{(1-t_1)(1-t_2)} x \right) \\ |t_1| < 1, |t_2| < 1.$$

Using (4), we obtain the following theorem at once.

**Theorem 2.1.** *Corresponding to an identically nonvanishing function  $\Phi_\mu(\xi_1, \xi_2, \dots, \xi_s)$  of  $s$  (real or complex) variables  $\xi_1, \xi_2, \dots, \xi_s$  ( $s \in \mathbb{N} = \mathbb{N}_0 \setminus \{0\}$ ) and of (complex) order  $\mu$ , let*

$$\Delta^{(1)}(\xi_1, \xi_2, \dots, \xi_s; \omega) \quad : \quad = \sum_{l=0}^{\infty} a_l \Phi_{\mu+\psi_l}(\xi_1, \xi_2, \dots, \xi_s) \omega^l \\ a_l \neq 0.$$

Suppose also that

$$\Theta_{n, N_1, N_2}^{(1)}(x; \xi_1, \xi_2, \dots, \xi_s; \zeta, \rho) \quad : \quad = \sum_{k=0}^{\left[ \frac{n}{N_1} \right]} \sum_{l=0}^{\left[ \frac{k}{N_2} \right]} \frac{a_l \Phi_{\mu+\psi_l}(\xi_1, \xi_2, \dots, \xi_s)}{(n-N_2k)! (k-N_1l)!} \\ \times L_{n-N_2k, k-N_1l}^{(\alpha_1+\lambda_1l, \alpha_2+\lambda_2l; \beta+\lambda_3l)}(x) \zeta^k \rho^l \\ (n \in \mathbb{N})$$

then

$$(5) \quad \sum_{n=0}^{\infty} \Theta_{n, N_1, N_2}^{(1)} \left( x; \xi_1, \xi_2, \dots, \xi_s; \frac{t_2}{t_1^{N_2}}, \frac{\gamma}{t_2^{N_1}} \right) t_1^n \\ = \Delta^{(1)} \left( \xi_1, \xi_2, \dots, \xi_s; \frac{\gamma}{(1-t_1)^{\lambda_1} (1-t_2)^{\lambda_2}} \exp \left( \frac{\lambda_3(t_1+t_2-t_1t_2)}{(1-t_1)(1-t_2)} x \right) \right) G_1^*(x; t_1, t_2) \\ |t_1| < 1, |t_2| < 1,$$

provided that each member of (5) exists.

The notation  $[n/q]$  means the greatest integer less than or equal to  $n/q$ .

*Proof.* Substituting  $\Theta_{n,N_1,N_2}^{(1)}(x; \xi_1, \xi_2, \dots, \xi_s; \zeta, \rho)$  into the left hand side of (5), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \Theta_{n,N_1,N_2}^{(1)} \left( x; \xi_1, \xi_2, \dots, \xi_s; \frac{t_2}{t_1^{N_2}}, \frac{\gamma}{t_2^{N_1}} \right) t_1^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\left\lfloor \frac{n}{N_1} \right\rfloor} \sum_{l=0}^{\left\lfloor \frac{k}{N_2} \right\rfloor} \frac{a_l \Phi_{\mu+\psi l}(\xi_1, \xi_2, \dots, \xi_s)}{(n - N_2 k)! (k - N_1 l)!} \left( \frac{t_2}{t_1^{N_2}} \right)^k \left( \frac{\gamma}{t_2^{N_1}} \right)^l \\ & \quad \times L_{n-N_2 k, k-N_1 l}^{(\alpha_1+\lambda_1 l, \alpha_2+\lambda_2 l; \beta+\lambda_3 l)}(x) t_1^n. \end{aligned}$$

Now taking  $n \rightarrow n + N_2 k$

$$\begin{aligned} & \sum_{n=0}^{\infty} \Theta_{n,N_1,N_2}^{(1)} \left( x; \xi_1, \xi_2, \dots, \xi_s; \frac{t_2}{t_1^{N_2}}, \frac{\gamma}{t_2^{N_1}} \right) t_1^n \\ &= \sum_{n,k=0}^{\infty} \sum_{l=0}^{\left\lfloor \frac{k}{N_2} \right\rfloor} \frac{a_l \Phi_{\mu+\psi l}(\xi_1, \xi_2, \dots, \xi_s)}{n! (k - N_1 l)!} \gamma^l t_2^{k-N_1 l} L_{n,k-N_1 l}^{(\alpha_1+\lambda_1 l, \alpha_2+\lambda_2 l; \beta+\lambda_3 l)}(x) t_1^n. \end{aligned}$$

Writing  $k \rightarrow k + N_1 l$  and then using (4), we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \Theta_{n,N_1,N_2}^{(1)} \left( x; \xi_1, \xi_2, \dots, \xi_s; \frac{t_2}{t_1^{N_2}}, \frac{\gamma}{t_2^{N_1}} \right) t_1^n = \sum_{n,k,l=0}^{\infty} a_l \Phi_{\mu+\psi l}(\xi_1, \xi_2, \dots, \xi_s) \gamma^l \\ & \quad \times L_{n,k}^{(\alpha_1+\lambda_1 l, \alpha_2+\lambda_2 l; \beta+\lambda_3 l)}(x) \frac{t_1^n t_2^k}{n! k!} = \sum_{l=0}^{\infty} a_l \Phi_{\mu+\psi l}(\xi_1, \xi_2, \dots, \xi_s) \gamma^l \\ & \quad \times \frac{1}{(1-t_1)^{\alpha_1+\lambda_1 l+1} (1-t_2)^{\alpha_2+\lambda_2 l+1}} \exp \left( \frac{(\beta + \lambda_3 l) (t_1 + t_2 - t_1 t_2)}{(1-t_1)(1-t_2)} x \right) \\ &= \sum_{l=0}^{\infty} a_l \Phi_{\mu+\psi l}(\xi_1, \xi_2, \dots, \xi_s) \left( \frac{\gamma}{(1-t_1)^{\lambda_1} (1-t_2)^{\lambda_2}} \exp \left( \frac{\lambda_3 (t_1 + t_2 - t_1 t_2)}{(1-t_1)(1-t_2)} x \right) \right)^l \\ & \quad \cdot \frac{1}{(1-t_1)^{\alpha_1+1} (1-t_2)^{\alpha_2+1}} \exp \left( \frac{\beta (t_1 + t_2 - t_1 t_2)}{(1-t_1)(1-t_2)} x \right) \\ &= \Delta^{(1)} \left( \xi_1, \dots, \xi_s; \frac{\gamma}{(1-t_1)^{\lambda_1} (1-t_2)^{\lambda_2}} \exp \left( \frac{\lambda_3 (t_1 + t_2 - t_1 t_2)}{(1-t_1)(1-t_2)} x \right) \right) G_1^*(x; t_1, t_2). \end{aligned}$$

Whence the result.  $\square$

Now we consider  $r = 2$  cases of the generating relations (2) and (3), which are

$$\begin{aligned} (6) \quad G_2^*(x; t_1, t_2) &= \sum_{n,k=0}^{\infty} L_{n,k}^{(\alpha; \beta_1, \beta_2)}(x) \frac{t_1^n}{n!} \frac{t_2^k}{k!} \\ &= \frac{1}{(1-t_1-t_2)^{\alpha+1}} \exp \left( \frac{\beta_1 t_1 + \beta_2 t_2}{1-t_1-t_2} x \right) \\ & \quad |t_1 + t_2| < 1 \end{aligned}$$

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and

$$(7) \quad G_3^*(x; t_1, t_2) = \sum_{n,k=0}^{\infty} H_{n,k}^{(\delta; \alpha_1, \alpha_2)}(x) \frac{t_1^n}{n!} \frac{t_2^k}{k!} \\ = \exp \left( \left( \frac{\delta}{2} \right) (t_1^2 + t_2^2) + \alpha_1 t_1 + \alpha_2 t_2 + \delta x (t_1 + t_2) + \delta t_1 t_2 \right).$$

Applying the similar procedure used in the proof of Theorem 2.1, we get the following theorems for the multiple Laguerre *II* and multiple Hermite polynomials, respectively.

**Theorem 2.2.** *Corresponding to an identically nonvanishing function  $\Phi_\mu(\xi_1, \xi_2, \dots, \xi_s)$  of  $s$  (real or complex) variables  $\xi_1, \xi_2, \dots, \xi_s$  ( $s \in \mathbb{N} = \mathbb{N}_0 \setminus \{0\}$ ) and of (complex) order  $\mu$ , let*

$$\Delta^{(2)}(\xi_1, \xi_2, \dots, \xi_s; \omega) \quad : \quad = \sum_{l=0}^{\infty} a_l \Phi_{\mu+\psi_l}(\xi_1, \xi_2, \dots, \xi_s) \omega^l \\ a_l \neq 0.$$

Suppose also that

$$\Theta_{n, N_1, N_2}^{(2)}(x; \xi_1, \xi_2, \dots, \xi_s; \zeta, \rho) \quad : \quad = \sum_{k=0}^{\left[ \frac{n}{N_1} \right]} \sum_{l=0}^{\left[ \frac{k}{N_2} \right]} \frac{a_l \Phi_{\mu+\psi_l}(\xi_1, \xi_2, \dots, \xi_s)}{(n - N_2 k)! (k - N_1 l)!} \\ \times L_{n - N_2 k, k - N_1 l}^{(\alpha + \lambda_1 l; \beta_1 + \lambda_2 l, \beta_2 + \lambda_3 l)}(x) \zeta^k \rho^l \\ (n \in \mathbb{N})$$

then

$$(8) \quad \sum_{n=0}^{\infty} \Theta_{n, N_1, N_2}^{(2)} \left( x; \xi_1, \xi_2, \dots, \xi_s; \frac{t_2}{t_1^{N_1}}, \frac{\gamma}{t_2^{N_1}} \right) t_1^n \\ = \Delta^{(2)} \left( \xi_1, \xi_2, \dots, \xi_s; \frac{\gamma}{(1 - t_1 - t_2)^{\lambda_1}} \exp \left( \frac{\lambda_2 t_1 + \lambda_3 t_2}{1 - t_1 - t_2} x \right) \right) G_2^*(x; t_1, t_2)$$

provided that each member of (8) exists.

**Theorem 2.3.** *Corresponding to an identically nonvanishing function  $\Phi_\mu(\xi_1, \xi_2, \dots, \xi_s)$  of  $s$  (real or complex) variables  $\xi_1, \xi_2, \dots, \xi_s$  ( $s \in \mathbb{N} = \mathbb{N}_0 \setminus \{0\}$ ) and of (complex) order  $\mu$ , let*

$$\Delta^{(3)}(\xi_1, \xi_2, \dots, \xi_s; \omega) \quad : \quad = \sum_{l=0}^{\infty} a_l \Phi_{\mu+\psi_l}(\xi_1, \xi_2, \dots, \xi_s) \omega^l \\ a_l \neq 0.$$

Suppose also that

$$\Theta_{n, N_1, N_2}^{(3)}(x; \xi_1, \xi_2, \dots, \xi_s; \zeta, \rho) \quad : \quad = \sum_{k=0}^{\left[ \frac{n}{N_1} \right]} \sum_{l=0}^{\left[ \frac{k}{N_2} \right]} \frac{a_l \Phi_{\mu+\psi_l}(\xi_1, \xi_2, \dots, \xi_s)}{(n - N_2 k)! (k - N_1 l)!} \\ \times H_{n - N_2 k, k - N_1 l}^{(\delta + \lambda_1 l; \alpha_1 + \lambda_2 l, \alpha_2 + \lambda_3 l)}(x) \zeta^k \rho^l \\ (n \in \mathbb{N})$$

then

$$(9) \quad \sum_{n=0}^{\infty} \Theta_{n, N_1, N_2}^{(3)} \left( x; \xi_1, \xi_2, \dots, \xi_s; \frac{t_2}{t_1^{N_2}}, \frac{\gamma}{t_2^{N_1}} \right) t_1^n \\ = \Delta^{(3)} \left( \xi_1, \dots, \xi_s; \gamma \exp \left( \left( \frac{\lambda_1}{2} \right) (t_1^2 + t_2^2) + \lambda_2 t_1 + \lambda_3 t_2 + \lambda_1 x (t_1 + t_2) + \lambda_1 t_1 t_2 \right) \right) \\ \times G_3^*(x; t_1, t_2)$$

provided that each member of (9) exists.

In the following three theorem we let  $r \in \mathbb{N}$  to be arbitrary. For the multiple Laguerre  $I$  polynomials, we have:

**Theorem 2.4.** *Corresponding to an identically nonvanishing  $r$ -tuple function sequence  $\Phi_{\mu_1, \dots, \mu_r}(\xi_1, \xi_2, \dots, \xi_s)$  of  $s$  (real or complex) variables  $\xi_1, \xi_2, \dots, \xi_s$  ( $s \in \mathbb{N} = \mathbb{N}_0 \setminus \{0\}$ ) and of (complex) orders, let*

$$\Delta_{\psi_1, \dots, \psi_r, \mu_1, \dots, \mu_r}^{(1)}(\xi_1, \xi_2, \dots, \xi_s; \tau_1, \dots, \tau_r) \\ := \sum_{k_1=0}^{\infty} \dots \sum_{k_r=0}^{\infty} a_{k_1, \dots, k_r} \Phi_{\mu_1 + \psi_1 k_1, \dots, \mu_r + \psi_r k_r}(\xi_1, \xi_2, \dots, \xi_s) \tau_1^{k_1} \dots \tau_r^{k_r} \\ a_{k_1, \dots, k_r} \neq 0.$$

Suppose also that

$$\Lambda_{n_1, \dots, n_r, q_1, \dots, q_r}^{\mu, \lambda_1, \dots, \lambda_r, \psi_1, \dots, \psi_r}(x; \xi_1, \xi_2, \dots, \xi_s; \zeta_1, \dots, \zeta_r) \\ := \sum_{k_1=0}^{\left[ \frac{n_1}{q_1} \right]} \dots \sum_{k_r=0}^{\left[ \frac{n_r}{q_r} \right]} \frac{1}{(n_1 - q_1 k_1)! \dots (n_r - q_r k_r)!} a_{k_1, \dots, k_r} \\ \times L_{n_1 - q_1 k_1, \dots, n_r - q_r k_r}^{(\alpha_1 + \lambda_1 k_1, \dots, \alpha_r + \lambda_r k_r; \beta)}(x) \Phi_{\mu_1 + \psi_1 k_1, \dots, \mu_r + \psi_r k_r}(\xi_1, \xi_2, \dots, \xi_s) \zeta_1^{k_1} \dots \zeta_r^{k_r} \\ (n \in \mathbb{N})$$

then

$$(10) \quad \sum_{n_1=0}^{\infty} \dots \sum_{n_r=0}^{\infty} \Lambda_{n_1, \dots, n_r, q_1, \dots, q_r}^{\mu, \lambda_1, \dots, \lambda_r, \psi_1, \dots, \psi_r} \left( x; \xi_1, \xi_2, \dots, \xi_s; \frac{\nu_1}{t_1^{q_1}}, \dots, \frac{\nu_r}{t_r^{q_r}} \right) t_1^{n_1} \dots t_r^{n_r} \\ = \Delta_{\psi_1, \dots, \psi_r, \mu_1, \dots, \mu_r}^{(1)} \left( \xi_1, \xi_2, \dots, \xi_s; \frac{\nu_1}{(1 - t_1)^{\lambda_1}}, \dots, \frac{\nu_r}{(1 - t_r)^{\lambda_r}} \right) G_1(x; t_1, \dots, t_r); \\ |t_i| < 1; i = 1, 2, \dots, r;$$

provided that each member of (10) exists.

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*Proof.* Substituting  $\Lambda_{n_1, \dots, n_r, q_1, \dots, q_r}^{\mu, \lambda_1, \dots, \lambda_r, \psi_1, \dots, \psi_r}(x; \xi_1, \xi_2, \dots, \xi_s; \zeta_1 \dots \zeta_r)$  into the left hand side of (10), we get

$$\begin{aligned} & \sum_{n_1=0}^{\infty} \dots \sum_{n_r=0}^{\infty} \Lambda_{n_1, \dots, n_r, q_1, \dots, q_r}^{\mu, \lambda_1, \dots, \lambda_r, \psi_1, \dots, \psi_r} \left( x; \xi_1, \xi_2, \dots, \xi_s; \frac{\nu_1}{t_1^{q_1}}, \dots, \frac{\nu_r}{t_r^{q_r}} \right) t_1^{n_1} \dots t_r^{n_r} \\ &= \sum_{n_1, \dots, n_r=0}^{\infty} \sum_{k_1=0}^{\left[ \frac{n_1}{q_1} \right]} \dots \sum_{k_r=0}^{\left[ \frac{n_r}{q_r} \right]} \frac{1}{(n_1 - q_1 k_1)! \dots (n_r - q_r k_r)!} a_{k_1, \dots, k_r} L_{n_1 - q_1 k_1, \dots, n_r - q_r k_r}^{(\alpha_1 + \lambda_1 k_1, \dots, \alpha_r + \lambda_r k_r; \beta)}(x) \\ & \quad \times \Phi_{\mu_1 + \psi_1 k_1, \dots, \mu_r + \psi_r k_r}(\xi_1, \xi_2, \dots, \xi_s) \left( \frac{\nu_1}{t_1^{q_1}} \right)^{k_1} \dots \left( \frac{\nu_r}{t_r^{q_r}} \right)^{k_r} t_1^{n_1} \dots t_r^{n_r}. \end{aligned}$$

Taking  $n_1 \rightarrow n_1 + q_1 k_1, \dots, n_r \rightarrow n_r + q_r k_r$ , we obtain

$$\begin{aligned} & \sum_{n_1=0}^{\infty} \dots \sum_{n_r=0}^{\infty} \Lambda_{n_1, \dots, n_r, q_1, \dots, q_r}^{\mu, \lambda_1, \dots, \lambda_r, \psi_1, \dots, \psi_r} \left( x; \xi_1, \xi_2, \dots, \xi_s; \frac{\nu_1}{t_1^{q_1}}, \dots, \frac{\nu_r}{t_r^{q_r}} \right) \frac{t_1^{n_1}}{n_1!} \dots \frac{t_r^{n_r}}{n_r!} \\ &= \sum_{n_1, \dots, n_r=0}^{\infty} \sum_{k_1=0}^{\infty} \dots \sum_{k_r=0}^{\infty} a_{k_1, \dots, k_r} L_{n_1, \dots, n_r}^{(\alpha_1 + \lambda_1 k_1, \dots, \alpha_r + \lambda_r k_r; \beta)}(x) \\ & \quad \cdot \Phi_{\mu_1 + \psi_1 k_1, \dots, \mu_r + \psi_r k_r}(\xi_1, \xi_2, \dots, \xi_s) \nu_1^{k_1} \dots \nu_r^{k_r} \frac{t_1^{n_1}}{n_1!} \dots \frac{t_r^{n_r}}{n_r!} \\ &= \sum_{k_1=0}^{\infty} \dots \sum_{k_r=0}^{\infty} \left[ \sum_{n_1, \dots, n_r=0}^{\infty} L_{n_1, \dots, n_r}^{(\alpha_1 + \lambda_1 k_1, \dots, \alpha_r + \lambda_r k_r; \beta)}(x) \frac{t_1^{n_1}}{n_1!} \dots \frac{t_r^{n_r}}{n_r!} \right] \\ & \quad \cdot a_{k_1, \dots, k_r} \Phi_{\mu_1 + \psi_1 k_1, \dots, \mu_r + \psi_r k_r}(\xi_1, \xi_2, \dots, \xi_s) \nu_1^{k_1} \dots \nu_r^{k_r} \\ &= \sum_{k_1=0}^{\infty} \dots \sum_{k_r=0}^{\infty} \left[ \prod_{i=1}^r \frac{1}{(1 - t_i)^{\alpha_i + \lambda_i k_i + 1}} \right] \exp \left( \beta x \left( \frac{1}{(1 - t_1) \dots (1 - t_r)} - 1 \right) \right) \\ & \quad \cdot a_{k_1, \dots, k_r} \Phi_{\mu_1 + \psi_1 k_1, \dots, \mu_r + \psi_r k_r}(\xi_1, \xi_2, \dots, \xi_s) \nu_1^{k_1} \dots \nu_r^{k_r} \\ &= \sum_{k_1=0}^{\infty} \dots \sum_{k_r=0}^{\infty} \left[ \prod_{i=1}^r \frac{1}{(1 - t_i)^{\lambda_i k_i}} \right] a_{k_1, \dots, k_r} \Phi_{\mu_1 + \psi_1 k_1, \dots, \mu_r + \psi_r k_r}(\xi_1, \xi_2, \dots, \xi_s) \nu_1^{k_1} \dots \nu_r^{k_r} \\ & \quad \cdot \left[ \prod_{i=1}^r \frac{1}{(1 - t_i)^{\alpha_i + 1}} \right] \exp \left( \beta x \left( \frac{1}{(1 - t_1) \dots (1 - t_r)} - 1 \right) \right) \\ &= \Delta_{\psi_1, \dots, \psi_r, \mu_1, \dots, \mu_r}^{(1)} \left( \xi_1, \xi_2, \dots, \xi_s; \frac{\nu_1}{(1 - t_1)^{\lambda_1}}, \dots, \frac{\nu_r}{(1 - t_r)^{\lambda_r}} \right) \\ & \quad \cdot \left[ \prod_{i=1}^r \frac{1}{(1 - t_i)^{\alpha_i + 1}} \right] \exp \left( \beta x \left( \frac{1}{(1 - t_1) \dots (1 - t_r)} - 1 \right) \right). \end{aligned}$$

Whence the result.  $\square$

In a similar manner, using (2) and (3), we led fairly easily to:

**Theorem 2.5.** Corresponding to an identically nonvanishing  $r$ -tuple function sequence  $\Phi_{\mu_1, \dots, \mu_r}(\xi_1, \xi_2, \dots, \xi_s)$  of  $s$  (real or complex) variables  $\xi_1, \xi_2, \dots, \xi_s$  ( $s \in$

$\mathbb{N} = \mathbb{N}_0 \setminus \{0\}$  and of (complex) orders, let

$$\begin{aligned} & \Delta_{\psi_1, \dots, \psi_r, \mu_1, \dots, \mu_r}^{(2)}(\xi_1, \xi_2, \dots, \xi_s; \tau_1, \dots, \tau_r) \\ &:= \sum_{k_1=0}^{\infty} \dots \sum_{k_r=0}^{\infty} a_{k_1, \dots, k_r} \Phi_{\mu_1 + \psi_1 k_1, \dots, \mu_r + \psi_r k_r}(\xi_1, \xi_2, \dots, \xi_s) \tau_1^{k_1} \dots \tau_r^{k_r} \\ & \quad a_{k_1, \dots, k_r} \neq 0. \end{aligned}$$

Suppose also that

$$\begin{aligned} & \Omega_{n_1, \dots, n_r, q_1, \dots, q_r}^{\mu, \lambda_1, \dots, \lambda_r, \psi_1, \dots, \psi_r}(x; \xi_1, \xi_2, \dots, \xi_s; \zeta_1 \dots \zeta_r) \\ &:= \sum_{k_1=0}^{\left[\frac{n_1}{q_1}\right]} \dots \sum_{k_r=0}^{\left[\frac{n_r}{q_r}\right]} \frac{1}{(n_1 - q_1 k_1)! \dots (n_r - q_r k_r)!} a_{k_1, \dots, k_r} \\ & \times L_{n_1 - q_1 k_1, \dots, n_r - q_r k_r}^{(\alpha; \beta_1 + \lambda_1 k_1, \dots, \beta_r + \lambda_r k_r)}(x) \Phi_{\mu_1 + \psi_1 k_1, \dots, \mu_r + \psi_r k_r}(\xi_1, \xi_2, \dots, \xi_s) \zeta_1^{k_1} \dots \zeta_r^{k_r} \\ & \quad (n \in \mathbb{N}) \end{aligned}$$

then

$$\begin{aligned} (11) \quad & \sum_{n_1=0}^{\infty} \dots \sum_{n_r=0}^{\infty} \Omega_{n_1, \dots, n_r, q_1, \dots, q_r}^{\mu, \lambda_1, \dots, \lambda_r, \psi_1, \dots, \psi_r} \left( x; \xi_1, \xi_2, \dots, \xi_s; \frac{\nu_1}{t_1^{q_1}}, \dots, \frac{\nu_r}{t_r^{q_r}} \right) t_1^{n_1} \dots t_r^{n_r} \\ &= \Delta_{\psi_1, \dots, \psi_r, \mu_1, \dots, \mu_r}^{(2)} \left( \xi_1, \xi_2, \dots, \xi_s; \nu_1 \exp\left(\frac{\lambda_1 t_1}{1 - t_1 - \dots - t_r}\right), \dots, \nu_r \exp\left(\frac{\lambda_r t_r}{1 - t_1 - \dots - t_r}\right) \right) \\ & \quad \times G_2(x; t_1, \dots, t_r); \\ & \quad |t_1 + t_2 + \dots + t_r| < 1, \end{aligned}$$

provided that each member of (11) exists.

**Theorem 2.6.** Corresponding to an identically nonvanishing  $r$ -tuple function sequence  $\Phi_{\mu_1, \dots, \mu_r}(\xi_1, \xi_2, \dots, \xi_s)$  of  $s$  (real or complex) variables  $\xi_1, \xi_2, \dots, \xi_s$  ( $s \in \mathbb{N} = \mathbb{N}_0 \setminus \{0\}$ ) and of (complex) orders, let

$$\begin{aligned} & \Delta_{\psi_1, \dots, \psi_r, \mu_1, \dots, \mu_r}^{(3)}(\xi_1, \xi_2, \dots, \xi_s; \tau_1, \dots, \tau_r) \\ &:= \sum_{k_1=0}^{\infty} \dots \sum_{k_r=0}^{\infty} a_{k_1, \dots, k_r} \Phi_{\mu_1 + \psi_1 k_1, \dots, \mu_r + \psi_r k_r}(\xi_1, \xi_2, \dots, \xi_s) \tau_1^{k_1} \dots \tau_r^{k_r} \\ & \quad a_{k_1, \dots, k_r} \neq 0. \end{aligned}$$

Suppose also that

$$\begin{aligned} & \Psi_{n_1, \dots, n_r, q_1, \dots, q_r}^{\mu, \lambda_1, \dots, \lambda_r, \psi_1, \dots, \psi_r}(x; \xi_1, \xi_2, \dots, \xi_s; \zeta_1^{k_1} \dots \zeta_r^{k_r}) = \sum_{k_1=0}^{\left[\frac{n_1}{q_1}\right]} \dots \sum_{k_r=0}^{\left[\frac{n_r}{q_r}\right]} \frac{1}{(n_1 - q_1 k_1)! \dots (n_r - q_r k_r)!} \\ & \times a_{k_1, \dots, k_r} H_{n_1 - q_1 k_1, \dots, n_r - q_r k_r}^{(\delta; \alpha_1 + \lambda_1 k_1, \dots, \alpha_r + \lambda_r k_r)}(x) \Phi_{\mu_1 + \psi_1 k_1, \dots, \mu_r + \psi_r k_r}(\xi_1, \xi_2, \dots, \xi_s) \zeta_1^{k_1} \dots \zeta_r^{k_r} \\ & \quad (n \in \mathbb{N}) \end{aligned}$$

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then

$$(12) \quad \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \Psi_{n_1, \dots, n_r, q_1, \dots, q_r}^{\mu, \lambda_1, \dots, \lambda_r, \psi_1, \dots, \psi_r} \left( x; \xi_1, \xi_2, \dots, \xi_s; \frac{\nu_1}{t_1^{q_1}}, \dots, \frac{\nu_r}{t_r^{q_r}} \right) t_1^{n_1} \cdots t_r^{n_r} \\ = \Delta_{\psi_1, \dots, \psi_r, \mu_1, \dots, \mu_r}^{(3)} (\xi_1, \xi_2, \dots, \xi_s; \nu_1 e^{\lambda_1 t_1}, \dots, \nu_r e^{\lambda_r t_r}) G_3(x; t_1, \dots, t_r).$$

provided that each member of (12) exists.

### 3. APPLICATIONS OF THE MAIN RESULTS

When the multivariable function  $\Phi_{\mu + \psi k}(\xi_1, \dots, \xi_s)$  ( $k \in \mathbb{N}_0, s \in \mathbb{N}$ ) (or  $\Phi_{\mu_1 + \psi_1 k_1, \dots, \mu_r + \psi_r k_r}(\xi_1, \xi_2, \dots, \xi_s)$ ) can be expressed by means of several simpler functions of one and more variables then one can give further applications of Theorems 2.1, 2.2 and 2.3 (of Theorems 2.4, 2.5 and 2.6). We start with the following illustrative example.

**Example 3.1.** Taking  $s = 2, \xi_1 = y, \xi_2 = z, \mu = 0, \psi = 1, \Phi_l(y, z) = g_l^{(\beta; \gamma)}(y, z), a_l = 1$  and  $\lambda_1 = \lambda_2 = \lambda_3 = 0$  in Theorem 2.2, we have

$$\Delta^{(2)}(y, z; \tau) = \frac{1}{(1 - y\tau)^{\beta} (1 - z\tau)^{\gamma}}$$

where  $g_l^{(\beta; \gamma)}(y, z)$  are the Lagrange polynomials (see [9], [5]). Hence, we obtain the following generating function between the multiple Laguerre II polynomials and the Lagrange polynomials

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{N_1}\right]} \sum_{l=0}^{\left[\frac{k}{N_2}\right]} \frac{g_l^{(\beta; \gamma)}(y, z) L_{n - N_2 k, k - N_1 l}^{(\alpha; \beta_1, \beta_2)}(x)}{(n - N_2 k)! (k - N_1 l)!} \tau^l t_1^{n - N_2 k} t_2^{k - N_1 l} \\ = \frac{1}{(1 - y\tau)^{\beta} (1 - z\tau)^{\gamma}} \frac{1}{(1 - t_1 - t_2)^{\alpha+1}} \exp \left( \frac{\beta_1 t_1 + \beta_2 t_2}{1 - t_1 - t_2} x \right); \\ |\tau| < \min \left\{ |x|^{-1}, |y|^{-1} \right\}, |t_1 + t_2| < 1.$$

It should be noted here that, one should use Theorem 2.2 in order to give bilinear, bilateral, mixed multilateral finite-series relationship and generating functions between the polynomials of which contain one summation symbol in its generating relation and the multiple Laguerre II polynomials.

On the other hand, one can not give any bilinear, bilateral, mixed multilateral finite-series relationship and generating functions between the polynomials of which contain more than one summation symbol in its generating relation and the multiple Laguerre II polynomials by using Theorem 2.2. But one can use Theorem 2.5 to achieve this problem.

**Example 3.2.** Choosing  $s = 1, \xi_1 = x, \mu_1 = \dots = \mu_r = 0, \psi_1 = \dots = \psi_r = 1, \Phi_{k_1, \dots, k_r}(x) = L_{n_1, \dots, n_r}^{(\alpha_1, \dots, \alpha_r; \beta)}(x)$  and

$$a_{k_1, \dots, k_r} = \frac{1}{k_1! \cdots k_r!},$$

we get from (1) that

$$\Delta_{1, \dots, 1; 0, \dots, 0}^{(1)}(x; \tau_1, \dots, \tau_r) = \left[ \prod_{i=1}^r \frac{1}{(1 - t_i)^{\alpha_i+1}} \right] \exp \left( \beta x \left( \frac{1}{(1 - t_1) \cdots (1 - t_r)} - 1 \right) \right),$$

where  $|t_i| < 1; i = 1, 2, \dots, r$ . Therefore, as a consequence of Theorem 2.5, we get following bilateral generating relation between multiple Laguerre I and II polynomials:

$$\begin{aligned} & \sum_{n_1, \dots, n_r=0}^{\infty} \sum_{k_1=0}^{\left[\frac{n_1}{q_1}\right]} \dots \sum_{k_r=0}^{\left[\frac{n_r}{q_r}\right]} \frac{L_{n_1-q_1k_1, \dots, n_r-q_rk_r}^{(\alpha; \beta_1+\lambda_1k_1, \dots, \beta_r+\lambda_rk_r)}(x) L_{k_1, \dots, k_r}^{(\alpha_1, \dots, \alpha_r; \beta)}(x)}{(n_1-q_1k_1)! \dots (n_r-q_rk_r)! k_1! \dots k_r!} \prod_{i=1}^r v_i^{k_i} t_i^{n_i-q_i k_i} \\ &= \frac{1}{(1-t_1-\dots-t_r)^{\alpha+1}} \exp\left(\frac{\beta_1 t_1 + \dots + \beta_r t_r}{1-t_1-\dots-t_r} x\right) \left[ \prod_{i=1}^r \frac{1}{\left(1 - \nu_i \exp\left(\frac{\lambda_i t_i}{1-t_1-\dots-t_r}\right)\right)^{\alpha_i+1}} \right] \\ & \quad \times \exp\left(\beta x \left( \frac{1}{\left(1 - \nu_1 \exp\left(\frac{\lambda_1 t_1}{1-t_1-\dots-t_r}\right)\right) \dots \left(1 - \nu_r \exp\left(\frac{\lambda_r t_r}{1-t_1-\dots-t_r}\right)\right)} - 1 \right)\right) \\ & \quad |t_1 + t_2 + \dots + t_r| < 1, \left( \left| \nu_i \exp\left(\frac{\lambda_i t_i}{1-t_1-\dots-t_r}\right) \right| < 1; i = 1, 2, \dots, r \right). \end{aligned}$$

Moreover, for each suitable choice of the coefficients  $a_k$  (or  $a_{k_1, \dots, k_r}$ ), if the multi-variable function  $\Phi_{\mu+\psi k}(\xi_1, \dots, \xi_s) (k \in \mathbb{N}_0, s \in \mathbb{N})$  (or  $\Phi_{\mu_1+\psi_1 k_1, \dots, \mu_r+\psi_r k_r}(\xi_1, \dots, \xi_s)$ ) is expressed as an appropriate product of several simpler functions, Theorems 2.1, 2.2 and 2.3 (or Theorems 2.4, 2.5 and 2.6) can be shown to yield various classes of mixed multilateral generating functions for the multiple Laguerre and multiple Hermite polynomials.

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# A viscosity iterative scheme for inverse-strongly accretive operators in Banach spaces\*

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## Abstract

In this paper is to introduce a new iterative scheme for finding solutions of a variational inequality for inverse-strongly accretive mappings with a viscosity approximation method in Banach spaces. We obtain a strong convergence theorem in Banach spaces under some parameters controlling conditions. Our results extend and improve the recent results of Yan Hao [Iterative algorithms for inverse-strongly accretive mappings with applications, *J. Appl. Math. Comput.*, 31, 193–202, 2009.], Yeol Je Cho et al., [Strong convergence of an iterative algorithm for accretive operators in Banach spaces, *J. Comput. Anal. and Appl.*, vol. 10, no. 1, 113–125, 2008.] and many others.

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## 1 Introduction

Let  $E$  be a real Banach space with norm  $\|\cdot\|$  and inner product  $\langle \cdot, \cdot \rangle$ ,  $C$  be a nonempty closed convex subset of  $E$  and  $A$  be a monotone operator of  $C$  into  $H$ . The *variational inequality problem*, denote by  $VI(C, A)$ , is to find  $x^* \in C$  such that

$$\langle Ax^*, x - x^* \rangle \geq 0,$$

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for all  $x \in C$ . In the case when  $C = H$ ,  $VI(H, A) = A^{-1}0$  holds, where

$$A^{-1}0 = \{x^* \in H : Ax^* = 0\}.$$

An element of  $A^{-1}0$  is called a *zero point* of  $A$ . Recall that a mapping  $A$  is said to be  $\beta$ -*inverse-strongly monotone*, if there exists a positive real number  $\beta > 0$  such that

$$\langle Ax - Ay, x - y \rangle \geq \beta \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Let  $C$  be a nonempty closed and convex subset of a Banach space  $E$ . An operator  $A$  of  $C$  into  $E$  is said to be *accretive* if there exists  $j(x - y) \in J(x - y)$  such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0$$

for all  $x, y \in C$ . An operator  $A$  of  $C$  into  $E$  is said to be  $\beta$ -*inverse strongly accretive* if, for any  $\beta > 0$ ,

$$\langle Ax - Ay, J(x - y) \rangle \geq \beta \|Ax - Ay\|^2$$

for all  $x, y \in C$ . Evidently, the definition of the inverse strongly accretive operator is based on that of the inverse strongly monotone operator. Recall also that self mapping  $f : C \rightarrow C$  is a contraction on  $C$  if there exists a constant  $\alpha \in (0, 1)$  and  $x, y \in C$  such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|.$$

An interesting problem to extend the above results to find a solution of the variational inequality for an inverse-strongly accretive mappings in Banach spaces. Aoyama et al. [1] introduced the following iteration scheme for an inverse-strongly accretive operator in Banach spaces  $E$ :

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n), \end{cases} \quad (1.1)$$

for all  $n \geq 1$  where  $C \subset E$  and  $Q_C$  is a sunny nonexpansive retraction from  $E$  onto  $C$ . They proved a weak convergence theorem in Banach spaces. Hao [4] introduced the following iteration scheme for an inverse-strongly accretive operator in Banach spaces  $E$ :

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n), \end{cases} \quad (1.2)$$

for all  $n \geq 1$  where  $C \subset E$  and  $Q_C$  is a sunny nonexpansive retraction from  $E$  onto  $C$ . They proved a strong convergence theorem in Banach spaces. Cho et al. [3] introduced the following iteration scheme for an inverse-strongly accretive operator in Banach spaces  $E$  for any fixed  $u \in C$ :

$$\begin{cases} x_0 = u \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(x_n - \lambda_n A x_n), \end{cases} \quad (1.3)$$

for all  $n \geq 1$  where  $C \subset E$  and  $Q_C$  is a sunny nonexpansive retraction from  $E$  onto  $C$ . They proved a strong convergence theorem in Banach spaces.

In this paper, motivated and inspired by the idea of Yan Hao [4] and Yeol Je Cho et al. [3], we will introduce a viscosity iterative scheme for an inverse-strongly accretive operator in Banach spaces as follows:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Q_C(x_n - \lambda_n A x_n), \quad (1.4)$$

we shall prove a strong convergent theorem under some parameters controlling conditions.

## 2 Preliminaries

Let  $D$  be a subset of  $C$  and  $Q : C \rightarrow D$ . Then  $Q$  is said to *sunny* if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever  $Qx + t(x - Qx) \in C$  for  $x \in C$  and  $t \geq 0$ . A subset  $D$  of  $C$  is said to be a *sunny nonexpansive retract* of  $C$  if there exists a sunny nonexpansive retraction  $Q$  of  $C$  onto  $D$ . A mapping  $Q : C \rightarrow C$  is called a *retraction* if  $Q^2 = Q$ . If a mapping  $Q : C \rightarrow C$  is a retraction, then  $Qz = z$  for all  $z$  is in the range of  $Q$ .

The following result describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

**Proposition 1**([6]) *Let  $E$  be a smooth Banach space and let  $C$  be a nonempty subset of  $E$ . Let  $Q : E \rightarrow C$  be a retraction and let  $J$  be the normalized duality mapping on  $E$ . Then the following are equivalent:*

- (i)  $Q$  is sunny and nonexpansive;
- (ii)  $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy) \rangle, \forall x, y \in E$ ;
- (iii)  $\langle x - Qx, J(y - Qx) \rangle \leq 0, \forall x \in E, y \in C$ .

**Proposition 2**([5]) *Let  $C$  be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space  $E$  and let  $T$  be a nonexpansive mapping of  $C$  into itself with  $F(T) \neq \emptyset$ . Then the set  $F(T)$  is a sunny nonexpansive retract of  $C$ .*

Let  $E$  be a Banach space and let  $E^*$  be the dual space of  $E$  and  $\langle \cdot, \cdot \rangle$  denote the pairing between  $E$  and  $E^*$ . For  $q > 1$ , the *generalized duality mapping*  $J_q : E \rightarrow 2^{E^*}$  is defined by

$$J_q(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}$$

for all  $x \in E$ . In particular, if  $q = 2$ , the mapping  $J_2$  is called the *normalized duality mapping* and, usually, write  $J_2 = J$ . Further, we have the following properties of the generalized duality mapping  $J_q$ :

- (i)  $J_q(x) = \|x\|^{q-2} J_2(x)$  for all  $x \in E$  with  $x \neq 0$ ;
- (ii)  $J_q(tx) = t^{q-1} J_q(x)$  for all  $x \in E$  and  $t \in [0, \infty)$ ;
- (iii)  $J_q(-x) = -J_q(x)$  for all  $x \in E$ .

Let  $U = \{x \in E : \|x\| = 1\}$ . A Banach space  $E$  is said to *uniformly convex* if, for any  $\epsilon \in (0, 2]$ , there exists  $\delta > 0$  such that, for any  $x, y \in U$ ,  $\|x - y\| \geq \epsilon$  implies  $\|\frac{x+y}{2}\| \leq 1 - \delta$ .

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space  $E$  is said to be *smooth* if the limit  $\lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t}$  exists for all  $x, y \in U$ . It is also said to be *uniformly smooth* if the limit is attained uniformly for  $x, y \in U$ . The *modulus of smoothness* of  $E$  is defined by

$$\rho(\tau) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\right\},$$

where  $\rho : [0, \infty) \rightarrow [0, \infty)$  is a function. It is known that  $E$  is uniformly smooth if and only if  $\lim_{\tau \rightarrow 0} \frac{\rho(\tau)}{\tau} = 0$ . Let  $q$  be a fixed real number with  $1 < q \leq 2$ . A Banach space  $E$  is said to be  *$q$ -uniformly smooth* if there exists a constant  $c > 0$  such that  $\rho(\tau) \leq c\tau^q$  for all  $\tau > 0$ .

Note that typical examples of both uniformly convex and uniformly smooth Banach spaces are  $L^p$ , where  $p > 1$ . More precisely,  $L^p$  is  $\min\{p, 2\}$ -uniformly smooth for every  $p > 1$ . Note also that no Banach space is  $q$ -uniformly smooth for  $q > 2$ ; see [8] for more details.

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We need the following lemmas for proving our main results.

**Lemma 2.1.** ([8]) *Let  $E$  be a real 2-uniformly smooth Banach space with the best smooth constant  $K$ . Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx \rangle + 2\|Ky\|^2, \quad \forall x, y \in E.$$

**Lemma 2.2.** ([7]) *Let  $\{x_n\}$  and  $\{y_n\}$  be bounded sequences in a Banach space  $X$  and let  $\{\beta_n\}$  be a sequence in  $[0, 1]$  with  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ . Suppose  $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$  for all integers  $n \geq 0$  and  $\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$ . Then,  $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$ .*

**Lemma 2.3.** ([9]) *Assume  $\{a_n\}$  is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 0$$

*where  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  and  $\{\delta_n\}$  is a sequence in  $\mathbb{R}$  such that*

$$(1) \sum_{n=1}^{\infty} \alpha_n = \infty$$

$$(2) \limsup_{n \rightarrow \infty} \frac{\delta_n}{\alpha_n} \leq 0 \text{ or } \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

*Then  $\lim_{n \rightarrow \infty} a_n = 0$ .*

The following Lemma is characterized by the set of solutions of variational inequality by using sunny nonexpansive retractions.

**Lemma 2.4.** ([1]) *Let  $C$  be a nonempty closed convex subset of a smooth Banach space  $E$ . Let  $Q_C$  be a sunny nonexpansive retraction from  $E$  onto  $C$  and let  $A$  be an accretive operator of  $C$  into  $E$ . Then, for all  $\lambda > 0$ ,*

$$VI(C, A) = F(Q(I - \lambda A)).$$

**Lemma 2.5.** ([2]) *Let  $E$  be a uniformly convex Banach space,  $C$  a nonempty closed convex subset of  $E$  and  $T : C \rightarrow C$  a nonexpansive mapping. Then  $I - T$  is demi-closed at zero.*

### 3 Main results

In this section, we prove a strong convergence theorem.

**Theorem 3.1.** *Let  $E$  be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant  $K$  and  $C$  a nonempty closed convex subset of  $E$ . Let  $Q_C$  be a sunny nonexpansive retraction from  $E$  onto  $C$  and  $A : C \rightarrow E$  be an  $\beta$ -inverse-strongly accretive operator with  $VI(C, A) \neq \emptyset$ . Let  $f$  be a contraction of  $C$  into itself with coefficient  $\alpha \in (0, 1)$ . Suppose the sequences  $\{\alpha_n\}, \{\beta_n\}$  and  $\{\gamma_n\}$  in  $(0, 1)$  satisfy  $\alpha_n + \beta_n + \gamma_n = 1$ ,  $n \geq 1$  and  $\{\lambda_n\}$  is a sequence in  $[a, b]$  for some  $a, b$  with  $0 < a < b < \frac{\beta}{K^2}$ . The following conditions are satisfied:*

(i).  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;

(ii).  $\lim_{n \rightarrow \infty} (\lambda_{n+1} - \lambda_n) = 0$ ;

(iii).  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ .

*For arbitrary given  $x_1 \in C$ , the sequences  $\{x_n\}$  generated by (1.4). Then  $\{x_n\}$  converges strongly to  $Q_{VI(C, A)}x$ .*

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**Proof.** First, we observe that  $I - \lambda_n A$  is nonexpansive. Let  $x, y \in C$ , from the assumption  $\lambda_n \in [a, b]$  and Lemma 2.1, we have

$$\begin{aligned}
 \|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 &= \|(x - y) - \lambda_n(Ax - Ay)\|^2 \\
 &\leq \|x - y\|^2 - 2\lambda_n \langle Ax - Ay, J(x - y) \rangle + 2K^2 \lambda_n^2 \|Ax - Ay\|^2 \\
 &\leq \|x - y\|^2 - 2\lambda_n \beta \|Ax - Ay\|^2 + 2K^2 \lambda_n^2 \|Ax - Ay\|^2 \\
 &= \|x - y\|^2 + 2\lambda_n (\lambda_n K^2 - \beta) \|Ax - Ay\|^2 \\
 &\leq \|x - y\|^2.
 \end{aligned}$$

Next, we prove that  $\{x_n\}$  bounded. Let  $p \in VI(C, A)$ , from Lemma 2.4, we see that  $p = Q_C(p - \lambda_n Ap)$ , for each  $n \geq 1$ . Put  $y_n = Q_C(x_n - \lambda_n Ax_n)$ , we have

$$\begin{aligned}
 \|y_n - p\| &= \|Q_C(x_n - \lambda_n Ax_n) - Q_C(p - \lambda_n Ap)\| \\
 &\leq \|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap)\| \\
 &= \|(I - \lambda_n A)x_n - (I - \lambda_n A)p\| \\
 &\leq \|x_n - p\|.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \|x_{n+1} - p\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - p\| \\
 &\leq \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|y_n - p\| \\
 &\leq \alpha \alpha_n \|x_n - p\| + \alpha_n \|f(p) - p\| + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| \\
 &\leq (1 - \alpha_n + \alpha \alpha_n) \|x_n - p\| + \alpha_n \|f(p) - p\| \\
 &= (1 - \alpha_n(1 - \alpha)) \|x_n - p\| + \alpha_n(1 - \alpha) \frac{\|f(p) - p\|}{1 - \alpha} \\
 &\leq \max\{\|x_1 - p\|, \frac{\|f(p) - p\|}{1 - \alpha}\}.
 \end{aligned}$$

This implies that  $\{x_n\}$  bounded, so are  $\{f(x_n)\}$  and  $\{y_n\}$ .

Next, we show that  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ . Notice that

$$\begin{aligned}
 \|y_{n+1} - y_n\| &= \|Q_C(x_{n+1} - \lambda_{n+1} Ax_{n+1}) - Q_C(x_n - \lambda_n Ax_n)\| \\
 &\leq \|(x_{n+1} - \lambda_{n+1} Ax_{n+1}) - (x_n - \lambda_n Ax_n)\| \\
 &= \|(x_{n+1} - \lambda_{n+1} Ax_{n+1}) - (x_n - \lambda_{n+1} Ax_n) + (\lambda_n - \lambda_{n+1}) Ax_n\| \\
 &\leq \|(I - \lambda_{n+1} A)x_{n+1} - (I - \lambda_{n+1} A)x_n\| + |\lambda_n - \lambda_{n+1}| \|Ax_n\| \\
 &\leq \|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}| \|Ax_n\|,
 \end{aligned}$$

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Setting  $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$ , we see that  $z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}$ , then we have

$$\begin{aligned}
 \|z_{n+1} - z_n\| &= \left\| \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \right\| \\
 &= \left\| \frac{\alpha_{n+1}f(x_{n+1}) + \beta_{n+1}x_{n+1} + \gamma_{n+1}y_{n+1} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - \beta_n x_n}{1 - \beta_n} \right\| \\
 &= \left\| \frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}y_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_{n+1}f(x_n)}{1 - \beta_{n+1}} + \frac{\alpha_{n+1}f(x_n)}{1 - \beta_{n+1}} - \frac{\gamma_{n+1}y_n}{1 - \beta_{n+1}} + \frac{\gamma_{n+1}y_n}{1 - \beta_{n+1}} \right. \\
 &\quad \left. - \frac{\alpha_n f(x_n) + \gamma_n y_n}{1 - \beta_n} \right\| \\
 &= \left\| \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(f(x_{n+1}) - f(x_n)) + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(y_{n+1} - y_n) + \left(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\right)f(x_n) \right. \\
 &\quad \left. + \left(\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}\right)y_n \right\| \\
 &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|f(x_{n+1}) - f(x_n)\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}\|y_{n+1} - y_n\| + \left|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\right|\|f(x_n)\| \\
 &\quad + \left|\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}\right|\|y_n\| \\
 &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|x_{n+1} - x_n\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}\|y_{n+1} - y_n\| + \left|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\right|(\|f(x_n)\| + \|y_n\|) \\
 &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|x_{n+1} - x_n\| + \left|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\right|(\|f(x_n)\| + \|y_n\|) + \|y_{n+1} - y_n\| \\
 &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|x_{n+1} - x_n\| + \left|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\right|(\|f(x_n)\| + \|y_n\|) \\
 &\quad + \|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}|\|Ax_n\|.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|x_{n+1} - x_n\| + \left|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\right|(\|f(x_n)\| + \|y_n\|) \\
 &\quad + |\lambda_n - \lambda_{n+1}|\|Ax_n\|.
 \end{aligned}$$

It follow from the condition (i), (ii) and (iii), which implies that

$$\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Applying Lemma 2.2, we obtain  $\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0$  and also

$$\|x_{n+1} - x_n\| = (1 - \beta_n)\|z_n - x_n\| \rightarrow 0$$

as  $n \rightarrow \infty$ . Therefore, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (3.1)$$

On the other hand, we have

$$\begin{aligned}
 \|x_n - y_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\
 &= \|x_n - x_{n+1}\| + \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - y_n\| \\
 &= \|x_n - x_{n+1}\| + \|\alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)y_n - y_n\| \\
 &= \|x_n - x_{n+1}\| + \|\alpha_n(f(x_n) - y_n) + \beta_n(x_n - y_n)\| \\
 &\leq \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - y_n\| + \beta_n\|x_n - y_n\|.
 \end{aligned}$$

It follows that

$$(1 - \beta_n)\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - y_n\|,$$

and hence

$$\|x_n - y_n\| \leq \frac{1}{(1 - \beta_n)}\|x_n - x_{n+1}\| + \frac{\alpha_n}{(1 - \beta_n)}\|f(x_n) - y_n\|.$$

From the condition (i), (iii) and (3.1), then we obtain that

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \quad (3.2)$$

Next, we show that  $\limsup_{n \rightarrow \infty} \langle (f - I)z, J(x_n - z) \rangle \leq 0$ , where  $z = Q_{VI(C,A)}x$ ,  $VI(C,A)$  is a sunny nonexpansive retraction of  $C$  onto  $VI(C,A)$ . We can choose a sequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that

$$\limsup_{n \rightarrow \infty} \langle (f - I)z, J(x_n - z) \rangle = \lim_{k \rightarrow \infty} \langle (f - I)z, J(x_{n_k} - z) \rangle. \quad (3.3)$$

Since  $\{x_{n_k}\}$  is bounded, there exists a subsequence  $\{x_{n_{k_j}}\}$  of  $\{x_{n_k}\}$  which converges weakly to  $p$ . Without loss of generality, we can assume that  $x_{n_k} \rightharpoonup p$ . Next, we show that  $p \in VI(C,A)$ . From the assumption, we see that control sequence  $\{\lambda_{n_k}\}$  is bounded. So, there exists a subsequence  $\{\lambda_{n_{k_j}}\}$  converges to  $\lambda_0$ . We may assume, without loss of generality, that  $\lambda_{n_k} \rightarrow \lambda_0$ . Observe that

$$\begin{aligned} \|Q_C(x_{n_k} - \lambda_0 A x_{n_k}) - x_{n_k}\| &\leq \|Q_C(x_{n_k} - \lambda_0 A x_{n_k}) - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \\ &\leq \|(x_{n_k} - \lambda_0 A x_{n_k}) - (x_{n_k} - \lambda_{n_k} A x_{n_k})\| + \|y_{n_k} - x_{n_k}\| \\ &\leq M\|\lambda_{n_k} - \lambda_0\| + \|y_{n_k} - x_{n_k}\|, \end{aligned}$$

where  $M$  is as appropriate constant such that  $M \geq \sup_{n \geq 1} \{\|A x_n\|\}$ . It follows from (3.2) that

$$\lim_{k \rightarrow \infty} \|Q_C(x_{n_k} - \lambda_0 A x_{n_k}) - x_{n_k}\| = 0.$$

On the other hand, we know that  $Q_C(I - \lambda_0 A)$  is nonexpansive. Indeed, for  $x, y \in C$ , from Lemma 2.1, we see that

$$\begin{aligned} \|Q_C(I - \lambda_0 A)x - Q_C(I - \lambda_0 A)y\|^2 &\leq \|(I - \lambda_0 A)x - (I - \lambda_0 A)y\|^2 \\ &= \|(x - y) - \lambda_0(Ax - Ay)\|^2 \\ &\leq \|x - y\|^2 - 2\lambda_0 \langle Ax - Ay, J(x - y) \rangle + 2K^2\lambda_0^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\lambda_0\beta\|Ax - Ay\|^2 + 2K^2\lambda_0^2\|Ax - Ay\|^2 \\ &= \|x - y\|^2 + 2\lambda_0(\lambda_0 K^2 - \beta)\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

It follows from Lemma 2.5 that  $p \in F(Q_C(I - \lambda_0 A))$ . By using Lemma 2.4, we can obtain that  $p \in F(Q_C(I - \lambda_0 A)) = VI(C,A)$ . From (3.3), we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle (f - I)z, J(x_n - z) \rangle &= \lim_{k \rightarrow \infty} \langle (f - I)z, J(x_{n_k} - z) \rangle \\ &= \lim_{k \rightarrow \infty} \langle (f - I)z, J(p - z) \rangle \leq 0 \end{aligned} \quad (3.4)$$



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Finally, we show that  $\{x_n\}$  converges strongly to  $z = Q_{VI(C,A)}x$ . Observe that

$$\begin{aligned}
 \|x_{n+1} - z\|^2 &= \langle x_{n+1} - z, J(x_{n+1} - z) \rangle \\
 &= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - z, J(x_{n+1} - z) \rangle \\
 &= \langle \alpha_n (f(x_n) - z) + \beta_n (x_n - z) + \gamma_n (y_n - z), J(x_{n+1} - z) \rangle \\
 &= \alpha_n \langle f(x_n) - f(z), J(x_{n+1} - z) \rangle + \alpha_n \langle f(z) - z, J(x_{n+1} - z) \rangle + \beta_n \langle x_n - z, J(x_{n+1} - z) \rangle \\
 &\quad + \gamma_n \langle y_n - z, J(x_{n+1} - z) \rangle \\
 &\leq \alpha_n \alpha_n \|x_n - z\| \|x_{n+1} - z\| + \alpha_n \langle f(z) - z, J(x_{n+1} - z) \rangle + \beta_n \|x_n - z\| \|x_{n+1} - z\| \\
 &\quad + \gamma_n \|x_n - z\| \|x_{n+1} - z\| \\
 &= \frac{\alpha_n \alpha_n + \beta_n + \gamma_n}{2} (\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n \langle f(z) - z, J(x_{n+1} - z) \rangle \\
 &= \frac{\alpha_n \alpha_n + 1 - \alpha_n}{2} (\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n \langle f(z) - z, J(x_{n+1} - z) \rangle \\
 &= \frac{1 - \alpha_n(1 - \alpha)}{2} (\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n \langle f(z) - z, J(x_{n+1} - z) \rangle \\
 &\leq \frac{1 - \alpha_n(1 - \alpha)}{2} \|x_n - z\|^2 + \frac{1}{2} \|x_{n+1} - z\|^2 + \alpha_n \langle f(z) - z, J(x_{n+1} - z) \rangle,
 \end{aligned}$$

which implies that

$$\|x_{n+1} - z\|^2 \leq (1 - \alpha_n(1 - \alpha))\|x_n - z\|^2 + 2\alpha_n \langle f(z) - z, J(x_{n+1} - z) \rangle. \quad (3.5)$$

Now, from (i), (3.4) and applying Lemma 2.3 to (3.5), we get  $\|x_n - z\| \rightarrow 0$  as  $n \rightarrow \infty$ . This completes the proof.  $\square$

**Corollary 3.2.** [3, Theorem 3.1,] Let  $E$  be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant  $K$  and  $C$  a nonempty closed convex subset of  $E$ . Let  $Q_C$  be a sunny nonexpansive retraction from  $E$  onto  $C$  and  $A : C \rightarrow E$  be an  $\beta$ -inverse-strongly accretive mapping with  $VI(C, A) \neq \emptyset$ . Suppose the sequences  $\{\alpha_n\}, \{\beta_n\}$  and  $\{\gamma_n\}$  in  $(0, 1)$  satisfy  $\alpha_n + \beta_n + \gamma_n = 1$ ,  $n \geq 1$  and  $\{\lambda_n\}$  is a sequence in  $[a, b]$  for some  $a, b$  with  $0 < a < b < \frac{\beta}{K^2}$ . The following conditions are satisfied:

- (i).  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;
- (ii).  $\lim_{n \rightarrow \infty} (\lambda_{n+1} - \lambda_n) = 0$ ;
- (iii).  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ .

For arbitrary given  $x_1 \in C$ , the sequences  $\{x_n\}$  generated by (1.3). Then  $\{x_n\}$  converges strongly to  $Q_{VI(C,A)}x$ .

**Proof.** Taking  $f(x) = x_1 := u$  for all  $x \in C$  in (1.4), we can conclude the desired conclusion easily. This completes the proof.  $\square$

**Corollary 3.3.** [4, Theorem 3.1,] Let  $E$  be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant  $K$  and  $C$  a nonempty closed convex subset of  $E$ . Let  $Q_C$  be a sunny nonexpansive retraction from  $E$  onto  $C$  and  $A : C \rightarrow E$  be an  $\beta$ -inverse-strongly accretive mapping with  $VI(C, A) \neq \emptyset$ . Suppose the sequences  $\{\alpha_n\}, \{\beta_n\}$  and  $\{\gamma_n\}$  in  $(0, 1)$  satisfy  $\alpha_n + \beta_n + \gamma_n = 1$ ,  $n \geq 1$  and  $\{\lambda_n\}$  is a sequence in  $[a, b]$  for some  $a, b$  with  $0 < a < b < \frac{\beta}{K^2}$ . The following conditions are satisfied:

- (i).  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;
- (ii).  $\lim_{n \rightarrow \infty} (\lambda_{n+1} - \lambda_n) = 0$ ;
- (iii).  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ .

For arbitrary given  $x_1 \in C$ , the sequences  $\{x_n\}$  generated by (1.2). Then  $\{x_n\}$  converges strongly to  $Q_{VI(C,A)}x$ .

**Proof.** Taking  $f(x) = x_1$  for all  $x \in C$  and  $\beta_n = 0$ , for all  $n \in \mathbb{N}$ , in (1.4), we can conclude the desired conclusion easily. This completes the proof.  $\square$

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# On Boundedness of a Certain Non-linear Differential Equation of Third Order

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## Abstract

This paper presents sufficient conditions for all solutions as well as their derivatives up to second order to  $x''' + a(t)f(x')x'' + b(t, x, x') + c(t)g(x) = e(t, x, x', x'')$  to be bounded. What is more, it shows that all the solutions are in  $L^p[0, \infty)$  under somewhat more restrictive conditions.

## 1 Introduction

In this paper we consider the following non-linear third order differential equation of the form

$$x''' + a(t)f(x')x'' + b(t, x, x') + c(t)g(x) = e(t, x, x', x'') \quad (1)$$

where the functions  $a$ ,  $f$ ,  $b$ ,  $c$ ,  $g$  and  $e$  are continuous, besides, the functions  $a$ ,  $f$  and  $c$  are differentiable related to the given arguments.

We discuss the boundedness of solutions of Eq.(1) and whether the solutions are also in  $L^p[0, \infty)$  with sufficient conditions. Incidentally, by  $L^p$ -solutions we mean that  $\int_0^\infty |x(t)|^p dt < \infty$ .

So far, the boundedness of solutions of various second and third order non-linear ordinary differential equations have been discussed by many authors in the literature. The readers can refer to the book of Reissig[20] as a survey, the papers and the books of [1-33] and the references therein. By the way, let us say that Liapunov's second method has been used for the investigation of the boundedness of solutions of non-linear differential equations effectively and it is still being used. But, some authors obtained their results about the boundedness of solutions of the mentioned various order differential equations via non-Liapunov sense, which is not

usual (see [6],[9],[13-16],[18],[31] and [32]). Specifically, Kroopnick [13-16] put many valuable studies in second order in this sense. To the best of our knowledge, in 1972, Kroopnick began to investigate the properties of solutions of the second order differential equations in the cited [13] by standart method, integral test. Later, the author has got some results about the boundedness of solutions of different types of the second order differential equations (see [14],[15] and [16]) in the same manner which looks like simple and usefel but rarely has been encountered in third order, see Ogundare[19] including the equation of the form

$$x''' + a(t)f(x')x'' + b(t)g(x)x' + c(t, x) = e(t),$$

where  $e(t)$  is square integrable.

Here, the mentioned studies of Kroopnick inspire us to be able to acquire effective results in our work for the third order without considering Liapunov method.

Therefore, throughout this paper presenting three theorems in sufficient conditions, first we will show the boundedness of all solutions of Eq.(1) as well as their derivatives up to  $2^{nd}$  order and that the derivatives are elements of  $L^2[0, \infty)$  in first theorem. Next, the same results will be obtained again in second theorem with small but radical changes in assumptions of the first theorem. Lastly, under more restrictive conditions, that all the solutions to Eq.(1) are in  $L^p[0, \infty)$  will be shown.

## 2 Main Results

**Theorem 1** Assume that

- i)  $b(t, x, y)y \geq 0$ ,  $b(t, x, y)z \geq 0$ ,  $|e(t, x, y, z)| \leq q(t)$  for all  $t \geq 0$ ,  $x$ ,  $y$ ,  $z$  where  $\int_0^t |q(s)| ds < \infty$  and  $x = x(t)$ ,  $y = x'(t)$ ,  $z = x''(t)$ ,
- ii)  $G(x) = \int_0^x g(s)ds \rightarrow \infty$  as  $|x| \rightarrow \infty$ ,
- iii)  $c(t) \geq c_0 > 0$ ,  $c'(t) \leq 0$ ,  $a(t) > a_0 > 0$ ,  $f(y) > f_0 > 0$ ,  $g(x) > g_0 > 0$ ,  $y(t) \geq \int y^2(s)ds$ , where  $c_0$ ,  $a_0$ ,  $f_0$  and  $g_0$  are some constants.

Then all solutions of Eq.(1) are bounded as well as their derivatives up to  $2^{nd}$  order and the derivatives are elements of  $L^2$ -solutions.

**Proof.** First, by standart existence theory, it is obvious that there exist at least one solution, which is local on  $[0, t)$ . But global existence of all solutions may be mentioned in the case we can show the solutions are bounded.

Upon multiplying Eq.(1) by  $x''(t)$  and integrating from 0 to  $t$ , we have

$$\begin{aligned} \int_0^t x'''(s)x''(s)ds &+ \int_0^t a(s)f(x'(s))x''(s)^2ds \\ &+ \int_0^t b(s, x(s), x'(s))x''(s)ds + \int_0^t c(s)g(x(s))x''(s)ds \\ &= \int_0^t e(s, x(s), x'(s), x''(s))x''(s)ds. \end{aligned}$$

Using the assumptions in Theorem 1, it follows that

$$\begin{aligned} \frac{x''(t)^2}{2} + a_0 f_0 \int_0^t x''(s)^2 ds + \int_0^t b(s, x(s), x'(s))x''(s)ds + c_0 g_0 x'(t) \\ \leq \int_0^t |q(s)| |x''(s)| ds + \frac{x''(0)^2}{2}. \end{aligned} \quad (2)$$

Applying the mean value theorem for integral to the first term on the RHS of (2), we have

$$\begin{aligned} \frac{x''(t)^2}{2} + a_0 f_0 \int_0^t x''(s)^2 ds + \int_0^t b(s, x(s), x'(s))x''(s)ds + c_0 g_0 x'(t) \\ \leq |x''(t^*)| \int_0^t |q(s)| ds + \frac{x''(0)^2}{2}, \end{aligned}$$

where  $0 < t^* < t$ .

If  $|x''(t)|$  becomes unbounded, LHS approaches  $\infty$  faster than RHS, which is impossible. So,  $|x''(t)|$  must stay bounded. Evidently,  $|x'(t)|$  also must remain bounded. And again from the assumptions of Theorem 1, the term  $c_0 g_0 x'(t)$  on LHS becomes  $c_0 g_0 \int x'(s)^2 ds$  since  $c_0 g_0 \int x'(s)^2 ds \leq c_0 g_0 x'(t)$ .

Since the terms  $a_0 f_0 \int_0^t x''(s)^2 ds$  and  $c_0 g_0 \int x'(s)^2 ds$  are bounded by RHS,  $x'(t)$  and  $x''(t)$  are obtained as square integrable.

Now, let us multiply Eq.(1) by  $x'(t)$  and integrate from 0 to  $t$ , then we have

$$\begin{aligned} \int_0^t x'''(s)x'(s)ds + \int_0^t a(s)f(x'(s))x''(s)x'(s)ds \\ + \int_0^t b(s, x(s), x'(s))x'(s)ds + \int_0^t c(s)g(x(s))x'(s)ds \\ = \int_0^t e(s, x(s), x'(s), x''(s))x'(s)ds. \end{aligned} \quad (3)$$

And applying the integrating by parts to the first and fourth terms on the LHS of (3), we see that

$$\begin{aligned}
 x'(t)x''(t) &+ \int_0^t a(s)f(x'(s))x''(s)x'(s)ds + \int_0^t b(s, x(s), x'(s))x'(s)ds \\
 &+ c(t)G(x(t)) - \int_0^t c'(s)G(x(s))ds \\
 &\leq \int_0^t |q(s)| |x'(s)| ds + \int_0^t x''(s)^2 ds \\
 &+ |x'(0)x''(0)| + |c(0)G(x(0))|.
 \end{aligned} \tag{4}$$

Applying the mean value theorem for integral to the first term on the RHS of (4)

$$\begin{aligned}
 x'(t)x''(t) &+ \int_0^t a(s)f(x'(s))x''(s)x'(s)ds \\
 &+ \int_0^t b(s, x(s), x'(s))x'(s)ds + c(t)G(x(t)) - \int_0^t c'(s)G(x(s))ds \\
 &\leq |x'(t^*)| \int_0^t |q(s)| ds + \int_0^t x''(s)^2 ds \\
 &+ |x'(0)x''(0)| + |c(0)G(x(0))|,
 \end{aligned} \tag{5}$$

where  $0 < t^* < t$ .

Since  $|x'(t)|$  are also bounded and  $x''(t)$  is square integrable, the RHS of (5) is finite. Therefore,  $|x(t)|$  must also remain bounded. This completes the proof of Theorem 1. ■

**Theorem 2** *The hypotheses are the same as Theorem 1 except that  $c'(t) \geq 0$ . In addition,  $a'(t) \leq 0$ . Then the same results in Theorem 1 are valid.*

**Proof.** First of all, change of the conditions in Theorem 2 makes no difference for the boundedness and  $L^2$ -solutions of  $x'(t)$  and  $x''(t)$ . But, the boundedness of  $x(t)$  may require some more analysis.

Letting  $F(y) = \int_0^y uf(u)du$ , where  $y = x'(t)$ , we have another integration for the second term on LHS of (5) and taking the RHS of (5) as  $K$ , we have

$$\begin{aligned}
 x'(t)x''(t) &+ a(t)F(y(t)) - \int_0^t a'(s)F(y(s))ds + \int_0^t b(s, x(s), x'(s))x'(s)ds \\
 &+ c(t)G(x(t)) - \int_0^t c'(s)G(x(s))ds \\
 &\leq |K| + a(0)F(y_0).
 \end{aligned} \tag{6}$$

Specifically, the inequality (6) gives that

$$c(t)G(x(t)) \leq \alpha + \int_0^t c'(s)G(x(s))ds$$

where  $\alpha = |K| + M^2 + 3a(0) \sup_{-M \leq y \leq M} F(y)$  and  $M$  is the bound of  $|x'(t)|$  and  $|x''(t)|$  on  $[0, \infty)$ . Then we can write

$$c(t)G(x(t)) \leq \alpha + \int_0^t \frac{c'(s)}{c(s)} c(s)G(x(s))ds. \quad (7)$$

By Gronwall-Reid-Bellman inequality, (7) leads

$$\begin{aligned} c(t)G(x(t)) &\leq \alpha \left( \exp \int_0^t \frac{c'(s)}{c(s)} ds \right) \\ c(t)G(x(t)) &\leq \alpha \frac{c(t)}{c(0)} \\ G(x(t)) &\leq \frac{\alpha}{c(0)}. \end{aligned}$$

Since  $G(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$ ,  $x(t)$  must stay finite on  $[0, \infty)$ . ■

**Remark 1** Theorem 2 is still true under the assumptions  $a_1 \geq a(t) \geq a_0 > 0$  and  $a'(t) \in L^1[0, \infty)$  since  $\left| \int_0^t a'(s)F(y(s))ds \right| \leq \sup_{-M \leq y \leq M} F(y) \int_0^\infty |a'(s)| ds$ . This implies all terms on the LHS of (6) are bounded. And so, the inequality (7) still holds even if  $\alpha$  will be different.

**Theorem 3** Under hypotheses of Theorem 1, we assume that  $xg(x) \geq N|x|^p$  where  $N > 0$ , then  $\int_0^\infty |x(s)|^p ds < \infty$ .

**Proof.** Multiply Eq.(1) by  $x(t)$  and integrate from 0 to  $t$  where we integrate by parts the first term on LHS, we have

$$\begin{aligned} &x(t)x''(t) + \frac{x'(0)^2}{2} + \int_0^t a(s)f(x'(s))x''(s)x(s)ds \\ &+ \int_0^t b(s, x(s), x'(s))x(s)ds + \int_0^t c(s)g(x(s))x(s)ds \\ &\leq \int_0^t |q(s)||x(s)|ds + \frac{x'(t)^2}{2} + \left| x(0)x''(0) \right|. \end{aligned} \quad (8)$$

Applying the mean value theorem for integral to the first term on the RHS of (8), we obtain

$$\begin{aligned} & x(t)x''(t) + \frac{x'(0)^2}{2} + \int_0^t a(s)f(x'(s))x''(s)x(s)ds \\ & + \int_0^t b(s, x(s), x'(s))x(s)ds + \int_0^t c(s)g(x(s))x(s)ds \\ & \leq |x(t^*)| \int_0^t |q(s)| ds + \frac{x'(t)^2}{2} + |x(0)x''(0)| \end{aligned} \quad (9)$$

where  $0 < t^* < t$ .

Since  $|x(t)|$  and  $|x'(t)|$  are finite from Theorem 1, the LHS of (9) is also bounded, that is, so is the term  $\int_0^t c(s)g(x(s))x(s)ds$ . From the above assumption, we have

$$\int_0^t c(s)g(x(s))x(s)ds \geq c_0 N \int_0^t |x(s)|^p ds \leq \beta,$$

where  $\beta$  is the RHS of (9).

Consequently,  $x(t)$  is an element of  $L^p[0, \infty)$ . ■

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# SOME IDENTITIES OF THE GENERALIZED TWISTED BERNOULLI NUMBERS AND POLYNOMIALS OF HIGHER ORDER

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**Abstract** The purpose of this paper is to derive some identities of the higher order generalized twisted Bernoulli numbers and polynomials attached to  $\chi$  from the properties of the  $p$ -adic invariant integral. We give some interesting identities for the power sums and the generalized twisted Bernoulli numbers and polynomials of higher order using the symmetric properties of the  $p$ -adic invariant integral.

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## 1. INTRODUCTION AND PRELIMINARIES

Let  $p$  be a fixed prime number. Throughout this paper, the symbol  $\mathbb{Z}$ ,  $\mathbb{Z}_p$ ,  $\mathbb{Q}_p$ , and  $\mathbb{C}_p$  denote the ring of rational integers, the ring of  $p$ -adic integers, the field of  $p$ -adic rational numbers, and the completion of algebraic closure of  $\mathbb{Q}_p$ , respectively. Let  $\mathbb{N}$  be the set of natural numbers and  $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ . Let  $\nu_p$  be the normalized exponential valuation of  $\mathbb{C}_p$  with  $|p|_p = p^{-\nu_p(p)} = p^{-1}$ .

Let  $UD(\mathbb{Z}_p)$  be the space of uniformly differentiable function on  $\mathbb{Z}_p$ . For  $f \in UD(\mathbb{Z}_p)$ , the  $p$ -adic invariant integral on  $\mathbb{Z}_p$  is defined as

$$I(f) = \int_{\mathbb{Z}_p} f(x) dx = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x). \quad (1.1)$$

(see [4-5]). From (1.1), we note that

$$I(f_1) = I(f) + f'(0), \quad (1.2)$$

where  $f'(0) = \frac{df(x)}{dx}|_{x=0}$  and  $f_1(x) = f(x+1)$ . For  $n \in \mathbb{N}$ , let  $f_n(x) = f(x+n)$ . Then we can derive the following equation from (1.2).

$$I(f_n) = I(f) + \sum_{i=0}^{n-1} f'(i), \quad (\text{see [4-5]}). \quad (1.3)$$

Let  $d$  be a fixed positive integer. For  $n \in \mathbb{N}$ , let

$$\begin{aligned} X &= X_d = \varprojlim_{\mathbb{N}} \mathbb{Z}/dp^N \mathbb{Z}, \quad X_1 = \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} (a + dp \mathbb{Z}_p), \\ a + dp^N \mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned}$$

where  $a \in \mathbb{Z}$  lies in  $0 \leq a < dp^N$ . It is easy to see that

$$\int_X f(x)dx = \int_{\mathbb{Z}_p} f(x)dx, \quad \text{for } f \in UD(\mathbb{Z}_p). \quad (1.4)$$

The ordinary Bernoulli polynomials  $B_n(x)$  are defined as

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

and the Bernoulli numbers  $B_n$  are defined as  $B_n = B_n(0)$  (see [1-19]).

For  $n \in \mathbb{N}$ , let  $T_p$  be the  $p$ -adic locally constant space defined by

$$T_p = \bigcup_{n \geq 1} \mathbb{C}_{p^n} = \lim_{n \rightarrow \infty} \mathbb{C}_{p^n},$$

where  $\mathbb{C}_{p^n} = \{\omega | \omega^{p^n} = 1\}$  is the cyclic group of order  $p^n$ . It is well known that the twisted Bernoulli polynomials are defined as

$$\frac{t}{\xi e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\xi}(x) \frac{t^n}{n!}, \quad \xi \in T_p,$$

and the twisted Bernoulli numbers  $B_{n,\xi}$  are defined as  $B_{n,\xi} = B_{n,\xi}(0)$  (see [14-18]).

Let  $\chi$  be the Dirichlet's character with conductor  $d \in \mathbb{N}$ . Then we have

$$\int_X \chi(x) \xi^x e^{xt} dx = \frac{t \sum_{a=0}^{d-1} \chi(a) \xi^a e^{at}}{\xi^d e^{dt} - 1}. \quad (1.5)$$

It is known that the generalized twisted Bernoulli numbers attached to  $\chi$ ,  $B_{n,\chi,\xi}$ , are defined as

$$\frac{t \sum_{a=0}^{d-1} \chi(a) \xi^a e^{at}}{\xi^d e^{dt} - 1} = \sum_{n=0}^{\infty} B_{n,\chi,\xi} \frac{t^n}{n!}, \quad \xi \in T_p. \quad (1.6)$$

The generalized twisted Bernoulli polynomials attached to  $\chi$ ,  $B_{n,\chi,\xi}(x)$ , are defined as

$$\frac{t \sum_{a=0}^{d-1} \chi(a) \xi^a e^{at}}{\xi^d e^{dt} - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi,\xi}(x) \frac{t^n}{n!}, \quad \xi \in T_p, \quad (1.7)$$

(see [13], [16]). From (1.5), (1.6) and (1.7), we derive that

$$\int_X \chi(x) \xi^x x^n dx = B_{n,\chi,\xi} \quad \text{and} \quad \int_X \chi(y) \xi^y (x+y)^n dy = B_{n,\chi,\xi}(x). \quad (1.8)$$

By (1.3) and (1.4), it is easy to see that for  $n \in \mathbb{N}$ ,

$$\int_X f(x+n)dx = \int_X f(x)dx + \sum_{i=0}^{n-1} f'(i), \quad (1.9)$$

where  $f'(i) = \frac{df(x)}{dx}|_{x=i}$ . From (1.9), it follows that

$$\begin{aligned} & \frac{1}{t} \left( \int_X \chi(x) \xi^{nd+x} e^{(nd+x)t} dx - \int_X \chi(x) \xi^x e^{xt} dx \right) \\ &= \frac{nd \int_X \chi(x) \xi^x e^{xt} dx}{\int_X \xi^{ndx} e^{ndxt} dx} = \frac{\xi^{nd} e^{ndt} - 1}{\xi^d e^{dt} - 1} \left( \sum_{i=0}^{d-1} \chi(i) \xi^i e^{it} \right) = \sum_{k=0}^{\infty} \left( \sum_{l=0}^{nd-1} \chi(l) \xi^l l^k \right) \frac{t^k}{k!}. \end{aligned} \quad (1.10)$$

For  $k \in \mathbb{Z}_+$ , let us define the  $p$ -adic functional  $T_{k,\chi,\xi}(n)$  as follows:

$$T_{k,\chi,\xi}(n) = \sum_{l=0}^n \chi(l) \xi^l l^k. \quad (1.11)$$

Let  $k, n, d \in \mathbb{N}$ . By (1.10) and (1.11), we see that

$$\int_X \chi(x) \xi^{nd+x} (nd+x)^k dx - \int_X \chi(x) \xi^x x^k dx = k T_{k-1,\chi,\xi}(nd-1). \quad (1.12)$$

From (1.8) and (1.12), we have that

$$\frac{\xi^{nd} B_{k,\chi,\xi}(nd) - B_{k,\chi,\xi}}{k} = T_{k-1,\chi,\xi}(nd-1). \quad (1.13)$$

For  $w_1, w_2, d \in \mathbb{N}$ , we note that

$$\begin{aligned} & \frac{d \int_X \int_X \chi(x_1) \chi(x_2) \xi^{w_1 x_1 + w_2 x_2} e^{(w_1 x_1 + w_2 x_2)t} dx_1 dx_2}{\int_X \xi^{dw_1 w_2 x} e^{dw_1 w_2 xt} dx} \\ &= \frac{t(\xi^{dw_1 w_2} e^{dw_1 w_2 t} - 1)}{(\xi^{w_1 d} e^{w_1 dt} - 1)(\xi^{w_2 d} e^{w_2 dt} - 1)} \left( \sum_{a=0}^{d-1} \chi(a) \xi^{w_1 a} e^{w_1 at} \right) \left( \sum_{b=0}^{d-1} \chi(b) \xi^{w_2 b} e^{w_2 bt} \right). \end{aligned} \quad (1.14)$$

In the next section, we will consider the extension of (1.14) related to the generalized twisted Bernoulli numbers and polynomials of higher order attached to  $\chi$ .

The generalized twisted Bernoulli polynomials of order  $k$  attached to  $\chi$ ,  $B_{n,\chi,\xi}^{(k)}(x)$ , are defined as

$$\left( \frac{t \sum_{a=0}^{d-1} \chi(a) \xi^a e^{at}}{\xi^d e^{dt} - 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi,\xi}^{(k)}(x) \frac{t^n}{n!}, \quad \xi \in T_p, \quad (1.15)$$

and  $B_{n,\chi,\xi}^{(k)} = B_{n,\chi,\xi}^{(k)}(0)$  are called the generalized twisted Bernoulli numbers of order  $k$  attached to  $\chi$ . When  $k = 1$ , the polynomials and numbers are called the generalized twisted Bernoulli polynomials and numbers attached to  $\chi$ , respectively (see [12-22]).

The authors of this paper have studied various identities for the Bernoulli and the Euler polynomials by the symmetric properties of the  $p$ -adic invariant integrals (see [6-8], [10]). T. Kim [6] established interesting identities by the symmetric properties of the  $p$ -adic invariant integrals and some relationships between the power sums and the Bernoulli polynomials. In [8], Kim et al. gave some identities of symmetry for the generalized Bernoulli polynomials. The twisted Bernoulli polynomials and numbers are very important in several field of mathematics and physics, and so have been studied by many authors (cf. [9-22]). Recently, Kim-Hwang [10] obtained some relations between the power sum polynomials and twisted Bernoulli polynomials.

In this paper, we extend our results to the generalized twisted Bernoulli numbers and polynomials of higher order attached to  $\chi$ . The purpose of this paper is to derive some identities of the higher order generalized twisted Bernoulli numbers and polynomials attached to

$\chi$  from the properties of the  $p$ -adic invariant integral. In Section 2, we give interesting identities for the power sums and the generalized twisted Bernoulli numbers and polynomials of higher order using the symmetric properties for the  $p$ -adic invariant integral.

## 2. SOME IDENTITIES OF THE GENERALIZED TWISTED BERNOULLI NUMBERS AND POLYNOMIALS OF HIGHER ORDER

Let  $w_1, w_2, d \in \mathbb{N}$ . For  $\xi \in T_p$ , we set

$$\begin{aligned} Y(m, \chi, \xi | w_1, w_2) &= \left( \frac{d \int_{X^m} \left( \prod_{i=1}^m \chi(x_i) \right) \xi^{\left( \sum_{i=1}^m x_i \right) w_1} e^{\left( \sum_{i=1}^m x_i + w_2 x \right) w_1 t} dx_1 \cdots dx_m}{\int_X \xi^{dw_1 w_2 x} e^{dw_1 w_2 x t} dx} \right) \\ &\quad \times \left( \int_{X^m} \left( \prod_{i=1}^m \chi(x_i) \right) \xi^{\left( \sum_{i=1}^m x_i \right) w_2} e^{\left( \sum_{i=1}^m x_i + w_1 y \right) w_2 t} dx_1 \cdots dx_m \right), \end{aligned} \quad (2.1)$$

where

$$\int_{X^m} f(x_1, \dots, x_m) dx_1 \cdots dx_m = \underbrace{\int_X \cdots \int_X}_{m\text{-times}} f(x_1, \dots, x_m) dx_1 \cdots dx_m.$$

In (2.1), we note that  $Y(m, \chi, \xi; w_1, w_2)$  is symmetric in  $w_1, w_2$ . From (2.1), we derive that

$$\begin{aligned} Y(m, \chi, \xi | w_1, w_2) &= \left( \int_{X^m} \left( \prod_{i=1}^m \chi(x_i) \right) \xi^{\left( \sum_{i=1}^m x_i \right) w_1} e^{\left( \sum_{i=1}^m x_i \right) w_1 t} dx_1 \cdots dx_m \right) e^{w_1 w_2 x t} \\ &\quad \times \left( \frac{d \int_X \chi(x) \xi^{w_2 x} e^{w_2 x t} dx}{\int_X \xi^{dw_1 w_2 x} e^{dw_1 w_2 x t} dx} \right) \\ &\quad \times \left( \int_{X^{m-1}} \left( \prod_{i=1}^{m-1} \chi(x_i) \right) \xi^{\left( \sum_{i=1}^{m-1} x_i \right) w_2} e^{\left( \sum_{i=1}^{m-1} x_i \right) w_2 t} dx_1 \cdots dx_{m-1} \right) e^{w_1 w_2 y t}. \end{aligned} \quad (2.2)$$

From (1.10) and (1.11), it follows that

$$\frac{dw_1 \int_X \chi(x) \xi^x e^{xt} dx}{\int_X \xi^{dw_1 x} e^{dw_1 x t} dx} = \sum_{i=0}^{w_1 d - 1} \chi(i) \xi^i e^{it} = \sum_{k=0}^{\infty} T_{k, \chi, \xi}(w_1 d - 1) \frac{t^k}{k!}. \quad (2.3)$$

By (1.15), we also see that

$$\begin{aligned} &e^{w_1 w_2 x t} \left( \int_{X^m} \left( \prod_{i=1}^m \chi(x_i) \right) \xi^{\left( \sum_{i=1}^m x_i \right) w_1} e^{\left( \sum_{i=1}^m x_i \right) w_1 t} dx_1 \cdots dx_m \right) \\ &= \left( \frac{w_1 t}{\xi^{dw_1} e^{dw_1 t} - 1} \sum_{a=0}^{d-1} \chi(a) \xi^{w_1 a} e^{aw_1 t} \right)^m e^{w_1 w_2 x t} = \sum_{n=0}^{\infty} B_{n, \chi, \xi^{w_1}}^{(m)}(w_2 x) \frac{w_1^n t^n}{n!}. \end{aligned} \quad (2.4)$$

By (2.2), (2.3) and (2.4), we have that

$$\begin{aligned} & Y(m, \chi, \xi | w_1, w_2) \\ &= \left( \sum_{l=0}^{\infty} B_{l, \chi, \xi^{w_1}}^{(m)}(w_2 x) \frac{w_1^l t^l}{l!} \right) \left( \frac{1}{w_1} \sum_{k=0}^{\infty} T_{k, \chi, \xi^{w_2}}(w_1 d - 1) \frac{w_2^k t^k}{k!} \right) \left( \sum_{i=0}^{\infty} B_{i, \chi, \xi^{w_2}}^{(m-1)}(w_1 y) \frac{w_2^i t^i}{i!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{j=0}^n \binom{n}{j} w_2^j w_1^{n-j-1} B_{n-j, \chi, \xi^{w_1}}^{(m)}(w_2 x) \sum_{k=0}^j \binom{j}{k} T_{k, \chi, \xi^{w_2}}(w_1 d - 1) B_{j-k, \chi, \xi^{w_2}}^{(m-1)}(w_1 y) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.5)$$

From the symmetry of  $Y(m, \chi, \xi | w_1, w_2)$  in  $w_1$  and  $w_2$ , we see that

$$\begin{aligned} & Y(m, \chi, \xi | w_1, w_2) \\ &= \sum_{n=0}^{\infty} \left( \sum_{j=0}^n \binom{n}{j} w_1^j w_2^{n-j-1} B_{n-j, \chi, \xi^{w_2}}^{(m)}(w_1 x) \sum_{k=0}^j \binom{j}{k} T_{k, \chi, \xi^{w_1}}(w_2 d - 1) B_{j-k, \chi, \xi^{w_2}}^{(m-1)}(w_2 y) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.6)$$

Comparing the coefficients on the both sides of (2.5) and (2.6), we obtain an identity for the generalized twisted Bernoulli polynomials of higher order as follows.

**Theorem 1.** *Let  $d, w_1, w_2 \in \mathbb{N}$ . For  $n \in \mathbb{Z}_+$  and  $m \in \mathbb{N}$ , we have*

$$\begin{aligned} & \sum_{j=0}^n \binom{n}{j} w_2^j w_1^{n-j-1} B_{n-j, \chi, \xi^{w_1}}^{(m)}(w_2 x) \sum_{k=0}^j \binom{j}{k} T_{k, \chi, \xi^{w_2}}(w_1 d - 1) B_{j-k, \chi, \xi^{w_2}}^{(m-1)}(w_1 y) \\ &= \sum_{j=0}^n \binom{n}{j} w_1^j w_2^{n-j-1} B_{n-j, \chi, \xi^{w_2}}^{(m)}(w_1 x) \sum_{k=0}^j \binom{j}{k} T_{k, \chi, \xi^{w_1}}(w_2 d - 1) B_{j-k, \chi, \xi^{w_1}}^{(m-1)}(w_2 y). \end{aligned}$$

**Remark 1.** *Taking  $m = 1$  and  $y = 0$  in (2.7) derives the following identity :*

$$\begin{aligned} & \sum_{j=0}^n \binom{n}{j} w_2^j w_1^{n-j-1} B_{n-j, \chi, \xi^{w_1}}(w_2 x) T_{j, \chi, \xi^{w_2}}(w_1 d - 1) \\ &= \sum_{j=0}^n \binom{n}{j} w_1^j w_2^{n-j-1} B_{n-j, \chi, \xi^{w_2}}(w_1 x) T_{j, \chi, \xi^{w_1}}(w_2 d - 1). \end{aligned} \quad (2.7)$$

Moreover, if we take  $x = 0$  and  $y = 0$  in Theorem 1, then we have the following identity for the generalized twisted Bernoulli numbers of higher order.

**Corollary 2.** *Let  $d, w_1, w_2 \in \mathbb{N}$ . For  $n \in \mathbb{Z}_+$  and  $m \in \mathbb{N}$ , we have*

$$\begin{aligned} & \sum_{j=0}^n \binom{n}{j} w_2^j w_1^{n-j-1} B_{n-j, \chi, \xi^{w_1}}^{(m)} \sum_{k=0}^j \binom{j}{k} T_{k, \chi, \xi^{w_2}}(w_1 d - 1) B_{j-k, \chi, \xi^{w_2}}^{(m-1)} \\ &= \sum_{j=0}^n \binom{n}{j} w_1^j w_2^{n-j-1} B_{n-j, \chi, \xi^{w_2}}^{(m)} \sum_{k=0}^j \binom{j}{k} T_{k, \chi, \xi^{w_1}}(w_2 d - 1) B_{j-k, \chi, \xi^{w_1}}^{(m-1)}. \end{aligned}$$

We also note that taking  $m = 1$  in Corollary 2 shows the following identity :

$$\begin{aligned} & \sum_{j=0}^n \binom{n}{j} w_2^j w_1^{n-j-1} B_{n-j, \chi, \xi^{w_1}} T_{j, \chi, \xi^{w_2}}(w_1 d - 1) \\ &= \sum_{j=0}^n \binom{n}{j} w_1^j w_2^{n-j-1} B_{n-j, \chi, \xi^{w_2}} T_{j, \chi, \xi^{w_1}}(w_2 d - 1). \end{aligned} \quad (2.8)$$

Now we will derive another interesting identities for the generalized twisted Bernoulli numbers and polynomials of higher order. From (1.15), (2.2) and (2.3), we can derive that

$$\begin{aligned}
 & Y(m, \chi, \xi | w_1, w_2) \\
 &= \frac{1}{w_1} \left( \sum_{i=0}^{w_1 d-1} \chi(i) \xi^{w_2 i} \int_{X^m} \left( \prod_{i=1}^m \chi(x_i) \right) \xi^{\left( \sum_{i=1}^m x_i \right) w_1} e^{\left( \sum_{i=1}^m x_i + \frac{w_2}{w_1} i + w_2 x \right) w_1 t} dx_1 \cdots dx_m \right) \quad (2.9) \\
 & \quad \times \left( \int_{X^{m-1}} \left( \prod_{i=1}^{m-1} \chi(x_i) \right) \xi^{\left( \sum_{i=1}^{m-1} x_i \right) w_2} e^{\left( \sum_{i=1}^{m-1} x_i + w_1 y \right) w_2 t} dx_1 \cdots dx_{m-1} \right) \\
 &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \binom{n}{k} w_1^{k-1} w_2^{n-k} B_{n-k, \chi, \xi^{w_2}}^{(m-1)}(w_1 y) \sum_{i=0}^{w_1 d-1} \chi(i) \xi^{w_2 i} B_{k, \chi, \xi^{w_1}}^{(m)} \left( w_2 x + \frac{w_2}{w_1} i \right) \right) \frac{t^n}{n!}.
 \end{aligned}$$

From the symmetry property of  $Y(m, \chi, \xi | w_1, w_2)$  in  $w_1$  and  $w_2$ , we see that

$$\begin{aligned}
 & Y(m, \chi, \xi | w_1, w_2) \quad (2.10) \\
 &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \binom{n}{k} w_2^{k-1} w_1^{n-k} B_{n-k, \chi, \xi^{w_1}}^{(m-1)}(w_2 y) \sum_{i=0}^{w_2 d-1} \chi(i) \xi^{w_1 i} B_{k, \chi, \xi^{w_2}}^{(m)} \left( w_1 x + \frac{w_1}{w_2} i \right) \right) \frac{t^n}{n!}.
 \end{aligned}$$

Comparing the coefficients on the both sides of (2.9) and (2.10), we obtain the following theorem which shows the relationship between the power sums and the generalized twisted Bernoulli polynomials.

**Theorem 3.** Let  $d, w_1, w_2 \in \mathbb{N}$ . For  $n \in \mathbb{Z}_+$  and  $m \in \mathbb{N}$ , we have

$$\begin{aligned}
 & \sum_{k=0}^n \binom{n}{k} w_1^{k-1} w_2^{n-k} B_{n-k, \chi, \xi^{w_2}}^{(m-1)}(w_1 y) \sum_{i=0}^{w_1 d-1} \chi(i) \xi^{w_2 i} B_{k, \chi, \xi^{w_1}}^{(m)} \left( w_2 x + \frac{w_2}{w_1} i \right) \\
 &= \sum_{k=0}^n \binom{n}{k} w_2^{k-1} w_1^{n-k} B_{n-k, \chi, \xi^{w_1}}^{(m-1)}(w_2 y) \sum_{i=0}^{w_2 d-1} \chi(i) \xi^{w_1 i} B_{k, \chi, \xi^{w_2}}^{(m)} \left( w_1 x + \frac{w_1}{w_2} i \right).
 \end{aligned}$$

**Remark 2.** Let  $m = 1$  and  $y = 0$  in Theorem 3. Then it follows that

$$w_1^{n-1} \sum_{i=0}^{w_1 d-1} \chi(i) B_{n, \chi, \xi^{w_1}} \left( w_2 x + \frac{w_2}{w_1} i \right) = w_2^{n-1} \sum_{i=0}^{w_2 d-1} \chi(i) B_{n, \chi, \xi^{w_2}} \left( w_1 x + \frac{w_1}{w_2} i \right). \quad (2.11)$$

Moreover, if we take  $x = 0$  and  $y = 0$  in Theorem 3, then we have the following identity for the generalized twisted Bernoulli numbers of higher order.

**Corollary 4.** Let  $d, w_1, w_2 \in \mathbb{N}$ . For  $n \in \mathbb{Z}_+$  and  $m \in \mathbb{N}$ , we have

$$\begin{aligned}
 & \sum_{k=0}^n \binom{n}{k} w_1^{k-1} w_2^{n-k} B_{n-k, \chi, \xi^{w_2}}^{(m-1)} \sum_{i=0}^{dw_1-1} \chi(i) \xi^{w_2 i} B_{k, \chi, \xi^{w_1}}^{(m)} \left( \frac{w_2}{w_1} i \right) \\
 &= \sum_{k=0}^n \binom{n}{k} w_2^{k-1} w_1^{n-k} B_{n-k, \chi, \xi^{w_1}}^{(m-1)} \sum_{i=0}^{dw_2-1} \chi(i) \xi^{w_1 i} B_{k, \chi, \xi^{w_2}}^{(m)} \left( \frac{w_1}{w_2} i \right).
 \end{aligned}$$

If we take  $m = 1$  in Corollary 4, we derive the identity for the generalized twisted Bernoulli numbers : for  $d, w_1, w_2 \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ ,

$$w_1^{n-1} \sum_{i=0}^{dw_1-1} \chi(i) \xi^{w_2 i} B_{n, \chi, \xi^{w_1}} \left( \frac{w_2}{w_1} i \right) = w_2^{n-1} \sum_{i=0}^{dw_2-1} \chi(i) \xi^{w_1 i} B_{n, \chi, \xi^{w_2}} \left( \frac{w_1}{w_2} i \right). \quad (2.12)$$



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## Closed images of certain locally compact spaces \*

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### Abstract

We show that (1) a space is a closed  $L$ -image of a paracompact locally compact space if and only if it has a point-countable  $HPC$   $k$ -system; (2) a regular space is a closed  $L$ -image of a locally compact metric space if and only if it has a point-countable  $HPC$   $mk$ -system if and only if it is a  $k'$ -space with a point-countable compact  $k$ -network; (3) a regular space is a closed image of a locally compact metric space if and only if it is a Fréchet space with a point-countable weak-compact  $k$ -network.

**Keywords and phrases:** Closed maps;  $L$ -maps;  $k$ -systems;  $mk$ -systems;  $k'$ -space; Fréchet spaces; Compact  $k$ -networks; Weak-compact  $k$ -networks.

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### 1. Introduction and definitions

Some characterizations for certain quotient images (or closed images) of paracompact locally compact spaces are obtained by means of  $k$ -systems (see [1,10,11,17,18]). On the other hand, some characterizations for certain quotient images (or closed images) of locally compact metric spaces are also obtained by means of  $mk$ -systems or compact  $k$ -networks (see [10,15,20,22,24]). In this paper, we introduce the concept of weak-compact  $k$ -networks and give new characterizations for closed images of certain locally compact spaces.

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Let  $\mathcal{P}$  be a cover of a space  $X$ .  $\mathcal{P}$  is called a  $k$ -network [5] for  $X$  if whenever  $K \subset U$  with  $K$  compact and  $U$  open in  $X$ , then  $K \subset \bigcup \mathcal{P}' \subset U$  for some finite  $\mathcal{P}' \subset \mathcal{P}$ . If  $\mathcal{P}$  is composed of compact subsets of  $X$ , then  $\mathcal{P}$  is called a compact  $k$ -network [22] for  $X$ . If the closure of each element of  $\mathcal{P}$  is compact in  $X$ , then  $\mathcal{P}$  is called a weak-compact  $k$ -network for  $X$ .  $\mathcal{P}$  is called a  $k$ -cover [21] for  $X$  if every compact  $K \subset X$  is covered by some finite  $\mathcal{P}' \subset \mathcal{P}$ . Obviously, every  $k$ -network is a  $k$ -cover.

$X$  is determined by  $\mathcal{P}$  if  $A \subset X$  is closed in  $X$  if and only if  $A \cap P$  is relatively closed in  $P$  for every  $P \in \mathcal{P}$ . If each element of  $\mathcal{P}$  is compact (resp. compact metric) in  $X$ , then  $\mathcal{P}$  is called a  $k$ -system [4] (resp.  $mk$ -system [22]) for  $X$ .

$X$  is called a  $k$ -space (resp. a sequential space) if it is determined by the cover consisting of all compact (resp. all compact metric) subsets of  $X$ .  $X$  is called a  $k_\omega$ -space if  $X$  has a countable  $k$ -system.

Let  $\mathcal{P}$  be a family of subsets of a space  $X$ .  $\mathcal{P}$  is called closure-preserving if  $\overline{\bigcup \mathcal{P}'} = \bigcup \{\overline{P} : P \in \mathcal{P}'\}$  for each  $\mathcal{P}' \subset \mathcal{P}$ .  $\mathcal{P}$  is called  $HPC$  (ie. hereditarily closure-preserving) if a family  $\{H(P) : P \in \mathcal{P}\}$  is closure-preserving for each  $\{H(P) : H(P) \subset P \in \mathcal{P}\}$ .  $\mathcal{P}$  is called  $\sigma$ - $HPC$  if  $\mathcal{P} = \bigcup \{\mathcal{P}_n : n \in N\}$  and each  $\mathcal{P}_n$  is  $HPC$ .  $\mathcal{P}$  is called point-countable  $HPC$  (resp. point-countable  $\sigma$ - $HPC$ ) if  $\mathcal{P}$  is point-countable and  $HPC$  (resp.  $\sigma$ - $HPC$ ).

$X$  is called a  $k'$ -space [1] if  $x \in \overline{A} \subset X$ , then  $x \in \overline{A \cap K}$  for some compact  $K \subset X$ .  $X$  is called a countably bi- $k$ -space [1] if whenever  $(F_n)$  is a decreasing sequence of subsets of  $X$  with a common cluster point  $x$ , then there exists a decreasing sequence  $(A_n)$  of subsets of  $X$  such that  $x \in \overline{F_n \cap A_n}$  for any  $n \in N$ ,  $K = \bigcap \{A_n : n \in N\}$  is compact in  $X$  and each neighborhood of  $K$  contains some  $A_n$ .  $X$  is called a  $\sigma$ -space if  $X$  has a  $\sigma$ -locally finite network.

Let  $P$  be a topology property.  $X$  is called a locally  $P$  space if for each  $x \in X$ , there exists a open neighborhood  $V$  of  $x$  in  $X$  such that the subspace  $\overline{V}$  has property  $P$ .

Let  $f : X \rightarrow Y$  be a map.  $f$  is called a  $L$ -map if for each  $y \in Y$ ,  $f^{-1}(y)$  is a Lindelöf subspace of  $X$ .  $f$  is called an  $s$ -map if for each  $y \in Y$ ,  $f^{-1}(y)$  is separable in  $X$ .  $f$  is called a countably bi-quotient map if whenever  $y \in Y$  and  $\mathcal{U}$  is a countable family of open subsets of  $X$  such that  $f^{-1}(y) \subset \bigcup \mathcal{U}$ , then  $y \in \text{Int}(f(\bigcup \mathcal{U}^*))$  for some finite  $\mathcal{U}^* \subset \mathcal{U}$ .

Every locally compact space and every Fréchet space are a  $k'$ -space. Every  $k'$ -space can be characterized as a pseudo-open image of a paracompact locally compact space, every  $k'$ -space is a  $k$ -space. Every locally compact space and every first countable space are a countably

bi- $k$ -space. Every countably bi- $k$ -space can be characterized as a countably bi-quotient image of a paracompact  $M$ -space, every countably bi- $k$ -space is a  $k$ -space. Every locally  $k_\omega$ -space with the Lindelöf property is a  $k_\omega$ -space.

Open maps and perfect maps are countably bi-quotient, countably bi-quotient maps and closed map are pseudo-open, and pseudo-open map is quotient.

In this paper, all spaces are assumed to be  $T_2$ , all maps are assumed to be continuous and onto. For two families  $\mathcal{A}$  and  $\mathcal{B}$  of subsets of a space  $X$ , denote  $\{A \cap B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$  by  $\mathcal{A} \wedge \mathcal{B}$ .

## 2. Closed $L$ -images of paracompact locally compact spaces

**Lemma 2.1** [11]. A space is a paracompact locally  $k_\omega$ -space if and only if it has a  $\sigma$ -locally finite  $k$ -system.

**Theorem 2.2.** The following are equivalent for a space  $X$ :

- (1)  $X$  is a closed  $L$ -image of a paracompact locally compact space.
- (2)  $X$  has a point-countable  $HPC$   $k$ -system.
- (3)  $X$  has a point-countable  $k$ -system and a  $HPC$   $k$ -system.
- (4)  $X$  is a paracompact, locally  $k_\omega$ -,  $k'$ -space.
- (5)  $X$  is a  $k'$ -space with a point-countable  $\sigma$ - $HPC$   $k$ -system.

**Proof.** (1)  $\implies$  (2). Suppose  $X$  is a closed  $L$ -image of a paracompact locally compact space. Let  $f : Z \longrightarrow X$  is a closed  $L$ -map, where  $Z$  is a paracompact locally compact space. By Lemma 1 in [11],  $Z$  has a locally finite  $k$ -system  $\mathcal{F}$ . Put  $\mathcal{P} = \{f(F) : F \in \mathcal{F}\}$ . Since quotient maps preserve  $k$ -systems, then  $\mathcal{P}$  is a  $k$ -system for  $X$ . Since  $f$  is closed, then  $\mathcal{P}$  is  $HCP$ . By  $f$  is a  $L$ -map,  $\mathcal{P}$  is point-countable in  $X$ . Thus  $X$  has a point-countable  $HPC$   $k$ -system.

(2)  $\implies$  (3) is obvious.

(3)  $\implies$  (4). Suppose  $X$  has a point-countable  $k$ -system and a  $HPC$   $k$ -system. Since  $X$  has a  $HPC$   $k$ -system, then  $X$  is a closed image of a paracompact locally compact space by Theorem 2 in [11]. Let  $f : Z \longrightarrow X$  is a closed map, where  $Z$  is a paracompact locally compact space. Obviously  $X$  is a countable bi- $k$ -space. Because  $X$  has a point-countable  $k$ -system, then  $X$  is a paracompact locally  $k_\omega$ -space by Theorem 13 in [9]. Since pseudo-open images of paracompact locally compact spaces are  $k'$ -spaces, then  $X$  is a  $k'$ -space. Thus  $X$  is a paracompact, locally  $k_\omega$ -,  $k'$ -space.

(4)  $\implies$  (5) holds by Lemma 2.1.

(5)  $\implies$  (2). Suppose  $\mathcal{P}$  is a point-countable  $\sigma$ -HCP  $k$ -system for a  $k'$ -space  $X$ . Let  $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in N\}$ , where each  $\mathcal{P}_n$  is HCP in  $X$ . We can assume that each  $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ . For each  $n \in N$ , set  $X_n = \bigcup \mathcal{P}_n$ , then  $\{X_n : n \in N\}$  is a rising closed cover of  $X$ . Put

$$\begin{aligned}\mathcal{F}_1 &= \mathcal{P}_1, \\ \mathcal{F}_n &= \{\overline{P \setminus X_{n-1}} : P \in \mathcal{P}_n\} \quad (n > 1), \\ \mathcal{F} &= \bigcup\{\mathcal{F}_n : n \in N\}.\end{aligned}$$

As in the proof of Theorem 2 (1)  $\implies$  (2) in [11], we can prove that  $\mathcal{F}$  is a HCP  $k$ -system for  $X$ . For  $n > 1$  and  $P \in \mathcal{P}_n$ ,  $\overline{P \setminus X_{n-1}} \subset \overline{P} = P$ . Because  $\mathcal{P}$  is point-countable in  $X$ , then  $\mathcal{F}$  is point-countable in  $X$ . Thus  $X$  has a point-countable HPC  $k$ -system.

(2)  $\implies$  (1). Suppose  $X$  has a point-countable HPC  $k$ -system for  $X$ . Let  $\mathcal{P}$  is a point-countable HCP  $k$ -system. Put  $M = \bigoplus\{P : P \in \mathcal{P}\}$ , the  $M$  is a paracompact locally compact space. Let  $f : M \rightarrow X$  be the natural map. Since  $\mathcal{P}$  is HCP, then  $f$  is closed. Because  $\mathcal{P}$  is a point-countable in  $X$ , then  $f$  is a  $L$ -map. Hence  $X$  is a closed  $L$ -image of a paracompact locally compact space.

### 3. Closed $L$ -images of locally compact metric spaces

From the proof of Proposition 2.1 in [19], the following holds.

**Lemma 3.1.** Suppose that  $\mathcal{P}$  is a point-countable cover of  $X$ . Then  $\mathcal{P}$  is a  $mk$ -system if and only if  $X$  is a  $k$ -space and  $\mathcal{P}$  is a  $k$ -cover consisting of compact metric subspaces.

**Lemma 3.2** [24]. The following are equivalent for a  $k'$ -space  $X$ :

- (1)  $X$  has a  $\sigma$ -locally finite  $mk$ -system.
- (2)  $X$  has a point-countable  $mk$ -system.

**Lemma 3.3** [24]. The following are equivalent for a  $k'$ -space  $X$ :

- (1)  $X$  has a  $\sigma$ -locally finite compact  $k$ -network.
- (2)  $X$  has a point-countable compact  $k$ -network.

**Theorem 3.4.** The following are equivalent for a regular space  $X$ :

- (1)  $X$  is a closed  $L$ -image of a locally compact metric space.
- (2)  $X$  is a pseudo-open  $L$ -image of a locally compact metric space.
- (3)  $X$  is a  $k'$ -space with a point-countable  $mk$ -system.

- (4)  $X$  has a point-countable  $HPC$   $mk$ -system.
- (5)  $X$  is a  $k'$ -space with a point-countable compact  $k$ -network.
- (6)  $X$  is a  $k'$ -space with a point-countable  $k$ -cover consisting of compact metric subspaces.

**Proof.** (1)  $\implies$  (2) is clear.

(2)  $\implies$  (3). Suppose  $X$  is the image of a locally compact metric space  $M$  under a pseudo-open  $L$ -map  $f$ . Since  $M$  is locally compact, then there exists compact subset  $K_m$  of  $M$  such that  $m \in \text{int}(K_m)$  for each  $m \in M$ . By  $M$  is paracompact,  $\{\text{int}(K_m) : m \in M\}$  has a locally finite closed refinement  $\mathcal{F}$ . Let  $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ . For each  $\alpha \in \Lambda$ ,  $F_\alpha \subset \text{int}(K_{m_\alpha})$  for some  $m_\alpha \in M$ . By  $F_\alpha \subset K_{m_\alpha}$ ,  $F_\alpha$  is compact in  $M$ . Thus  $\mathcal{F}$  is a locally finite closed cover consisting of compact metric subspaces. Put  $\mathcal{P} = \{P_\alpha : \alpha \in \Lambda\}$ , here  $P_\alpha = f(F_\alpha)$ . We shall show that  $\mathcal{P}$  is a  $k$ -system for  $X$ . Suppose  $A$  is a subset of  $X$  such that  $A \cap P_\alpha$  is closed in the subspace  $P_\alpha$  for any  $\alpha \in \Lambda$ . If  $A$  is not closed in  $X$ . Since  $f$  is quotient,  $f^{-1}(A)$  is not closed in  $M$ . By  $M$  is a  $k$ -space,  $f^{-1}(A) \cap K$  is not closed in the subspace  $K$  for some compact  $K \subset M$ . So  $f^{-1}(A) \cap K$  is not closed in  $M$ . Because  $\mathcal{F}$  is locally finite in  $M$ , then  $K \subset \bigcup \{F_\alpha : \alpha \in \Lambda'\}$  for some finite  $\Lambda' \subset \Lambda$ . Thus  $f^{-1}(A) \cap F_\alpha$  is not closed in  $M$  for some  $\alpha \in \Lambda'$ . Since  $f^{-1}(A \cap P_\alpha) = f^{-1}(A) \cap F_\alpha$ , then  $A \cap P_\alpha$  is not closed in  $X$ . So  $A \cap P_\alpha$  is not closed in the subspace  $P_\alpha$ , a contradiction. Hence  $\mathcal{P}$  is a  $k$ -system for  $X$ . Because continuous maps preserve compact metric spaces, then  $\mathcal{P}$  is a  $mk$ -system for  $X$ . By  $f$  is a  $L$ -map,  $\mathcal{P}$  is point-countable in  $X$ . Since pseudo-open maps preserve  $k'$ -spaces, thus  $X$  is a  $k'$ -space with point-countable  $mk$ -system.

(3)  $\implies$  (4). Suppose  $X$  is a  $k'$ -space with a point-countable  $mk$ -system, then  $X$  has a  $\sigma$ -locally finite  $mk$ -system by Lemma 3.2. Let  $\mathcal{P}$  is a  $\sigma$ -locally finite  $mk$ -system for  $X$ . Denote  $\mathcal{P} = \bigcup \{\mathcal{P}_n : n \in N\}$ , where each  $\mathcal{P}_n$  is locally finite in  $X$ . We can assume that each  $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ . For each  $n \in N$ , set  $X_n = \bigcup \mathcal{P}_n$ , then  $\{X_n : n \in N\}$  is a rising closed cover of  $X$ . Put

$$\begin{aligned}\mathcal{F}_1 &= \mathcal{P}_1, \\ \mathcal{F}_n &= \{\overline{P \setminus X_{n-1}} : P \in \mathcal{P}_n\} \quad (n > 1), \\ \mathcal{F} &= \bigcup \{\mathcal{F}_n : n \in N\}.\end{aligned}$$

As in the proof of Theorem 2 (1)  $\implies$  (2) in [11], we can prove that  $\mathcal{F}$  is a  $HCP$   $mk$ -system for  $X$ . Because  $\mathcal{P}$  is point-countable in  $X$ , then  $\mathcal{F}$  is point-countable in  $X$ . Thus  $X$  has a point-countable  $HPC$   $mk$ -system.

(4)  $\implies$  (1). Suppose  $\mathcal{P}$  is a point-countable *HPC mk*-system. Put  $M = \bigoplus\{P : P \in \mathcal{P}\}$ , the  $M$  is a locally compact metric space. Let  $f : M \rightarrow X$  be the natural map, then  $f$  is a closed  $L$ -map. Hence  $X$  is a closed  $L$ -image of a locally compact metric space.

(1)  $\implies$  (5). Suppose  $X$  is the image of a locally compact metric space  $M$  under a closed  $L$ -map  $f$ . Let  $\mathcal{B}$  be a  $\sigma$ -locally finite base for  $M$ . Since  $M$  is locally compact, set  $\mathcal{B}' = \{B \in \mathcal{B} : \overline{B} \text{ is compact in } M\}$ , then  $\mathcal{B}'$  also is a  $\sigma$ -locally finite base for  $M$ . Because  $M$  is a locally separable metric space, then there is a family  $\{M_\alpha : \alpha \in A\}$  of separable metric spaces such that  $M = \bigoplus\{M_\alpha : \alpha \in A\}$  (see [6, 4.4F]). For each  $\alpha \in A$ ,  $M_\alpha$  is both open and closed in  $M$ , and  $M_\alpha$  is a Lindelöf subspace of  $M$ . Put

$$\mathcal{P}_\alpha = \{M_\alpha \cap \overline{B} : B \in \mathcal{B}'\},$$

then  $\mathcal{P}_\alpha$  is a countable compact  $k$ -network for  $M_\alpha$ . So  $\mathcal{P} = \cup\{\mathcal{P}_\alpha : \alpha \in A\}$  is a locally-countable compact  $k$ -network for  $M$ . Put

$$\mathcal{F} = \{f(P) : P \in \mathcal{P}\}.$$

Since  $f$  is compact-covering and compact-covering maps preserve compact  $k$ -networks,  $\mathcal{F}$  is a compact  $k$ -network for  $X$ . By  $f$  is a  $L$ -map,  $\mathcal{F}$  is a point-countable in  $X$ . Because pseudo-open maps preserve  $k'$ -spaces, then  $X$  is a  $k'$ -space. Hence  $X$  is a  $k'$ -space with point-countable compact  $k$ -network.

(3)  $\iff$  (5) holds by Corollary 1 in [22].

(5)  $\implies$  (6). Suppose  $X$  is a  $k'$ -space with a point-countable compact  $k$ -network, then  $X$  has a  $\sigma$ -locally finite compact  $k$ -network by Lemma 3.3. So  $X$  is a  $\sigma$ -space. Let  $\mathcal{P}$  is a  $\sigma$ -locally finite compact  $k$ -network for  $X$ . Obviously  $\mathcal{P}$  is a point-countable  $k$ -cover for  $X$ . Because a countably compact  $\sigma$ -space is metrizable, then any compact subspaces of  $X$  are metrizable. So every element of  $\mathcal{P}$  is metrizable. Thus  $X$  is a  $k'$ -space with a point-countable  $k$ -cover consisting of compact metric subspaces.

(6)  $\implies$  (3) holds by Lemma 3.1.

**Corollary 3.5** [20]. A regular space  $X$  is a closed  $s$ -image of a locally compact metric space if and only if it is a Fréchet space with a point-countable compact  $k$ -network.

#### 4. Closed images of locally compact metric spaces

**Lemma 4.1** [15]. A regular space is a closed image of a locally compact metric space



if and only if it is a Fréchet space with a point-countable  $k$ -network, and each of its closed first-countable subspace is locally compact.

**Lemma 4.2.** For a space  $X$ , the following hold.

(1) Let  $\mathcal{P}$  be a point-countable  $k$ -network for  $X$ . If a point  $x$  in  $X$  has a countable local base, then for each neighborhood  $U$  of  $x$ , there exists a finite subset  $\mathcal{F}$  of  $\mathcal{P}$  such that  $x \in (\cup \mathcal{F})^\circ \subset \cup \mathcal{F} \subset U$  [15].

(2) A countable compact  $k$ -space with a point-countable space  $k$ -network is compact and metrizable [8].

(3) Let  $f : X \rightarrow Y$  be a perfect map, let  $Y$  be a first-countable space, if each  $f^{-1}(y)$  is a first-countable subset of  $X$ , then  $X$  is a first-countable space [16].

**Lemma 4.3.** Suppose  $X$  is a  $k$ -space with a point-countable  $k$ -network, then  $X$  has a point-countable weak-compact  $k$ -network if and only if each closed first-countable subspace of  $X$  is locally compact.

**Proof.** Necessity. Suppose  $E$  is a closed first-countable subspace of  $X$ . Pick  $x \in E$ . Let  $\mathcal{P}$  be a point-countable weak-compact  $k$ -network for  $X$ . Put  $\mathcal{F} = \{P \cap E : P \in \mathcal{P}\}$ , then  $\mathcal{F}$  is a point-countable  $k$ -network for  $E$ . By Lemma 4.2(1), there exists a finite subset  $\mathcal{R}$  of  $\mathcal{F}$  such that  $x \in \text{int}_E(\cup \mathcal{R})$ . So  $\mathcal{R} = \{P \cap E : P \in \mathcal{P}'\}$  for some finite subset  $\mathcal{P}'$  of  $\mathcal{P}$ . Since  $\cup \{\overline{P} \cap E : P \in \mathcal{P}'\}$  is compact subspace of  $E$  and  $x \in \text{int}_E(\cup \{\overline{P} \cap E : P \in \mathcal{P}'\})$ . Hence  $E$  is locally compact.

Sufficiency. With loss of generality we can assume that  $\mathcal{P}$  is a point-countable  $k$ -network for  $X$  which is closed under finite intersections. Put  $\mathcal{F} = \{P \in \mathcal{P} : \overline{P} \text{ is compact in } X\}$ . Then  $\mathcal{F}$  is a weak-compact  $k$ -network. In fact, let  $K \subset U$  with  $K$  compact and  $U$  open in  $X$ .  $\mathcal{P}$  is a point-countable  $k$ -network for  $X$ , by Miščenko's Lemma ([14, Lemma 3.3.10]) there are at most  $\omega$  minimal (i.e. not containing proper subcover) finite covers of  $K$  by members of  $\mathcal{P}$ , say  $\{\mathcal{B}_i\}$ . Put  $\mathcal{L}_n = \bigwedge_{i \leq n} \mathcal{B}_i$ , and  $L_n = \cup \mathcal{L}_n$  for each  $n \in N$ . Then  $\{L_n\}_{n \in N}$  is a decreasing set of  $K$  in  $X$ . For each  $n \in N$ , if no  $\overline{L_n}$  is compact in  $X$ , thus no  $\overline{L_n}$  is countable compact in  $X$  by Lemma 4.2(2). Thus  $\overline{L_n}$  contains an infinite closed discrete subset  $D_n$ . Define  $C = K \cup (\cup_{n \in N} D_n)$ , and endow  $C$  with the subspace topology of  $X$ . Then no neighborhood of  $K$  in  $C$  is countable compact. By the compactness of  $K$ ,  $C$  is not locally countable compact, so  $C$  is not locally compact. Let  $f : C \rightarrow C/K$  be a natural quotient map. Then  $f$  is a perfect map and  $C/K$  is first-countable (In fact,  $C/K$  is a metric space). By

the Lemma 4.2(3),  $C$  is first-countable, so  $C$  is a non-locally compact, closed first-countable subset of  $X$ , a contradiction. Thus,  $\overline{L_m}$  is compact in  $X$  for some  $m \in N$ . Choose  $n \geq m$  with  $K \subset L_n \subset U$ . This shows that  $\mathcal{L}_n$  is a finite subset of  $\mathcal{F}$  and  $K \subset \cup \mathcal{L}_n \subset U$ . Thus  $\mathcal{F}$  is a weak-compact  $k$ -network for  $X$ .

By Lemma 4.1 and Lemma 4.3, the following holds.

**Theorem 4.4.** A regular space  $X$  is a closed image of a locally compact metric space if and only if  $X$  is a Fréchet space with a point-countable weak-compact  $k$ -network.

## 5. Examples

**Example 5.1.** Open finite-to-one images of paracompact locally compact spaces need not be closed  $L$ -images of paracompact locally compact spaces.

There are a paracompact locally compact space  $Z$  and open finite-to-one map  $f$  from  $Z$  onto a space  $X$  which is not paracompact (see [3, Example 5.11]). Because closed maps preserve paracompact spaces, then  $X$  is not a closed  $L$ -image of a paracompact locally compact space.

This example also illustrates:

$X$  is a pseudo-open  $L$ -image of a paracompact locally compact space  $\not\Rightarrow X$  is a closed  $L$ -image of a paracompact locally compact space.

**Example 5.2.** Quotient finite-to-one images of locally compact metric spaces need not be closed  $L$ -images of metric spaces.

Let

$$S = \left\{ \frac{1}{n} : n \in N \right\} \cup \{0\}, \quad X = [0, 1] \times S.$$

And let

$$Y = [0, 1] \times \left\{ \frac{1}{n} : n \in N \right\}$$

have the usual Euclidean topology as a subspace of  $[0, 1] \times S$ . Define a typical neighborhood of  $(t, 0)$  in  $X$  to be of the form

$$\{(t, 0)\} \cup \left( \bigcup_{k \geq n} V(t, 1/k) \right), \quad n \in N,$$

where  $V(t, 1/k)$  is a neighborhood of  $(t, 1/k)$  in  $[0, 1] \times \{1/k\}$ . Put

$$M = (\oplus_{n \in N} [0, 1] \times \{1/n\}) \oplus (\oplus_{t \in [0, 1]} \{t\} \times S),$$

and define  $f$  from  $M$  onto  $X$  such that  $f$  is an obvious map.

Then  $f$  is a compact-covering, quotient, two-to-one map from the locally compact metric space  $M$  onto a separable, regular, non-Lindelöf,  $k$ -space  $X$  (see [12, Example 2.8.16] or [8, Example 9.3]).

$X$  has no compact-countable  $k$ -network. In fact. Suppose  $\mathcal{P}$  is a compact-countable  $k$ -network for  $X$ . Put

$$\mathcal{F} = \{(t, 0) : t \in [0, 1]\} \cup \{P \cap Y : P \in \mathcal{P}\}.$$

Since  $[0, 1] \times \{0\}$  is a closed discrete subspace of  $X$ , then  $\mathcal{F}$  is a  $k$ -network for  $X$ . But  $Y$  is a  $\sigma$ -compact subspace of  $X$ . Thus  $\{P \cap Y : P \in \mathcal{P}\}$  is countable, and so  $\mathcal{F}$  is star-countable. Since a regular,  $k$ -space with a star-countable  $k$ -network is an  $\aleph_0$ -space (see [7]), then  $X$  is a Lindelöf space, a contradiction. Thus  $X$  has no compact-countable  $k$ -network. So  $X$  is not an  $\aleph$ -space. By Theorem 2.7.23 in [12],  $X$  is not a closed  $s$ -image of a metric space. Hence  $X$  is not a closed  $L$ -image of a metric space.

This example also illustrates:

$X$  is a quotient  $L$ -image of a locally compact metric space  $\not\Rightarrow X$  is a closed  $L$ -image of a locally compact metric space.

**Example 5.3.** Closed  $L$ -images of locally compact metric spaces need not be countably bi-quotient images of paracompact locally compact spaces. See Example 10.1 in [1].

**Example 5.4.** Countably bi-quotient images of paracompact locally compact spaces need not be closed  $L$ -images of paracompact locally compact spaces.

Let  $Q$  be the rational number set, and endow  $Q$  with the usual subspace topology. Since  $Q$  is first-countable and paracompact, then  $Q$  is a countably bi-quotient image of a paracompact locally compact space (see [1]). But  $Q$  is a non- $k_\omega$ -space with the Lindelöf property (see [2, Proposition 20]), then  $Q$  is not a locally  $k_\omega$ -space. So  $Q$  is not a closed  $L$ -image of a paracompact locally compact space by Theorem 2.1.

Two examples above illustrate:

A countably bi-quotient image of a paracompact locally compact space is each other independent of a closed  $L$ -image of a paracompact locally compact space.

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# **Journal of Computational Analysis and Applications**

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# Recurrent points and periodic points of graph maps<sup>\*</sup>

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**Abstract.** Let  $G$  be a graph and  $f \in C^0(G)$ . It is proved that  $\overline{P(f)} = G$  if  $\overline{R(f)} = G$  and  $P(f) \neq \emptyset$ . This result generalizes several corresponding results given in [3] and [10].

## 1 Introduction

In this paper, let  $\mathbb{N}$  denote the set of all positive integers. Write  $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ ,  $\mathbb{N}_n = \{1, 2, \dots, n\}$  and  $\mathbb{Z}_n = \{0\} \cup \mathbb{N}_n$  for any  $n \in \mathbb{N}$ . Let  $X$  be a topological space and  $C^0(X)$  the set of all continuous maps from  $X$  to  $X$ . For any  $x \in X$  and  $f \in C^0(X)$ ,  $O(x, f) = \{f^k(x) : k \in \mathbb{Z}_+\}$  is called the orbit of  $x$ . The set of periodic points of  $f$ , the set of recurrent points of  $f$ , the set of  $\omega$ -limit points for some  $x \in X$  and the set of non-wandering points of  $f$  ( for the definitions see [2] ) are denoted by  $P(f)$ ,  $R(f)$ ,  $\omega(x, f)$  and  $\Omega(f)$  respectively. Write  $\omega(f) = \cup_{x \in X} \omega(x, f)$  and  $EP(f) = \{x : f^k(x) \in P(f) \text{ for some } k \in \mathbb{N}\}$ , which are called  $\omega$ -limit set and the set of eventually periodic points of  $f$  respectively. It is well known that  $P(f) \subset R(f) \subset \omega(f) \subset \Omega(f)$ . For any  $A \subset X$ , let  $\text{Int}(A)$ ,  $\partial A$  and  $\overline{A}$  be the interior, the boundary and the closure of  $A$  respectively. For any finite subset  $A \subset X$ , denote by  $|A|$  the number of elements of  $A$ .

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A space  $A$  is called an **arc** if there is a homeomorphism  $h : [0, 1] \rightarrow A$ . A connected metric space  $G$  is called a **graph** if there exist finitely many arcs  $A_1, A_2, \dots, A_n$  in  $G$  such that  $G = \bigcup_{i=1}^{\infty} A_i$  and  $A_i \cap A_j = \partial A_i \cap \partial A_j$  for  $1 \leq i < j \leq n$ . Let  $S^1$  be the unit circle in the complex plane  $\mathbb{C}$ . A graph  $C$  is called a **circle** if it is homeomorphic to  $S^1$ . A graph  $T$  is called a **tree** if it contains no circle.

Let  $G$  be a graph,  $x \in G$ , and  $U$  be a neighborhood of  $x$  in  $G$  such that the closure  $\overline{U}$  is a tree. The number of connected components of  $\overline{U} - \{x\}$ , denoted by  $val_G(x)$  or  $val(x)$ , is called the **valence** of  $x$  in  $G$ .  $x$  is called an **endpoint** of  $G$  if  $Val_G(x) = 1$ ;  $x$  is called a **branch point** of  $G$  if  $Val_G(x) > 2$ . Let  $End(G)$  and  $Br(G)$  be the set of endpoints and the set of branch points of  $G$  respectively. Let  $V(G) = End(G) \cup Br(G)$ . Denote by  $E(G)$  the set of all connected components of  $G - V(G)$ . A finite set  $D(G) \supset V(G)$  is a set of vertices of  $G$  if for each simple closed curve  $S$  of  $G$ ,  $S \cap D(G) = S \cap V(G)$  when  $|S \cap V(G)| \geq 3$ , and  $|S \cap D(G)| \geq 3$  when  $|S \cap V(G)| < 3$ , that is, we add some artificial points with valence 2 as vertices. In this way, each edge (the closure of some connected component of  $G - D(G)$ ) is homeomorphic to  $[0, 1]$  and if  $I$  and  $J$  are two edges of  $G$ , then  $|I \cap J| \leq 1$ . For some edge  $I$  of  $G$  and any  $a, b \in I$ , we use  $[a, b]_I$  (or simply  $[a, b]$  if there is no confusion) to denote the smallest connected closed subset of  $I$  containing  $\{a, b\}$ . Define  $(x, y] = [x, y] - \{x\}$  and  $(x, y) = (x, y] - \{y\}$ . For any  $x \in G$  and any  $\epsilon > 0$ , write  $B(x, \epsilon) = \{y \in G : d(x, y) < \epsilon\}$ .

Let  $G$  be a graph.  $f : G \rightarrow G$  is said to be a **graph map** if  $f \in C^0(G)$ . Let  $f$  be a graph map,  $L$  be a connected component of  $G - D(G)$  and  $h$  be a homeomorphism from  $L$  to the open interval  $(0, 1) \subset \mathbb{R}$ . For any  $x, y \in L$ , we write  $x <_h y$  (resp.  $x \leq_h y$ ,  $x >_h y$ ,  $x \geq_h y$ ) if  $h(x) < h(y)$  (resp.  $h(x) \leq h(y)$ ,  $h(x) > h(y)$ ,  $h(x) \geq h(y)$ ). Let  $K = (a, b) \subset L$ .  $K$  is said to be **free** (with respect to  $f$ ) if no iterate of a point of  $K$  belongs to  $K$ .  $K$  is said to be **positive** (with respect to  $f$ ) in the sense of  $h$  if  $K$  is not free and whenever  $x \in K$  and  $f^n(x) \in K$  for some  $n \geq 1$  then  $x <_h f^n(x)$ .

Let  $(X, d)$  be a metric space and  $f \in C^0(X)$ .  $f$  is said to be **transitive** if for any non-empty open subsets  $U$  and  $V$  of  $X$  there exists  $k \in \mathbb{N}$  such that  $f^k(U) \cap V$  is non-empty.  $f$  is said to be **Chaotic in the sense of Devaney** if it satisfies the following three conditions: (i)  $f$  is transitive; (ii)  $\overline{P(f)} = X$ ; (iii)  $f$  is sensitive dependence on initial conditions. It is interesting to remark that sensitivity dependence on initial conditions is widely understood as being the central idea in chaos. However, it has been proved that (i) and (ii) implies (iii) (see [1]). Also, for interval maps, M. Vellekoop and R. Berglund ([10]) proved that  $f$  is transitive if and only if  $f$  is chaotic in the sense of Devaney. For continuous maps  $f$  of the circle, it was shown by E.M. Coven and I. Mulvey that

**Theorem A.**([3]) For continuous maps of the circle with periodic points, the centre is closure of the periodic points and the depth of the centre is at most two.

In this paper, we will have a further discussion about on relationship between transitivity and periodic points of graph maps. Our results generalize several corresponding results given in [3] and [10].

**Theorem 2.8.** *Let  $G$  be a graph and  $f \in C^0(G)$ . If  $\overline{R(f)} = G$  and  $P(f) \neq \emptyset$ , then  $\overline{P(f)} = G$ .*

**Theorem 2.9.** *Let  $G$  be a graph and  $f \in C^0(G)$ . If  $f$  is transitive and  $P(f) \neq \emptyset$ , then the centre is  $\overline{P(f)}$  and  $\overline{P(f)} = G$ .*

## 2 Transitive graph maps with $\overline{P(f)} \neq \emptyset$

**Lemma 2.1.** *Let  $G$  be a graph,  $f \in C^0(G)$ ,  $L$  be a connected component of  $G - D(G)$  and  $h$  be a homeomorphism from  $L$  to the open interval  $(0, 1) \subset \mathbb{R}$ . If  $K = (a, b) \subset L$  is positive in the sense of  $h$ , then for every  $x \in K$ , there exists  $y \in K$  and  $n \geq 1$  such that  $f^n(y) \in K$  and  $f^n(y) >_h x$ .*

**Proof.** Without loss of generality, we suppose that  $a <_h b$ . Choose any  $x_0 \in K$  such that  $f^n(x_0) \in K$  for some  $n \geq 1$  and  $x_0 <_h f^n(x_0)$ . We have  $f^n(b) \notin K$  and  $(a, x_0) \cap f^n([x_0, b]) = \emptyset$  since  $K$  is positive in the sense of  $h$ . Since  $f^n([x_0, b])$  is connected, we have  $(f^n(x_0), b) \subset f^n([x_0, b])$ . Thus for every  $x \in K$ , there exists  $y \in K$  and  $n \geq 1$  such that  $f^n(y) \in K$  and  $f^n(y) >_h x$ . This completes Lemma 2.1.  $\square$

**Lemma 2.2.** *Let  $G$  be a graph,  $f \in C^0(G)$ ,  $L$  be a connected component of  $G - D(G)$  and  $h$  be a homeomorphism from  $L$  to the open interval  $(0, 1) \subset \mathbb{R}$ . If  $K = (a, b) \subset L$  is positive in the sense of  $h$ ,  $a <_h x <_h y <_h b$  and  $n \in \mathbb{N}$ , then there is  $j \in \mathbb{N}$  such that  $f^{jn}(x) \notin (a, y]$ .*

The proof of Lemma 2.2 is easy, so we omitted it.  $\square$

**Lemma 2.3.** *Let  $Y$  be a connected subset of  $G$ . Then  $|\partial Y| \leq 3|D(G)|$ .*

**Proof.** It is obvious  $|\partial Y \cap D(G)| \leq |D(G)|$ . Set  $\mathfrak{L} = \{L : L \text{ be a connected component of } G - D(G)\}$ . Then  $|\partial Y \cap L| \leq 2$  for each  $L \in \mathfrak{L}$ . Since  $|\mathfrak{L}| \leq |D(G)|$ . Thus  $|\partial Y| \leq 3|D(G)|$ .  $\square$

**Proposition 2.4.** *Let  $G$  be a graph,  $f \in C^0(G)$ ,  $L$  be a connected component of  $G - D(G)$  and  $h$  be a homeomorphism from  $L$  to the open interval  $(0, 1) \subset \mathbb{R}$ . If  $K =$*

$(a, b) \subset L$  is positive in the sense of  $h$ ,  $a <_h c <_h b$ , and for each  $x \in (c, b)$ , there is  $n \geq 1$  such that  $f^n([c, x]) \cap [c, x] \neq \emptyset$ , then  $c$  is an eventually periodic point.

**Proof.** Since  $K$  is positive in the sense of  $h$ , there is an  $x_0 \in (c, b)$  such that no iterate of  $c$  lies in  $(a, x_0]$ .

**Claim.** For all  $j, k \in \mathbb{N}$ ,  $f^j(c) \notin \text{int}(f^k([c, x_0]))$  if  $f^j(c) \notin V(G)$ .

**Proof of Claim.** Assume that there are  $j, k \in \mathbb{N}$  such that  $f^j(c) \in \text{int}(f^k([c, x_0]))$  and  $f^j(c) \notin V(G)$ . Then, there exist  $z$  and  $z'$  in  $K$  such that  $c <_h z <_h z' <_h x_0$  and  $f^j(c) \in \text{int}(f^k([z, z']))$ . It follows from the continuity of  $f$  that there exists  $y \in (c, z)$  such that  $f^i([c, y]) \cap [c, x_0] = \emptyset$  for all  $1 \leq i \leq j$  and  $f^j([c, y]) \subset \text{int}(f^k([z, z']))$ . On the other hand, by the hypothesis, there is  $n \in \mathbb{N}$  and  $w$  such that  $c <_h w <_h f^n(w) <_h y$ . Then  $n > j$ . Since  $f^j(w) \in \text{int}(f^k([z, z']))$ , there is a  $u \in [z, z']$  such that  $f^k(u) = f^n(w)$ . Therefore,  $f^{n-j+k}(u) = f^{n-j}(f^k(u)) = f^n(w) \in [c, y]$ , which contradicts the fact that  $K$  is positive in the sense of  $h$ . This establishes the claim.

It follows by the claim that, for all  $j, k \in \mathbb{N}$ ,  $f^j(c) \notin \text{int}(f^k([c, x_0]))$  if  $f^j(c) \notin V(G)$ . Let  $n_0 = 4|D(G)| + 2$  and  $Y = f^{n_0}([c, x_0])$ . Then, by Lemma 2.3, we have that  $|\partial Y \cup D(G)| \leq 4|D(G)| < n_0$ . Since, for all  $j, k \in \mathbb{N}$ ,  $f^j(c) \notin \text{int}(f^k([c, x_0]))$  if  $f^j(c) \notin V(G)$ , there is  $1 \leq j < k \leq n_0$  such that  $f^j(c) = f^k(c)$ . Thus  $c$  is an eventually periodic point of  $f$ .  $\square$

**Lemma 2.5.** ([5, Theorem 3.1]) Let  $G$  be a graph and  $f \in C^0(G)$ . Then  $\Omega(f|\Omega(f)) = \overline{R(f)}$  and the depth of  $f$  is at most 2.

**Lemma 2.6.** ([7, Theorem 2.1]) Let  $G$  be a graph and  $f \in C^0(G)$ . Then  $\overline{R(f)} = R(f) \cup \overline{P(f)}$ .

**Lemma 2.7.** ([7, Lemma 2.2]) Let  $G$  be a graph,  $A \subset G$  be an arc with  $\partial A = \{w, w'\}$ , and  $f \in C^0(G)$ . Suppose that  $\text{Int}(A) \cap V(G) = \emptyset$ ,  $A \cap P(f) = \emptyset$ , and there exists  $\{x, y\} \subset \text{Int}(A)$  with  $x \in (w, y)$  and  $j, k \in \mathbb{N}$  such that  $\{f^j(x), f^k(y)\} \subset [x, y]$ .

- (1) If  $f^j(x) \in (x, f^k(y)]$ , then  $f^j(w) \in \text{Int}(A)$ .
- (2) If  $f^k(y) \in [x, f^j(x))$ , then  $\{f^j(w), f^{j+k}(w)\} \cap \text{Int}(A) \neq \emptyset$ .

**Theorem 2.8.** Let  $G$  be a graph and  $f \in C^0(G)$ . If  $\overline{R(f)} = G$  and  $P(f) \neq \emptyset$ , then  $\overline{P(f)} = G$ .

**Proof.** Suppose that  $\overline{P(f)} \neq G$ . Then  $G - \overline{P(f)}$  is a nonempty open set. Let  $U$  be a connected component of  $G - \overline{P(f)}$ . Write  $U_0 = \overline{U} - U$ . Then  $U_0 = \overline{U} \cap \overline{P(f)} \neq \emptyset$ . It follows from Lemma 2.6 that  $U \subset R(f)$ . Thus there exists  $n \in \mathbb{N}$  such that  $f^n(U) \cap U \neq \emptyset$ . Let  $n_0 = \min\{n \in \mathbb{N} : f^n(U) \cap U \neq \emptyset\}$ . Obviously,  $f^{n_0}(U) \cap U$  is connected, and

$f^{n_0}(U) \cap \overline{P(f)} = \emptyset$  (if a point of  $U$  gets mapped into  $\overline{P(f)}$  then it is not recurrent). Thus,  $f^{n_0}(U) \subset U$ , and hence  $f^{n_0}(\overline{U}) \subset \overline{U}$ . This with  $f^{n_0}(\overline{P(f)}) = \overline{P(f)}$  implies  $f^{n_0}(U_0) \subset U_0$ . Noting that  $U_0$  is finite, we have  $U_0 \cap P(f) \neq \emptyset$ . Let  $v \in U_0 \cap P(f)$  and  $m$  be the period of  $v$ . Write  $g = f^{mn_0}$ . Then

$$v \in \text{Fix}(g), \quad g(\overline{U}) \subset \overline{U}, \quad g(U) \subset U, \quad g(U_0) \subset U_0 \quad \text{and} \quad U \subset R(g).$$

Choose an arc  $K$  in  $G$  and a connected component of  $G - D(G)$  such that  $v \in \partial K$ ,  $K \subset \overline{L} \cap \overline{U}$  and  $K \cap U_0 = \{v\}$ . Let  $h$  be a homeomorphism from  $L$  to open interval  $(0, 1)$  such that  $v >_h x$  for any  $x \in \text{Int}(K)$ . Let  $w \in \text{Int}(K)$ , write  $\varepsilon_0 = \frac{1}{2}d(v, w)$ .

**Case 1.** There exists  $0 < \varepsilon_1 < \varepsilon_0$  such that whenever  $x \in \text{Int}(K) \cap B(v, \varepsilon_1)$  and  $f^n(x) \in \text{Int}(K) \cap B(v, \varepsilon_1)$  for some  $i \geq 1$  then  $x <_h f^i(x)$ . Write  $K_1 = \text{Int}(K) \cap B(v, \varepsilon_1)$ . Obviously, there exists  $i \in \mathbb{N}$  such that  $g^i(K_1) \cap K_1 \neq \emptyset$  since  $K_1 \subset U \subset R(g)$ . Thus,  $K_1$  is positive in the sense of  $h$ . Let  $c \in K_1$ . Since  $c \in R(g)$  and  $K_1$  is positive in the sense of  $h$ , for any  $x \in (c, v)$ , there is  $j \geq 1$  such that  $g^j([c, x]) \cap [c, x] \neq \emptyset$ . It follows from Proposition 2.4 that  $c$  is an eventually periodic point, which contradicts  $c \in R(g) - \overline{P(g)}$ .

**Case 2.** There exists  $0 < \varepsilon_2 < \varepsilon_0$  such that whenever  $x \in \text{Int}(K) \cap B(v, \varepsilon_2)$  and  $f^n(x) \in \text{Int}(K) \cap B(v, \varepsilon_2)$  for some  $i \geq 1$  then  $x >_h f^i(x)$ . Write  $K_2 = \text{Int}(K) \cap B(v, \varepsilon_2)$ . Obviously, there exists  $i \in \mathbb{N}$  such that  $g^i(K_2) \cap K_2 \neq \emptyset$  since  $K_2 \subset U \subset R(g)$ . Let  $s(y) = 1 - y$  for any  $y \in (0, 1)$  and  $h_1 = h \circ s$ . Then  $h_1$  is a homeomorphism from  $L$  to the open interval  $(0, 1)$  and  $K_2$  is positive in the sense of  $h_1$ . Given any  $c \in K_2$ . Since  $c \in R(g)$  and  $K_2$  is positive in the sense of  $h_1$ , for any  $x \in K_2$  with  $x >_{h_1} c$ , there is an  $j \geq 1$  such that  $g^j([c, x]) \cap [c, x] \neq \emptyset$ . It follows from Proposition 2.4 that  $c$  is an eventually periodic point, which contradicts  $c \in R(g) - \overline{P(g)}$ .

**Case 3.** There exist  $x, y \in \text{Int}(K)$  and  $j, k \in \mathbb{N}$  such that  $x >_h g^j(x) >_h g^k(y) >_h y$ . It follows by Lemma 2.7 that  $g^j(c) \in (c, y]$  for any  $c \in (x, v)$ . By the continuity of  $g^j$ , we have  $g^j[v, x] \supset [v, x]$ . Thus there exist  $x_0 = x, x_1, x_2, \dots \in (v, x]$  such that  $x_i \in (x_{i+1}, x_{i-1})$  and  $g^j(x_i) = x_{i-1}$  for each  $i \in \mathbb{N}$ . Since  $x \in R(g) = R(g^j)$ , there exists  $l \in \mathbb{N}$  with  $l > 3$  such that  $g^{lj}(x) \in (x_1, g^j(x))$ . Let  $a = x_{l-2}, b = g^j(x)$  and  $d = g^{lj}(x)$ . Then  $g^{(l-1)j}(a) = b$  and  $g^{(l-1)j}(b) = d$ . It follows by  $v \notin g^{(l-1)j}([a, b])$  that  $g^{(l-1)j}([a, b]) \supset [b, d]$ . Therefore,  $[b, d] \cap \text{Fix}(g^{(l-1)j}) \neq \emptyset$ , which contradicts  $U \subset G - \overline{P(f)} = G - \overline{P(g)}$ .

Thus, by Case 1-3, we have  $\overline{P(f)} = G$ . The proof of Theorem 2.8 is complete.  $\square$

By Theorem 2.8, we have the following Theorem 2.9, which generalizes several corresponding results given in [3] and [10].

**Theorem 2.9.** *Let  $G$  be a graph and  $f \in C^0(G)$ . If  $f$  is transitive and  $P(f) \neq \emptyset$ , then  $\overline{P(f)} = G$ .*

**Proof.** Since  $f$  is transitive, we have  $\Omega(f) = G$ . Thus  $\Omega(f|\Omega(f)) = G$ . It follows by Lemma 2.5 that  $\overline{R(f)} = G$ . Since  $P(f) \neq \emptyset$ , by Theorem 2.8, we have  $\overline{P(f)} = G$ .  $\square$

Combine Theorem 2.9 and [6, Corollary 5.3], we have

**Corollary 2.10.** *Let  $G$  be a graph and  $f \in C^0(G)$  be transitive.*

- (1) *If  $P(f) \neq \emptyset$  then  $f$  is chaotic in the sense of Devaney.*
- (2) *If  $P(f) = \emptyset$  then  $f$  is minimal and  $f$  is conjugate to an irrational rotation on the unit circle.*

**Corollary 2.11.** *Let  $G$  be a graph with  $Br(G) \neq \emptyset$  and  $f \in C^0(G)$ . Then  $\overline{R(f)} = G$  if and only if  $\overline{P(f)} = G$ .*

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# DIFFERENTIATION FOLLOWED BY COMPOSITION FROM BLOCH SPACES TO $Q_p$ SPACES

XIANGLING ZHU

## Abstract

Suppose that  $\varphi$  is an analytic self-map of the unit disk  $\mathbb{D}$ . The boundedness, compactness and the weak compactness of the operator  $C_\varphi Df = f'(\varphi)$  from Bloch spaces to  $Q_p$  spaces on  $\mathbb{D}$  are studied.

## 1. INTRODUCTION

Let  $\mathbb{D}$  be the unit disk of complex plane  $\mathbb{C}$ , and  $H(\mathbb{D})$  the class of functions analytic in  $\mathbb{D}$ . Recall that an  $f \in H(\mathbb{D})$  is said to belong to the Bloch space  $\mathcal{B}$  if

$$\|f\|_b = \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

With the norm  $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_b$ ,  $\mathcal{B}$  is a Banach space. Let  $\mathcal{B}_0$  be the space which consists of all  $f \in \mathcal{B}$  satisfying  $(1 - |z|^2) |f'(z)| \rightarrow 0$  as  $|z| \rightarrow 1$ . This space is called the little Bloch space.

For  $a \in \mathbb{D}$ , let  $g(z, a) = \log \frac{1}{|\sigma_a(z)|}$  be the Green function in  $\mathbb{D}$ , where  $\sigma_a(z) = (a - z)/(1 - \bar{a}z)$ . For  $p \geq 0$ ,  $f \in H(\mathbb{D})$ , we say that  $f$  belongs to the space  $Q_p$  if

$$\|f\|_{Q_p}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g^p(z, a) dA(z) < \infty, \quad (1)$$

where  $dA$  is the normalized Lebesgue area measure on  $\mathbb{D}$ . The space  $Q_{p,0}$  ( $0 < p < \infty$ ) consists of analytic functions  $f$  on  $\mathbb{D}$  for which

$$\lim_{|a| \rightarrow 1} \int_{\mathbb{D}} |f'(z)|^2 g^p(z, a) dA(z) = 0. \quad (2)$$

$Q_{p,0}$  is a closed subspace of  $Q_p$  and  $Q_p \subset \mathcal{B}$ .  $Q_p$  is a Banach space under the norm  $\|f\|_p = |f(0)| + \|f\|_{Q_p}$ . If  $p = 1$ ,  $Q_p = BMOA$ . When  $p = 0$ , then  $Q_p$  is the Dirichlet space  $\mathcal{D}$ . If  $p > 1$ ,  $Q_p = \mathcal{B}$ . For more on  $Q_p$  spaces, see [26].

Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . The composition operator  $C_\varphi$  with the symbol  $\varphi$  is defined by  $C_\varphi f = f \circ \varphi$  for  $f \in H(\mathbb{D})$  (see [3]).

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*Key words*. Composition operator, differentiation operator, Bloch space,  $Q_p$  space.

Let  $D$  be the differentiation operator. The operator  $C_\varphi D$  is defined by

$$C_\varphi Df = f'(\varphi), \quad f \in H(\mathbb{D}). \quad (3)$$

The operator  $C_\varphi D$ , and other products of compositions and differentiation operators have been studied in [4, 8, 9, 12, 14, 22, 23, 30]. For some other recently introduced products see [10, 11, 18, 21]. Composition and weighted composition operators between Bloch,  $Q_p$  and some other spaces of analytic functions, have been studied, e.g., in [2, 5, 7, 13, 16, 19, 20, 24, 25, 27].

Let  $L : X \rightarrow Y$  be a linear operator, where  $X$  and  $Y$  are Banach spaces. Then  $L$  is said to be weakly compact if for every bounded sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X$ ,  $(L(x_n))_{n \in \mathbb{N}}$  has a weakly convergent subsequence, i.e., there is a subsequence  $(x_{n_m})_{m \in \mathbb{N}}$  such that for every  $\Lambda \in Y^*$ ,  $\Lambda(L(x_{n_m}))_{m \in \mathbb{N}}$  converges.

Here we study the boundedness, compactness and weak compactness of the operator  $C_\varphi D$  from Bloch spaces  $\mathcal{B}$  or  $\mathcal{B}_0$  to the space  $Q_p$ .

Throughout this paper, constants are denoted by  $C$ , they are positive and may differ from one occurrence to the other. The notation  $A \asymp B$  means that there is a positive constant  $C$  such that  $B/C \leq A \leq CB$ .

## 2. MAIN RESULTS AND PROOFS

**Theorem 1.** *Let  $p \in [0, \infty)$  and  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . Then the following statements are equivalent:*

- (i)  $C_\varphi D : \mathcal{B} \rightarrow Q_p$  is bounded;
- (ii)  $C_\varphi D : \mathcal{B}_0 \rightarrow Q_p$  is bounded;
- (iii)

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^4} g^p(z, a) dA(z) < \infty. \quad (4)$$

*Proof.* (i)  $\Rightarrow$  (ii). This implication is obvious.

(ii)  $\Rightarrow$  (iii). Let  $f \in \mathcal{B}$ . Set  $f_s(z) = f(sz)$  for  $0 < s < 1$ , then we see that  $f_s \in \mathcal{B}_0$  and  $\|f_s\|_b \leq \|f\|_b$ . Thus, by the assumption for all  $f \in \mathcal{B}$  we have

$$\|C_\varphi Df_s\|_{Q_p} \leq \|C_\varphi D\| \|f_s\|_b \leq \|C_\varphi D\| \|f\|_b. \quad (5)$$

By [15] we know that there exist two Bloch functions  $f_1$  and  $f_2$  satisfying

$$\frac{1}{1 - |z|^2} \leq |f'_1(z)| + |f'_2(z)|, \quad z \in \mathbb{D}. \quad (6)$$

We choose  $h_1(z) = f_1(z) - zf'_1(0)$ ,  $h_2(z) = f_2(z) - zf'_2(0)$ . Since (see [29])

$$(1 - |z|^2)^2 |f''(z)| + |f'(0)| \asymp (1 - |z|^2) |f'(z)|, \quad (7)$$

it follows that  $h_1, h_2 \in \mathcal{B}$  and

$$(1 - |z|^2)^{-2} \leq |h''_1(z)| + |h''_2(z)|, \quad z \in \mathbb{D}. \quad (8)$$



Replacing  $f$  in (5) by  $h_1$  and  $h_2$  respectively, then

$$\begin{aligned}
& \int_{\mathbb{D}} \frac{|s\varphi'(z)|^2}{(1-|s\varphi(z)|^2)^4} g^p(z, a) dA(z) \\
& \leq 2 \int_{\mathbb{D}} (|h_1''(s\varphi(z))|^2 + |h_2''(s\varphi(z))|^2) |s\varphi'(z)|^2 g^p(z, a) dA(z) \\
& = 2 \int_{\mathbb{D}} (|(h_{1s}' \circ \varphi)'(z)|^2 + |(h_{2s}' \circ \varphi)'(z)|^2) g^p(z, a) dA(z) \\
& = 2\|C_\varphi D h_{1s}\|_{Q_p}^2 + 2\|C_\varphi D h_{2s}\|_{Q_p}^2 \leq \|C_\varphi D\|^2 (\|h_1\|_{\mathcal{B}}^2 + \|h_2\|_{\mathcal{B}}^2) < \infty \quad (9)
\end{aligned}$$

for all  $a \in \mathbb{D}$  and  $s \in (0, 1)$ . This estimate and Fatou's Lemma give (4).

(iii)  $\Rightarrow$  (i). For any  $f \in \mathcal{B}$ , by the following well-known estimate  $(1 - |z|^2)^2 |f''(z)| \leq C \|f\|_{\mathcal{B}}$ , (see, e.g., [29]) the implication follows.

The following lemma is proved as Proposition 3.11 in [3] or Lemma 3 in [17].

**Lemma 1.** *Let  $p \in [0, \infty)$  and  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . Then  $C_\varphi D : \mathcal{B} \rightarrow Q_p$  is compact if and only if for every bounded sequence  $\{f_n\}$  in  $\mathcal{B}$  converging to 0 uniformly on compacts of  $\mathbb{D}$  as  $n \rightarrow \infty$ ,  $\lim_{n \rightarrow \infty} \|C_\varphi D f_n\|_p = 0$ .*

**Lemma 2.** *Let  $p \geq 0$  and  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . If  $C_\varphi D : \mathcal{B}(\mathcal{B}_0) \rightarrow Q_p$  is compact, then for any  $\epsilon > 0$  there exists a  $\delta \in (0, 1)$ , such that for all  $f$  in  $\mathcal{B}_{\mathcal{B}}$  (or  $\mathcal{B}_{\mathcal{B}_0}$ ), the unit ball of  $\mathcal{B}$  (or  $\mathcal{B}_0$ ), and  $\delta < r < 1$  holds*

$$\sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} |f''(\varphi(z))|^2 |\varphi'(z)|^2 g^p(z, a) dA(z) < \epsilon. \quad (10)$$

*Proof.* We only give the proof for  $\mathcal{B}_0$ , since the proof for  $\mathcal{B}$  is similar. For  $f \in \mathcal{B}_{\mathcal{B}_0}$ , let  $f_s(z) = f(sz)$ ,  $0 < s < 1$ . Then  $f_s \in \mathcal{B}_{\mathcal{B}_0}$  and  $f_s \rightarrow f$  uniformly on compact subsets of  $\mathbb{D}$  as  $s \rightarrow 1$ . Since  $C_\varphi D$  is compact,

$$\|C_\varphi D f_s - C_\varphi D f\|_{Q_p} \rightarrow 0 \quad \text{as } s \rightarrow 1.$$

That is, for given  $\epsilon > 0$  there exists  $s \in (0, 1)$  such that

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_s''(\varphi(z)) - f''(\varphi(z))|^2 |\varphi'(z)|^2 g^p(z, a) dA(z) < \epsilon. \quad (11)$$

For  $0 < r < 1$ , using the triangle inequality and (11),

$$\sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} |f''(\varphi(z))|^2 |\varphi'(z)|^2 g^p(z, a) dA(z) \leq \epsilon + \|f_s''\|_\infty^2 \sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} |\varphi'(z)|^2 g^p(z, a) dA(z).$$

Set  $f_n(z) = z^n \in \mathcal{B}_0$ . Since  $C_\varphi D$  is compact,  $\lim_{n \rightarrow \infty} n^2 \|\varphi^{n-1}\|_p = 0$ . Thus, for given  $\epsilon > 0$  and  $\|f_s''\|_\infty^2 > 0$  there exists an  $N \in \mathbb{N}$  such that for  $n \geq N$ ,

$$\|f_s''\|_\infty^2 \cdot \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} n^2(n-1)^2 |\varphi^{n-2}(z)|^2 |\varphi'(z)|^2 g^p(z, a) dA(z) < \epsilon.$$

Hence, for  $0 < r < 1$ ,

$$\begin{aligned} & N^2(N-1)^2 \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\varphi^{N-2}(z)|^2 |\varphi'(z)|^2 g^p(z, a) dA(z) \\ & \geq N^2(N-1)^2 r^{2(N-2)} \sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} |\varphi'(z)|^2 g^p(z, a) dA(z). \end{aligned} \quad (12)$$

Therefore, for  $r \geq [N(N-1)]^{-\frac{1}{N-2}}$  we have

$$\|f_s''\|_\infty^2 \cdot \sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} |\varphi'(z)|^2 g^p(z, a) dA(z) < \epsilon.$$

Hence for any  $\epsilon > 0$  and  $f \in \mathbb{B}_{\mathcal{B}_0}$  there is a  $\delta = \delta(\epsilon, f)$  such that for  $\delta < r < 1$

$$\sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} |f''(\varphi(z))|^2 |\varphi'(z)|^2 g^p(z, a) dA(z) < \epsilon.$$

The rest of the proof can be completed by using the finite covering property of the set  $C_\varphi D(\mathbb{B}_{\mathcal{B}_0})$  which is relatively compact in  $Q_p$  (see, e.g., [2, 25]).  $\square$

The proof of the next lemma is similar to Lemma 4 in [10], so is omitted.

**Lemma 3.** Assume  $X = Q_p$  or  $Q_{p,0}$  and  $\varphi$  is an analytic self-map of  $\mathbb{D}$ . Then  $C_\varphi D : \mathcal{B}_0 \rightarrow X$  is weakly compact if and only if  $C_\varphi D : \mathcal{B}_0 \rightarrow X$  is compact.

**Theorem 2.** Let  $p \in [0, \infty)$  and  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . Then the following statements are equivalent:

- (i)  $C_\varphi D : \mathcal{B} \rightarrow Q_p$  is compact;
- (ii)  $C_\varphi D : \mathcal{B}_0 \rightarrow Q_p$  is compact;
- (iii)  $C_\varphi D : \mathcal{B}_0 \rightarrow Q_p$  is weakly compact;
- (iv)  $\varphi \in Q_p$  and

$$\limsup_{r \rightarrow 1} \sup_{a \in \mathbb{D}} \int_{|\varphi(z)| > r} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^4} g^p(z, a) dA(z) = 0. \quad (13)$$

*Proof.* (i)  $\Rightarrow$  (ii). It is obvious.

(ii)  $\Leftrightarrow$  (iii). It follows from Lemma 3.

(ii)  $\Rightarrow$  (iv). Assume  $C_\varphi D : \mathcal{B}_0 \rightarrow Q_p$  is compact. Choosing  $f = \frac{1}{2}z^2 \in \mathcal{B}_0$ , we obtain  $\varphi \in Q_p$ . Let  $f(z) = \frac{1}{4} \sum_{k=1}^{\infty} z^{2^k}$ . From [28], we see that  $f \in \mathbb{B}_{\mathcal{B}}$ . Choose a sequence  $\{\lambda_n\}$  in  $\mathbb{D}$  which converges to 1 as  $n \rightarrow \infty$ , and let  $f_n(z) = f(\lambda_n z)$  for  $n \in \mathbb{N}$ . Then,  $f_n \in \mathbb{B}_{\mathcal{B}_0}$  for all  $n \in \mathbb{N}$  and  $\|f_n\|_{\mathcal{B}} \leq C$ . Let  $f_{n,\theta}(z) = f_n(e^{i\theta}z)$ . Then  $f_{n,\theta} \in \mathbb{B}_{\mathcal{B}_0}$ . Replace  $f$  by  $f_{n,\theta}$  in (10) and then integrate both sides with respect to  $\theta$ . By Fubini's Theorem, Parseval's identity and the inequality  $2^k(2^k - 1) \geq (2^k - 2)^2$  ( $k > 2$ ), we obtain

$$\begin{aligned}
\epsilon &\geq \frac{1}{2\pi} \int_{|\varphi(z)|>r} \left( \int_0^{2\pi} |f_n''(e^{i\theta}\varphi(z))|^2 d\theta \right) |\varphi'(z)|^2 g^p(z, a) dA(z) \\
&= \frac{1}{32\pi} \int_{|\varphi(z)|>r} \int_0^{2\pi} \left| \sum_{k=1}^{\infty} 2^k (2^k - 1) (\lambda_n \varphi(z))^{2^k-2} e^{i\theta(2^k-2)} \right|^2 d\theta |\lambda_n|^4 |\varphi'(z)|^2 g^p(z, a) dA(z) \\
&= \frac{1}{16} \int_{|\varphi(z)|>r} \left( \sum_{k=1}^{\infty} [2^k (2^k - 1)]^2 |\lambda_n \varphi(z)|^{2(2^k-2)} \right) |\lambda_n|^4 |\varphi'(z)|^2 g^p(z, a) dA(z) \\
&\geq \frac{1}{16} \int_{|\varphi(z)|>r} \left( \sum_{k=1}^{\infty} (2^k - 2)^4 |\lambda_n \varphi(z)|^{2(2^k-2)} \right) |\lambda_n|^4 |\varphi'(z)|^2 g^p(z, a) dA(z). \quad (14)
\end{aligned}$$

Let  $F(r) = \sum_{k=1}^{\infty} (2^k - 2)^4 r^{2^{k+1}-4}$ . Since  $\log r \geq 2(r-1)$ ,  $r \in [\frac{1}{2}, 1)$ , we have  $r^{2^{k+1}-4} \geq \exp\{(2^{k+2} - 8)(r-1)\}$ ,  $r \in [1/2, 1)$ , so that

$$\begin{aligned}
F(r) &\geq \sum_{k=1}^{\infty} (2^k - 2)^4 \exp\{(2^{k+2} - 8)(r-1)\} \\
&= (1-r)^{-4} \sum_{k=1}^{\infty} [(2^k - 2)(1-r)]^4 \exp\{-4(2^k - 2)(1-r)\}. \quad (15)
\end{aligned}$$

Let  $t = (2^k - 2)(1-r)$ . Then the general term in series (15) is  $s(t) = t^4 e^{-4t}$ . It is easy to see that  $\sup_{t>0} t^4 e^{-4t} = e^{-4}$  is assumed for  $t = 1$  and  $s(t) = \frac{1}{8} e^{-2}$ . For  $r \in [3/4, 1)$ , we find  $k$  so that  $\frac{1}{2} \leq (2^k - 2)(1-r) \leq 1$ . For this  $k$ , we have

$$[(2^k - 2)(1-r)]^4 \exp\{-4((2^k - 2)(1-r))\} \geq \frac{e^{-2}}{8}.$$

Hence

$$F(r) \geq \frac{e^{-2}}{8} (1-r)^{-4}, \quad r \in [\frac{3}{4}, 1).$$

Therefore, for  $\delta < r < 1$  and for sufficient large  $n$ , (15) gives

$$\sup_{a \in \mathbb{D}} \int_{|\varphi(z)|>r} \frac{|\lambda_n|^4 |\varphi'(z)|^2}{(1 - |\lambda_n \varphi(z)|^2)^4} g^p(z, a) dA(z) < C\epsilon.$$

By Fatou's Lemma we get (13).

(iv)  $\Rightarrow$  (i). Assume that  $\varphi \in Q_p$  and (13) holds. Let  $\{f_n\}$  be a sequence in  $\mathbb{B}_{\mathcal{B}}$  which converges to 0 uniformly on compact subsets of  $\mathbb{D}$ . We need to show that  $\{C_{\varphi} f_n\} \rightarrow 0$  in  $Q_p$  space. From (13) for given  $\epsilon > 0$  there is an  $r \in (0, 1)$ , such that

$$\sup_{a \in \mathbb{D}} \int_{|\varphi(z)|>r} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^4} g^p(z, a) dA(z) < \epsilon.$$

Therefore

$$\begin{aligned} \int_{\mathbb{D}} |(C_{\varphi} D f_n)'(z)|^2 g^p(z, a) dA(z) &= \left\{ \int_{|\varphi(z)| \leq r} + \int_{|\varphi(z)| > r} \right\} |f_n''(\varphi(z))|^2 |\varphi'(z)|^2 g^p(z, a) dA(z) \\ &\leq \sup_{|w| \leq r} |f_n''(w)|^2 \|\varphi\|_{Q_p}^2 + \|f_n\|_{\mathcal{B}}^2 \epsilon. \end{aligned} \quad (16)$$

Since  $\{f_n\}$  converges to 0 uniformly on compacts of  $\mathbb{D}$ , we see that  $\{f_n'\}$  and  $\{f_n''\}$  also converges to 0 uniformly on compacts of  $\mathbb{D}$  by Cauchy's estimates. Thus  $\|C_{\varphi} D f_n\|_{Q_p} \rightarrow 0$  as  $n \rightarrow \infty$ , and consequently  $\|C_{\varphi} D f_n\|_p = \|C_{\varphi} D f_n\|_{Q_p} + |f_n'(\varphi(0))| \rightarrow 0$ , as  $n \rightarrow \infty$ . Hence  $C_{\varphi} D : \mathcal{B} \rightarrow Q_p$  is compact by Lemma 1.  $\square$

**Theorem 3.** Let  $p \in (0, \infty)$  and  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . Then the following statements are equivalent:

- (i)  $C_{\varphi} D : \mathcal{B}_0 \rightarrow Q_{p,0}$  is bounded;
- (ii)  $\varphi \in Q_{p,0}$  and

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^4} g^p(z, a) dA(z) < \infty. \quad (17)$$

*Proof.* (i)  $\Rightarrow$  (ii). If  $C_{\varphi} D : \mathcal{B}_0 \rightarrow Q_{p,0}$  is bounded, then  $C_{\varphi} D : \mathcal{B}_0 \rightarrow Q_p$  is bounded. By Theorem 1, (17) holds. Let  $f(z) = \frac{1}{2}z^2$ . Then it is easy to see that  $\varphi \in Q_{p,0}$ .

(ii)  $\Rightarrow$  (i). Assume that  $\varphi \in Q_{p,0}$  and (17) holds. By Theorem 1, we see that  $C_{\varphi} D : \mathcal{B}_0 \rightarrow Q_p$  is bounded. To prove that  $C_{\varphi} D : \mathcal{B}_0 \rightarrow Q_{p,0}$  is bounded, it suffices to prove that  $C_{\varphi} D f \in Q_{p,0}$  for any  $f \in \mathcal{B}_0$ . Let  $f \in \mathcal{B}_0$ . For every  $\epsilon > 0$ , we can choose  $\rho \in (0, 1)$  such that  $|f''(w)|(1 - |w|^2) < \epsilon$  for all  $w \in \mathbb{D} \setminus \rho\mathbb{D}$ . Then,

$$\begin{aligned} &\lim_{|a| \rightarrow 1} \int_{\mathbb{D}} |(C_{\varphi} D f)'(z)|^2 g^p(z, a) dA(z) \\ &= \lim_{|a| \rightarrow 1} \left( \int_{|\varphi(z)| > \rho} + \int_{|\varphi(z)| \leq \rho} \right) |f''(\varphi(z))|^2 |\varphi'(z)|^2 g^p(z, a) dA(z) \\ &\leq \epsilon \lim_{|a| \rightarrow 1} \int_{|\varphi(z)| > \rho} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^4} g^p(z, a) dA(z) + \frac{\|f\|_{\mathcal{B}}^2}{(1 - \rho^2)^4} \lim_{|a| \rightarrow 1} \int_{|\varphi(z)| \leq \rho} |\varphi'(z)|^2 g^p(z, a) dA(z). \end{aligned}$$

From the above inequality and by conditions (ii), we get the desired result.  $\square$

By modifying the proof of Theorem 4.2 of [13], we can prove the following result. We omit the details.

**Lemma 4.** Let  $p \in (0, \infty)$  and  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . Then  $C_{\varphi} D : \mathcal{B} \rightarrow Q_{p,0}$  is compact if and only if  $C_{\varphi} D : \mathcal{B} \rightarrow Q_{p,0}$  is bounded and

$$\lim_{|a| \rightarrow 1} \sup_{\|f\|_{\mathcal{B}} < 1} \int_{\mathbb{D}} |(C_{\varphi} D f)'(z)|^2 g^p(z, a) dA(z) = 0. \quad (18)$$

**Theorem 4.** Let  $p \in (0, \infty)$  and  $\varphi$  be an analytic self-map of  $\mathbb{D}$ . Then the following statements are equivalent:

- (i)  $C_\varphi D : \mathcal{B} \rightarrow Q_{p,0}$  is bounded;
- (ii)  $C_\varphi D : \mathcal{B} \rightarrow Q_{p,0}$  is compact;
- (iii)  $C_\varphi D : \mathcal{B}_0 \rightarrow Q_{p,0}$  is compact;
- (iv)  $C_\varphi D : \mathcal{B}_0 \rightarrow Q_{p,0}$  is weakly compact;
- (v)  $C_\varphi D(\mathcal{B}) \subset Q_{p,0}$ ;
- (vi)

$$I = \lim_{|a| \rightarrow 1} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^4} g^p(z, a) dA(z) = 0. \quad (19)$$

*Proof.* (iii)  $\Leftrightarrow$  (ii); (ii)  $\Rightarrow$  (i)  $\Rightarrow$  (v) are obvious. Lemma 3 gives (iii)  $\Leftrightarrow$  (iv).

(v)  $\Rightarrow$  (vi). Assume that  $C_\varphi D(\mathcal{B}) \subset Q_{p,0}$ . From the proof of Theorem 1, we can choose functions  $g_1, g_2 \in \mathcal{B}$  such that

$$(1 - |z|^2)^{-2} \leq |g_1''(z)| + |g_2''(z)|, \quad z \in \mathbb{D}.$$

Then we get  $C_\varphi D g_1, C_\varphi D g_2 \in Q_{p,0}$ . Therefore,

$$I \leq 2 \lim_{|a| \rightarrow 1} \int_{\mathbb{D}} (|g_1''(\varphi(z))|^2 + |g_2''(\varphi(z))|^2) |\varphi'(z)|^2 g^p(z, a) dA(z) = 0.$$

(vi)  $\Rightarrow$  (ii). Assume that (19) holds. By Theorem 1 we see that  $C_\varphi D : \mathcal{B} \rightarrow Q_p$  is bounded. We first prove that  $C_\varphi D : \mathcal{B} \rightarrow Q_{p,0}$  is bounded. It suffices to prove that  $C_\varphi D f \in Q_{p,0}$ . For any  $f \in \mathcal{B}$ , we have

$$\int_{\mathbb{D}} |(C_\varphi D f)'(z)|^2 g^p(z, a) dA(z) \leq \|f\|_{\mathcal{B}}^2 \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^4} g^p(z, a) dA(z). \quad (20)$$

Therefore (19) together with (20) imply that  $C_\varphi D : \mathcal{B} \rightarrow Q_{p,0}$  is bounded. Fix  $f \in \mathbb{B}_{\mathcal{B}}$ , the righthand side of (20) tends to 0, as  $|a| \rightarrow 1$  by (19). From Lemma 4, we see that  $C_\varphi D : \mathcal{B} \rightarrow Q_{p,0}$  is compact.

(iv)  $\Rightarrow$  (v). From Gantmacher's theorem we know that  $C_\varphi D : \mathcal{B}_0 \rightarrow Q_{p,0}$  is weakly compact if and only if  $(C_\varphi D)^{**}((\mathcal{B}_0)^{**}) \subset Q_{p,0}$ . By adopting the proof of Theorem 2 in [6], and using the facts  $((Q_{p,0})^*)^* = Q_p$  ( $0 < p < 2$ , see [1]),  $((\mathcal{B}_0)^*)^* = \mathcal{B}$  and  $(A^1)^* = \mathcal{B}$  we get  $(C_\varphi D)^{**}(f) = C_\varphi D(f)$  for every  $f \in \mathcal{B}$ . Hence  $(C_\varphi D)^{**}((\mathcal{B}_0)^{**}) = C_\varphi D(\mathcal{B}) \subset Q_{p,0}$ , as desired.  $\square$

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# Some Theorems and Examples of Cone Banach Spaces

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## Abstract

In this paper, by defining a cone norm  $\|\cdot\|_A$  on  $E$  over itself which behaves like the absolute value norm on  $R$ , we construct examples of cone Banach spaces. Namely, we define the  $m$ -Euclidian cone normed space  $E^m$ ,  $E^\infty$  and the space  $C_E(S)$  of continuous functions in cones, to generalize the Banach spaces  $R^m, l^\infty$  and  $C[a, b]$ , respectively. Some basic lemmas and theorems are also proved to help in the construction and in the proof of completeness of the above mentioned examples of cone normed spaces.

*Key words:* Cone metric space, cone normed space, cone Banach space, strongly minihedral cone, limit point in the sense of cone,  $A$ -property, generalized absolute value property, space  $E^m$ , space  $E^\infty$ .

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## 1 Introduction

In [1], cone metric spaces were introduced by means of a partial ordering " $\leq$ " on a Banach space  $(E, \|\cdot\|)$  via a cone  $P$ , where some fixed point theorems were proved to generalize the corresponding ones in metric spaces. Then the authors of this article and in [4], introduced the notion of cone normed space, where continuous linear operators and bounded linear operators between cone normed spaces were studied too. Recently, in [2], the authors proved that cone metric spaces are topological spaces and accordingly cone normed spaces are also topological spaces. Moreover, compactness, boundedness, closedness, first countability were discussed there.

In this paper we enrich the theory of cone normed spaces by constructing new examples of cone Banach spaces via a cone norm on  $E$  which behaves like the absolute value in  $R$ .

## 2 Preliminaries on Cone Metric Spaces and Cone Normed Spaces

In this section we review some basic definitions and theorems in cone metric spaces and cone normed spaces. For more details, we may refer the reader to the articles [1],[2],[4].

**Definition 1** [1] *Let  $E$  be a real Banach space with norm  $\|\cdot\|$  and  $P$  a subset of  $E$ . Then,  $P$  is called a cone if and only if*

*$P1)$   $P$  is closed, nonempty and  $P \neq \{0\}$*

*$P2)$   $a, b \in R$   $a, b \geq 0$ ;  $x, y \in P \Rightarrow ax + by \in P$*

*$P3)$   $x \in P$  and  $-x \in P \Rightarrow x = 0$*

Given a cone  $P \subset E$ , we define a partial ordering  $\leq$  with respect to  $P$  by  $x \leq y$  if and only if  $y - x \in P$ . We write  $x < y$  to indicate that  $x \leq y$  but  $x \neq y$ , while  $x << y$  will stand for  $y - x \in \text{Int}P$ . ( $\text{Int}P \cong \text{interior of } P$ ). The cone  $P$  is called normal if there is a number  $K > 0$ , such that for all  $x, y \in E$ ,  $0 \leq x \leq y$  implies  $\|x\| \leq K \|y\|$ , where  $K$  is called the normal constant of  $P$ . The cone  $P$  is called regular if every increasing sequence which is bounded from above is convergent. Equivalently the cone  $P$  is called regular if every decreasing sequence which is bounded from below is convergent [1].

$P$  is called minihedral cone if  $\sup\{x, y\}$  exists for all  $x, y \in E$ , and strongly minihedral if every subset of  $E$  which is bounded from above has a supremum. A norm  $\|\cdot\|$  on  $E$  is called monotonic if  $0 \leq x \leq y$  implies  $\|x\| \leq \|y\|$ , and semi-monotonic if  $\|x\| \leq K \|y\|$  for some  $K > 0$  and all  $x$  and  $y$  such that  $0 \leq x \leq y$  [3]. It is known in [3] that  $P$  is normal if and only if  $\|\cdot\|$  is semi-monotonic.

Throughout, we will assume that our cone  $P$  is strongly minihedral and hence every subset of  $E$  which is bounded below has infimum. Hence, In particular, every subset of  $P$  has infimum.

**Example 2** *Let  $E = R^2$  and  $P = \{(x, y) : x \geq 0, y \geq 0\}$ . It is easy to see that  $P$  is strongly minihedral in which clearly each bounded below subset of  $E$  has infimum.*

**Definition 3** [1] *A cone metric space is an ordered pair  $(X, d)$ , where  $X$  any*



set and  $d : X \times X \rightarrow E$  is a mapping satisfying:

d1)  $0 < d(x, y)$  for all  $x, y \in X$ , and  $d(x, y) = 0$  if and only if  $x = y$ .

d2)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ .

d3)  $d(x, y) \leq d(x, z) + d(z, y)$  for all  $x, y, z \in X$ .

**Definition 4** [1] Let  $(X, d)$  be a cone metric space, let  $\{x_n\}$  be a sequence in  $X$  and  $x \in X$ . If for any  $c \in E$  with  $c \gg 0$ , there is  $N$  such that for all  $n > N$ ,  $d(x_n, x) \ll c$  then  $\{x_n\}$  is said to be convergent with limit  $x$ . [ i.e.  $\lim_{n \rightarrow \infty} x_n = x$  or  $x_n \rightarrow x$  as  $n \rightarrow \infty$  ].

**Definition 5** [1] Let  $(X, d)$  be a cone metric space,  $\{x_n\}$  be a sequence in  $X$ , if for any  $c \in E$  with  $c \gg 0$ , there is  $N$  such that for all  $n, m > N$ ,  $d(x_m, x_n) \ll c$  then  $\{x_n\}$  is called a Cauchy sequence in  $X$ .

**Definition 6** [4] A cone normed space is an ordered pair  $(X, \|\cdot\|_c)$  where  $X$  is a vector space over  $R$  and  $\|\cdot\|_c : X \rightarrow (E, P, \|\cdot\|)$  is a function satisfying:

C1)  $0 < \|x\|_c$ , for all  $x \in X$ .

C2)  $\|x\|_c = 0$  if and only if  $x = 0$ .

C3)  $\|\alpha \cdot x\|_c = |\alpha| \|x\|_c$ , for each  $x \in X$  and  $\alpha \in R$ .

C4)  $\|x + y\|_c \leq \|x\|_c + \|y\|_c$ ,  $x, y \in X$ .

It is easy to see that each cone normed space is cone metric space. Namely, the cone metric is defined by  $d(x, y) = \|x - y\|_c$ .

According to what we mentioned above, we say that a sequence  $\{x_n\}$  of a cone normed space  $(X, \|\cdot\|_c)$  over  $(E, P, \|\cdot\|)$  is said to be convergent, if there exists  $x \in X$  such that for all  $c \gg 0$ ,  $c \in E$ , there exists  $n_0$  such that  $\|x_n - x\|_c \ll c$  for all  $n \geq n_0$ . Also, we say that  $\{x_n\}$  is Cauchy if for each  $c \gg 0$ , there exists  $n_0$  such that  $\|x_m - x_n\|_c \ll c$  for all  $m, n \geq n_0$ .

**Definition 7** [4] A cone normed space  $(X, \|\cdot\|_c)$  is called cone Banach space if every Cauchy sequence in  $X$  is convergent in  $X$ .

**Lemma 8** Let  $\{x_n\}$  be a Cauchy sequence in a cone metric space  $(X, d)$ , such that  $\lim_{n \rightarrow \infty} x_{k_n} = x$ . Then  $\lim_{n \rightarrow \infty} x_n = x$ .

**PROOF.** Let  $c \gg 0$ ,  $c \in E$  be given. Then by assumption find  $n_0$  such that  $d(x_n, x_m) \ll \frac{c}{2}$ ,  $\forall m, n \geq n_0$  and  $d(x_{k_n}, x) \ll \frac{c}{2}$ ,  $\forall n \geq n_0$ . Then for  $n \geq n_0$ ,

$$d(x_n, x) \leq d(x_n, x_{k_{n_0}}) + d(x_{k_{n_0}}, x) < \frac{c}{2} + \frac{c}{2} = c.$$

**Definition 9** [2] Let  $(X, d)$  be a cone metric space. Then  $A \subset X$  is called bounded above if there exists  $c \in E, c \gg 0$  such that  $d(x, y) \leq c$ , for all  $x, y \in A$ , and is called bounded if  $\delta(A) = \sup \{d(x, y) : x, y \in A\}$  exists in  $E$ . If the supremum does not exist we say that  $A$  is unbounded.

**Lemma 10** Every Cauchy sequence of a cone metric space  $(X, d)$  over a strongly minihedral cone, is bounded.

**PROOF.** For some  $c \gg 0$ , find  $n_0$  such that  $d(x_n, x_m) \leq c, \forall m, n \geq n_0$ . Let  $c' = \sup \{c, d(x_n, x_m) : m, n < n_0\}$  which exists, since  $P$  is strongly minihedral. That is,  $d(x_n, x_m) \leq c' \forall m, n$  and  $P$  is strongly minihedral implies that

$$\sup \{d(x_n, x_m) : m, n \in \mathbb{N}\} \quad (1)$$

exists, and hence,  $\{x_n\}$  is bounded.

### 3 Main Results

**Definition 11** [4] A cone norm  $\|\cdot\|_A : E \rightarrow E$  is said to satisfy the  $A$ -property (absolute value property) if  $-c \leq a \leq c$  if and only if  $\|a\|_A \leq c$ , for all  $a \in E$  and  $c \gg 0$ .

**Example 12** [4] Let  $E = R^2$  and  $P = \{(x, y) \in R^2 : x \geq 0, y \geq 0\}$ . Then the norm  $\|\cdot\|_A : E \rightarrow P$  defined by  $\|(x, y)\|_A = (|x|, |y|)$ , satisfies the  $A$ -property. Indeed  $a = (x, y)$ , satisfies  $\|a\|_A \leq c = (c_1, c_2), c_1, c_2 > 0$  if and only if  $(|x|, |y|) \leq (c_1, c_2)$  if and only if  $-c = (-c_1, -c_2) \leq a = (x, y) \leq (c_1, c_2) = c$ .

**Definition 13** A cone norm  $\|\cdot\|_A$  on  $E$  is said to have the generalized absolute value property if it satisfies:

A1) if there exists  $k_1 > 0$  such that  $0 \leq \|x\|_A \leq c, c \gg 0$  implies  $-k_1 c \leq x \leq k_1 c$  ( for all  $x \in X$ ).

A2) if there exists  $-k_1 > 0$  such that  $c \leq x \leq c, c \gg 0$  implies  $0 \leq \|x\|_A \leq k_2 c$  ( for all  $x \in X$ ).

**Example 14** Let  $E = R^2$  and  $P = \{(x, y) \in R^2 : x \geq 0, y \geq 0\}$ . Define the cone norm  $\|(x, y)\|_A = (\alpha |x|, \beta |y|)$ , where  $\alpha, \beta > 0$ . Then, one can easily show that the cone norm  $\|\cdot\|_A$  has the generalized absolute value property.

It is clear that if the  $\|\cdot\|_A$  on  $E$  satisfies the absolute value property, then it satisfies the generalized absolute value with  $k_1 = k_2 = 1$ .

**Definition 15** Let  $S$  be a subset of  $E$ . Then,

- a)  $S$  is called " $\|\cdot\|$ " bounded if  $\exists M > 0$  such that  $\|x\| \leq M$ , for all  $x \in S$ .
- b)  $S$  is called " $\leq$ " bounded if  $\exists c_1, c_2 \in E$  such that  $c_1 \leq x \leq c_2$ , for all  $x \in S$ .
- c)  $S$  is called " $\|\cdot\|_A$ " bounded if  $\sup_{x \in S} \|x\|_A$  exists in  $E$ .

**Remark 16** Let  $S$  be a subset of  $E$ . Then

- (1) If  $P$  is normal and  $S \subset P$  then " $\leq$ " boundedness implies " $\|\cdot\|$ " boundedness.
- (2) Assume  $\|\cdot\|_A$  has the generalized absolute value property and  $P$  is strongly minihedral. If  $S$  is " $\leq$ " bounded then it is " $\|\cdot\|_A$ " bounded.
- (3) If  $P$  is strongly minihedral then  $S \subset E$  is " $\|\cdot\|_A$ " bounded if and only if  $\exists c \gg 0$  such that  $\|x\|_A \leq c$ , for all  $x \in S$ .

**Corollary 17** Every Cauchy sequence of a cone normed space  $(X, \|\cdot\|_c)$  over a strongly minihedral cone is bounded. In particular every Cauchy sequence in  $(E, \|\cdot\|_A)$  is " $\|\cdot\|_A$ " bounded.

The proof follows by the Lemma 10 and that every cone normed space is cone metric space.

**Definition 18** Let  $\{x_n\}$  be an " $\leq$ " bounded sequence of  $E$ . The assumptions on our cone make it possible to define the limit superior of  $\{x_n\}$  by  $\limsup x_n = \inf_{n \geq 1} (\sup_{k \geq n} x_k)$  and the limit inferior of  $\{x_n\}$  by  $\liminf x_n = \sup_{n \geq 1} (\inf_{k \geq n} x_k)$ .

From the above definition, it follows that  $\left(\sup_{k \geq n} x_k\right) \downarrow_n \limsup x_n$  and  $\left(\inf_{k \geq n} x_k\right) \uparrow_n \liminf x_n$ . Of course, the monotonicity is given by means of the partial ordering " $\leq$ ".

**Definition 19** An element  $a \in E$  is said to be a limit point in the sense of cone ( or cluster point ) for a sequence  $\{a_n\}$  in  $E$ , if for every  $n \in \mathbb{N}$  and  $c \gg 0$ , there exists  $k > n$  (depending on  $c$  and  $n$ ) such that  $\|x_k - a\|_A \ll c$ .

$P$  is called normal with constant  $\alpha$  for the cone norm  $\|\cdot\|_A$ , if  $0 \leq x \leq y$  implies  $\|x\|_A \leq \alpha y$ , for all  $x, y \in E$ .

**Theorem 20** Every monotone increasing " $\leq$ " bounded sequence  $\{x_n\}$  of  $(E, P, \|\cdot\|_A)$  where  $P$  is strongly minihedral and normal in the cone norm  $\|\cdot\|_A$  with normal constant  $\alpha$  is " $\|\cdot\|_A$ " convergent (i.e. there exists  $x \in E$  such that for each  $c \gg 0$  there exists  $n_0$  such that  $\|x_k - x\|_A \ll c$  for all  $k \geq n_0$ ). Moreover, if every sequence in  $E$  which is bounded below has an infimum then every monotone decreasing " $\leq$ " bounded sequence is also " $\|\cdot\|_A$ " convergent.

**PROOF.** Assume  $\{x_i\}$  is increasing and bounded. Hence, there exists  $x \in E$  such that  $x = \sup \{x_i : i \in \mathbb{N}\}$ . We claim that  $\lim_{i \rightarrow \infty} x_i = x$ . To prove our claim let  $c \gg 0$ , then by the definition of supremum, there exists  $i_0$  such that  $x - \frac{c}{\alpha} < x_{i_0} \leq x$ . Thus, for  $i > i_0$  we have  $0 \leq x - x_i \leq x - x_{i_0} < \frac{c}{\alpha}$ , so that by the normality of  $P$  in the cone norm  $\|\cdot\|_A$  we have  $\|x_i - x\|_A \leq \alpha \frac{c}{\alpha} = c$ . Hence  $\lim_{i \rightarrow \infty} x_i = x$ . The proof of the other part can be done similarly, but by using the infimum definition.

**Remark 21** One has to note that when  $P$  is " $\|\cdot\|_A$ " normal with constant  $K$ , then  $x_n \rightarrow x$  in  $(E, \|\cdot\|_A)$  if and only if  $\|\|x_n - x\|_A\| \rightarrow 0$  as  $n \rightarrow \infty$ . This is clear by Lemma 1 in [1] and that each cone normed space is cone metric space.

**Theorem 22** Let  $\{x_n\}$  be a sequence in  $(E, \|\cdot\|_A)$ . Then an element  $x \in E$  is a limit point in the sense of cone metric space for  $\{x_n\}$  if and only if there exists a subsequence  $\{x_{k_n}\}$  of  $\{x_n\}$  such that  $\lim_{n \rightarrow \infty} x_{k_n} = x$ .

**PROOF.** Assume  $x$  is a limit point of  $\{x_n\}$ . For some fixed  $c_0 \gg 0$ , find  $k_1$  such that  $\|x_{k_1} - x\|_A \ll c_0$ . Now inductively, if  $k_1, k_2, \dots, k_n$  have been selected, then choose  $k_{n+1} > k_n$  such that  $\|x_{k_{n+1}} - x\|_A \ll \frac{c_0}{n+1}$ . Thus, we construct a sequence of natural numbers  $\{k_n\}$  such that  $k_1 < k_2 < \dots$  and  $0 \ll \|x_{k_n} - x\|_A \ll \frac{c_0}{n} \in P$ . Now, if  $c \gg 0$  is arbitrary given (by Lemma 2 in [2] and see also the proof of Lemma 3 in [2]) find  $n_0$  such that  $\frac{c_0}{n_0} \ll c$  and hence, for  $n > n_0$ , we have  $\|x_{k_n} - x\|_A \ll \frac{c_0}{n} \ll \frac{c_0}{n_0} \ll c$ . Therefore,  $\lim_{n \rightarrow \infty} x_{k_n} = x$ .

To prove the reverse implication consider a subsequence  $\{x_{k_n}\}$  of  $\{x_n\}$  such that  $\lim_{n \rightarrow \infty} x_{k_n} = x$ . Let  $m$  and  $c \gg 0$  be given. It must be shown that there exists  $p > m$  such that  $\|x_p - x\|_A \ll c$ . To this end, choose  $n_0$  such that  $\|x_{k_n} - x\|_A \ll c$  if  $n > n_0$ . Pick  $i_0 > \max\{n_0, m\}$ , and set  $p = k_{i_0}$ . But, then  $p > m$  (since  $k_i \geq i$ ), and  $\|x_p - x\|_A \ll c$ . The proof is finished.

**Definition 23** Let  $X$  be a vector space. A function  $\|\cdot\|_k : X \rightarrow R$  is called a  $k$ -norm if

$$k1) \ 0 \leq \|x\|_k < \infty \text{ for all } x \in X.$$

$$k2) \ \|x\|_k = 0 \text{ if and only if } x = 0.$$

$$k3) \ \|\lambda x\|_k = |\lambda| \|x\|_k \text{ for all } x \in X.$$

$$k4) \ \|x + y\|_k \leq k(\|x\|_k + \|y\|_k) \text{ for all } x, y \in X.$$

**Example 24** If  $P$  is  $\|\cdot\|_k$  normal with constant  $k$  and  $\|\cdot\|_c$  is a cone norm on  $X$  over  $(E, P, \|\cdot\|)$ , then it can be shown easily that  $\|\cdot\|_k \triangleq \|\|\cdot\|_c\|$  is a  $k$ -norm with constant  $k$ .

It is well known that regular cones are normal, for the proof see for example [5]. Theorem 20 states conditions under which  $\|\cdot\|_A$ -normality implies  $\|\cdot\|_A$ -regularity. Namely, under the above mentioned conditions we proved that every monotone " $\leq$ " bounded sequence is " $\|\cdot\|_A$ " convergent and hence, convergent in the scalar  $k$ -norm  $\|\|\cdot\|_A\|$  when our cone is normal.

**Theorem 25** If  $\{x_n\}$  is " $\leq$ " bounded sequence in  $(E, P, \|\cdot\|_A)$ , where we assume that  $P$  is strongly minihedral and  $\|\cdot\|_A$  has  $A$ -property. Then,  $\liminf x_n$  and  $\limsup x_n$  are the smallest and largest limit points of  $\{x_n\}$  in the sense of cone. In particular,  $\liminf x_n \leq \limsup x_n$ .

**PROOF.** Let  $x_n$  be a " $\leq$ " bounded sequence of  $E$ . ( since  $P$  is strongly minihedral cone ) then  $s = \limsup x_n$  exists. We shall show that  $s$  is the largest limit point in the sense of cone for  $\{x_n\}$ . The other case can be shown in a similar manner. We show that  $s$  is a limit point in the sense of cone firstly. To this end, let  $m \in \mathbb{N}$  and  $c \gg 0$ ,  $c \in E$ . Since  $\left\{ \sup_{k \geq n} x_k \right\} \downarrow_n s$ , there exists  $n > m$  such that  $s \leq \sup_{k \geq n} x_k < s + c$ . This implies the existence of some  $k \geq n > m$  such that  $s - c < x_k < s + c$ . Hence,  $s$  is a limit point in the sense of cone for  $\{x_n\}$ .

To finish the proof, we show that  $s$  is the largest limit point in the sense of cone. Let  $x$  be a limit point of  $\{x_n\}$ , and let  $c \gg 0$ . Then for each  $n \in \mathbb{N}$ , there exists  $m > n$  such that  $\|x_m - x\|_A < c$ . Since  $\|\cdot\|_A$  has the  $A$ -property then  $x - c < x_m < x + c$ . It follows that  $x - c < x_m < \sup_{k \geq n} x_k$  for each  $n$ ,

and so,  $x - c \leq \left( \inf_{n \geq 1} \sup_{k \geq n} x_k \right) = s$  for each  $c \gg 0$ . Thus  $x \leq s$ , and the proof is complete.

**Theorem 26** Assume  $\|\cdot\|_A$  has the  $A$ -property then a " $\leq$ " bounded sequence  $\{x_n\}$  in  $E$  is " $\|\cdot\|_A$ " convergent if and only if  $\limsup x_n = \liminf x_n = x$ . In this case,  $\lim_{n \rightarrow \infty} x_n = x$ .

**PROOF.** We assume  $\limsup x_n = \liminf x_n = x$  and show that  $\lim_{n \rightarrow \infty} x_n = x$ . Noting that  $\|\cdot\|_A$  has the  $A$ -property, the inequalities  $x_n - x \leq \sup_{k \geq n} x_k - \inf_{k \geq n} x_k$  and  $x - x_n \leq \sup_{k \geq n} x_k - \inf_{k \geq n} x_k$  imply that

$$\|x_n - x\|_A \leq \sup_{k \geq n} x_k - \inf_{k \geq n} x_k. \quad (2)$$

Since  $\lim_{n \rightarrow \infty} \left( \sup_{k \geq n} x_k - \inf_{k \geq n} x_k \right) = x - x = 0$ , it follows that  $\lim_{n \rightarrow \infty} x_n = x$  in the sense of cone, and the proof is complete.

It is to be noted that Theorem 26 is still valid, even when  $\|\cdot\|_A$  has the generalized absolute value property. It is just a matter of multiplying the right hand side of inequality (2) by  $k_2$ .

**Lemma 27** Let  $\|\cdot\|_A$  be a cone norm on  $E$  with the  $A$ -property and assume that  $P$  is strongly minihedral. Then, every  $\|\cdot\|_A$  Cauchy sequence is " $\leq$ " bounded.

**PROOF.** Let  $c \gg 0$  be fixed and find  $n_0$  such that

$$\|x_n - x_m\|_A \ll c \text{ for all } n, m > n_0 \quad (3)$$

Since  $\|\cdot\|_A$  has the  $A$ -property then (3) is equivalent to

$$-c \leq x_n - x_m \leq c \text{ for all } n, m \geq n_0 \quad (4)$$

From which it follows that

$$x_n \leq x_{n_0} + c \text{ for all } n \geq n_0 \quad (5)$$

and hence  $x_n \leq b$  where  $b = \sup \{x_{n_0} + c, x_1, x_2, \dots, x_{n_0-1}\}$  which exists by minihedrality of  $P$ . The proof is finished.

Now, as a generalization to the result obtained in [4] in Example 24, we obtain the following theorem.

**Theorem 28** *Let  $\|\cdot\|_A$  be a cone norm on  $E$  over  $E$  with the  $A$ -property and assume that  $P$  is strongly minihedral. Then, a sequence  $\{x_n\}$  in  $E$  is  $\|\cdot\|_A$  convergent if and only if it is  $\|\cdot\|_A$  Cauchy. That is,  $(E, \|\cdot\|_A)$  is a cone Banach space.*

**PROOF.** Clearly if  $x_n$  is  $\|\cdot\|_A$  convergent then it is  $\|\cdot\|_A$  Cauchy. Assume  $x_n$  is  $\|\cdot\|_A$  Cauchy. Then by Lemma 27 and that  $P$  is strongly minihedral we conclude that  $x = \limsup x_n$  exists and by Theorem 25 is a limit point in the sense of cone. Now, let  $\epsilon > 0$  be given and find  $n_1$  such that  $\|x_n - x_m\|_A < \frac{\epsilon}{2}$  for all  $m, n \geq n_1$ . By Theorem 22, there exists a subsequence  $\{x_{k_n}\}$  of  $\{x_n\}$  such that  $\lim_{n \rightarrow \infty} x_{k_n} = x$ . Hence, by Lemma 8,  $\lim_{n \rightarrow \infty} x_n = x$ .

**Remark 29** *One can easily infer that Theorem 25, and Lemma 27, are still valid when the cone norm  $\|\cdot\|_A$  has the generalized absolute value property and hence, Theorem 28, is still valid under the generalized absolute value property assumption.*

**Definition 30** *Let  $(E, \|\cdot\|_A)$  be a cone normed space over  $(E, P, \|\cdot\|)$  and  $m$  a positive integer. We define the  $m$ -Euclidean cone normed space ( or finite dimensional cone normed space) by*

$$E^m = \left\{ x = (x_1, x_2, \dots, x_m) : x_i \in E, \sup_{1 \leq i \leq m} \|x_i\|_A \text{ exists} \right\} \quad (6)$$

where always it is assume that  $\|\cdot\|_A$  has  $A$ -property ( or more generally, the generalized absolute value property ). On  $E^m$  define the following  $m$ -Euclidean cone norm  $\|x\|_e = \sup_{1 \leq i \leq m} \|x_i\|_A, x = (x_1, x_2, \dots, x_m) \in E^m$ .

The following essential lemma is helpful in proving that  $(E^m, \|\cdot\|_e)$  is a cone Banach space.

**Lemma 31** *If  $(X, \|\cdot\|_c)$  is a cone normed space over  $(E, \|\cdot\|, \|\cdot\|_A, P)$ , where  $\|\cdot\|_A$  has  $A$ -property. Then  $\|\cdot\|_c$  satisfies the " $\leq$ " inequality  $\|x\|_c - \|y\|_c \leq \|x - y\|_c$  and hence,  $\|\cdot\|_c$  is a uniformly continuous function from  $(X, \|\cdot\|_c)$  to  $(E, \|\cdot\|_A)$ .*

**PROOF.** For  $x, y \in X$  we have by the triangle inequality  $\|x\|_c = \|x - y + y\|_c \leq \|x - y\|_c + \|y\|_c$  or

$$\|x\|_c - \|y\|_c \leq \|x - y\|_c \quad (7)$$

also,

$$\|y\|_c = \|y - x + x\|_c \leq \|y - x\|_c + \|x\|_c = \|x - y\|_c + \|x\|_c \text{ or}$$

$$\|y\|_c - \|x\|_c \leq \|x - y\|_c \quad (8)$$

Since  $\|\cdot\|_A$  has the  $A$ -property then (7) and (8) implies that  $\|\|x\|_c - \|y\|_c\|_A \leq \|x - y\|_c$ .

**Remark 32** *If in Lemma 31, more generally, the cone norm  $\|\cdot\|_A$  has the generalized absolute value property, then the cone norm  $\|\cdot\|_c$  satisfies the inequality  $\|\|x\|_c - \|y\|_c\|_A \leq k_2 \|x - y\|_c$ . Hence,  $\|\cdot\|_c$  is a uniformly continuous function from  $(X, \|\cdot\|_c)$  to  $(E, \|\cdot\|_A)$ .*

**Theorem 33**  $(E^m, \|\cdot\|_e)$  is a cone Banach space.

**PROOF.** Using that  $\|\cdot\|_A$  is a cone norm on  $E$ , it can be easily seen that  $\|\cdot\|_e$  verifies the cone norm axioms. To show that  $(E^m, \|\cdot\|_e)$  is complete, let  $x_n = (x_1^n, x_2^n, \dots, x_m^n)$  be an " $\|\cdot\|_e$ " Cauchy sequence. Hence for  $c \gg 0, c \in E$ , find  $n_0$  such that  $\|x_n - x_k\|_e = \sup_{1 \leq i \leq m} \|x_i^n - x_i^k\|_A < \frac{c}{2}$ , (For all  $k, n \geq n_0$ ). Hence, for each  $1 \leq i \leq m$ , we have

$$\|x_i^n - x_i^k\|_A < \frac{c}{2} \quad (\text{For all } k, n \geq n_0) \quad (9)$$

This shows that  $\{x_i^n\}$  is a Cauchy sequence in  $(E, \|\cdot\|_A)$  for each  $i = 1, 2, \dots, m$ . By Theorem 28, for each  $i = 1, 2, \dots, m$ , find  $z_i \in E$  such that  $x_i^n$  converges to  $z_i$  in  $(E, \|\cdot\|_A)$ . Let  $z = (z_1, z_2, \dots, z_m)$ . We show that  $z \in E^m$  and  $x_n \rightarrow z$  in  $(E, \|\cdot\|_A)$ . Lemma 31 and that  $P$  is closed lead to  $\|x_i^n - z_i\|_A \leq \frac{c}{2}$  for all  $k \geq n_0$ . Since  $x_k = (x_1^k, x_2^k, \dots, x_m^k) \in E^m$ , there is  $c_k > 0$  such that  $\|x_j^k\|_A \leq c_k$  for all  $j = 1, 2, 3, \dots, m$  (i.e.  $\|x_k\|_A$  exists). Hence, by the triangle inequality for each  $j = 1, 2, 3, \dots, m$ , we have  $\|z_j\|_A \leq \|z_j - x_j^k\|_A + \|x_j^k\|_A \leq c + c_k$  for all  $k \geq n_0$ . The last inequality holds for every  $j$ , and the right-hand side does not involve  $j$  and hence,  $z = (z_1, z_2, \dots, z_m) \in E^m$ . Also, from (9) we obtain  $\|x_k - z\|_e = \sup_{1 \leq j \leq m} \|x_j^k - z_j\|_A \leq \frac{c}{2} < c$  for all  $k \geq n_0$ . This shows that  $x_n \rightarrow z$  in  $(E^m, \|\cdot\|_e)$ .

**Definition 34** Let  $(E, \|\cdot\|_A)$  be a cone Banach space over  $(E, P, \|\cdot\|)$ . Then the cone normed sequence space  $E^\infty$  is defined by

$$E^\infty = \left\{ x = (x_1, x_2, \dots) : \sup_i \|x_i\|_A \text{ exists in } E \right\} \quad (10)$$



with the norm  $\|x\|_\infty = \sup_i \|x_i\|_A$ . More generally the cone normed space  $E^\Delta$  where  $\Delta$  is any index set is defined by

$$E^\Delta = \left\{ x = (x_\alpha)_{\alpha \in \Delta} : x_\alpha \in E, \sup_{\alpha \in \Delta} \|x_\alpha\|_A \text{ exists in } E \right\} \quad (11)$$

**Theorem 35**  $(E^\infty, \|\cdot\|_\infty)$  is a cone Banach space.

**PROOF.** To show that  $\|\cdot\|_\infty$  is a cone norm is straightforward. Let  $x_n = (x_1^n, x_2^n, \dots, x_p^n, \dots)$  be an  $\|\cdot\|_\infty$  Cauchy sequence in  $E^\infty$ . Then for each  $c >> 0$  there exists  $n_0$  such that  $\|x_n - x_k\|_\infty = \sup_j \|x_j^n - x_j^k\|_A \leq c$  for all  $n, k \geq n_0$ . Hence, for every fixed  $j$ , we have

$$\|x_j^n - x_j^k\|_A \leq c \text{ for all } n, k \geq n_0. \quad (12)$$

From which it follows that, the sequence  $(x_j^1, x_j^2, \dots)$  is a Cauchy sequence in  $(E, \|\cdot\|_A)$  for all  $j = 1, 2, 3, \dots$ . It converges by Theorem 28, say,  $x_j^n \rightarrow z_j \in E$ . Define  $z = (z_1, z_2, \dots)$  and show that  $x_n \rightarrow z$  in  $(E^\infty, \|\cdot\|_\infty)$ . From (12) and Lemma 31, with  $k \rightarrow \infty$  we have

$$\|x_j^n - z_j\|_A \leq c \text{ for all } n \geq n_0. \quad (13)$$

Since  $x_n \in E^\infty$ , for each  $n$  find  $c_n >> 0$  such that  $\|x_j^n\|_A \leq c_n$  for all  $j$ . Hence, by the triangle inequality  $\|z_j\|_A \leq \|z_j - x_j^n\|_A + \|x_j^n\|_A \leq c + c_n$  for all  $n \geq n_0$ . This inequality holds for every  $j$ , and the right hand-side does not involve  $j$ . Hence, by using that  $E$  is strongly minihedral,  $z = (z_1, z_2, \dots) \in E^\infty$ . Also, from (13) we obtain  $\|x_n - z\|_\infty = \sup_j \|x_j^n - z_j\|_A \leq c$  for all  $n \geq n_0$ . This shows that  $x_n \rightarrow z$  in  $(E^\infty, \|\cdot\|_\infty)$ .

**Remark 36** As we proved that  $(E^\infty, \|\cdot\|_\infty)$  is a cone Banach, we can also easily prove that  $(E^\Delta, \|\cdot\|_\Delta)$  is cone Banach with the norm  $\|x\|_\Delta = \sup_{\alpha \in \Delta} \|x_\alpha\|_A$ ,  $x = (x_\alpha) \in E^\Delta$ .

**Theorem 37** Let  $S$  be a subset of a complete cone metric space  $(X, d)$ . Then  $S$  is closed if and only if  $S$  is complete.

**PROOF.** It is not difficult to prove that  $S$  is closed if and only if whenever  $x_n \in S$ ,  $x_n \rightarrow x$  then  $x \in S$ . Now, assume  $S$  is closed and let  $x_n \in S$  be a Cauchy sequence. Since  $(X, d)$  is complete, find  $x \in X$  such that  $x_n \rightarrow x$ . Since  $x_n \in S$  and  $S$  is closed then  $x \in S$ . Conversely, assume  $S$  is complete and let  $x_n \in S$  with  $x_n \rightarrow x$ . We have to show that  $x \in S$ . Since  $x_n \rightarrow x$ , then

$x_n$  is Cauchy sequence in  $S$  complete. Hence, find  $y \in S$  such that  $x_n \rightarrow y$ . Since limits are unique in cone metric spaces [4] then  $x = y \in S$ .

**Example 38** Let  $C_E$  be the space of all  $\|\cdot\|_A''$  convergent sequences  $x = (x_1, x_2, \dots)$  in  $(E, \|\cdot\|_A)$ . This is a subspace of  $E^\infty$ , since each  $\|\cdot\|_A''$  convergent sequence in  $E$  is  $\|\cdot\|_A''$  bounded by Lemma 10. To show that  $(C_E, \|\cdot\|_\infty)$  is a cone Banach space, by Theorem 37, it would be enough to prove that  $C_E$  is a (sequentially) closed subspace of  $E^\infty$ . Let  $x_n = (x_1^n, x_2^n, \dots)$  be a sequence in  $C_E$  such that  $x_n \rightarrow x = (x_1, x_2, \dots)$ . To show that  $x = (x_1, x_2, \dots)$  is a convergent sequence (i.e.  $x \in C_E$ ), by noting that  $(E, \|\cdot\|_A)$  is complete, it would be enough to prove that  $x = (x_1, x_2, \dots)$  is Cauchy sequence in  $(E, \|\cdot\|_A)$ . Let  $c \gg 0, c \in E$  be given. Find  $n_0 \in \mathbb{N}$  such that  $\|x_n - x\|_\infty = \sup_j \|x_j^n - x_j\|_A << \frac{c}{3}$  for all  $n \geq n_0$ . In particular,  $\|x_j^{n_0} - x_j\|_A << \frac{c}{3}$  for all  $j$ . Since  $x_{n_0} = (x_1^{n_0}, x_2^{n_0}, \dots) \in C_E$  then it is Cauchy in  $(E, \|\cdot\|_A)$ . Hence, find  $n_1 \in \mathbb{N}$  such that  $\|x_j^{n_0} - x_k^{n_0}\|_A << \frac{c}{3}$  for all  $j, k \geq n_1$ . Then the triangle inequality implies  $\|x_j - x_k\|_A \leq \|x_j - x_j^{n_0}\|_A + \|x_j^{n_0} - x_k^{n_0}\|_A + \|x_k^{n_0} - x_k\|_A << \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c$ . The proof is finished.

**Theorem 39** If  $f : (X, d) \rightarrow (Y, \rho)$  is a continuous function between cone metric spaces over  $E$  and  $S$  is a compact subset of  $X$ . Then  $f(S)$  is compact in  $(Y, \rho)$ .

The proof is the same as in topological spaces, since cone metric spaces are topological spaces or can be done directly by using the concept of sequential compactness in cone metric spaces and that  $f$  is sequentially continuous [2].

**Remark 40** Note that a subset  $S$  of a cone metric space  $(X, d)$  over  $E$  is bounded if and only if there exists  $y_0 \in X$  and  $c \gg 0, c \in E$  such that  $\sup_{x \in S} d(x, y_0)$  exists in  $E$ .

**Theorem 41** Let  $S$  be a compact subset of a cone metric space  $(X, d)$ . Then  $S$  is closed and bounded.

**PROOF.** From [2], we know that compactness is equivalent to sequential compactness in cone metric spaces and closedness is equivalent to sequential closedness. Hence, compactness implies closedness easily. Namely, if  $x_n \in S$ ,  $x_n \rightarrow x$ , then by compactness of  $S$  there exists a subsequence  $x_{n_k} \rightarrow y \in S$ . But also  $x_{n_k} \rightarrow x$ . But then by uniqueness of limit in cone metric spaces, we must have  $x = y \in S$ . Also, this can be proved by using that cone metric spaces are Hausdorff topological spaces. For the other part, assume  $S$  is unbounded. Hence, there exists  $y_0 \in S$  and  $c \gg 0, c \in E$  such that for each  $n \in \mathbb{N}$  there exists  $x_n \in S$  such that  $d(x_n, y_0) \geq nc$  for all  $n \in \mathbb{N}$  or  $d(x_n, y_0)$

is not comparable with  $nc$ . This sequence  $\{x_n\}$  can not have a convergent subsequence since convergent sequences are bounded.

Recall that in [4] a function  $f : (X, \|\cdot\|_{c_1}) \rightarrow (Y, \|\cdot\|_{c_2})$  between two cone normed spaces is continuous at  $x_0 \in X$  if for all  $c \gg 0$ , there exists  $b \gg 0$  such that  $\|f(x) - f(x_0)\|_{c_2} \leq c$  whenever  $x \in X$ ,  $\|x - x_0\|_{c_1} \leq b$ .

Given that cone normed spaces are cone metric spaces and cone metric spaces are topological spaces, it is easy to show that  $f$  is continuous at  $x_0$  if and only if  $f^{-1}(V)$  is open in  $(X, \|\cdot\|_{c_1})$  for any open  $V$  in  $(Y, \|\cdot\|_{c_2})$  containing  $f(x_0)$ .

The next example is given to generalize the Banach space  $C[a, b]$  of real-valued continuous functions on  $[a, b]$  with the norm  $\|f\| = \max_{x \in [a, b]} |f(x)|$ ,  $f \in C[a, b]$ .

**Definition 42** Let  $S$  be a compact subset of  $(E, \|\cdot\|_A)$  and  $f_n : S \rightarrow E$  be a sequence of continuous functions between cone normed spaces over  $E$ . We say that  $f_n$  converges uniformly to  $f : S \rightarrow E$  if for all  $c \gg 0, c \in E$  there exists  $n_0$  such that

$$\|f_n(x) - f(x)\|_A \leq c \quad \text{for all } n \geq n_0 \quad (14)$$

**Theorem 43** Let  $S$  be a compact subset of  $(E, \|\cdot\|_A)$ . If  $f_n : (S, \|\cdot\|_A) \rightarrow (E, \|\cdot\|_A)$  is a sequence of continuous functions such that  $f_n$  converges uniformly to  $f : S \rightarrow E$ . Then  $f$  is continuous.

**PROOF.** Assume  $f_n$  converges uniformly to  $f$  and let  $x_0 \in S$  and  $c \gg 0, c \in E$  be given. Find  $n_0$  such that

$$\|f_n(x) - f(x)\|_A \leq \frac{c}{3} \quad \text{for all } n \geq n_0, x \in S \quad (15)$$

By assumption, the function  $f_{n_0}$  is continuous at  $x_0$ , hence, find  $b \gg 0, b \in E$  such that

$$x \in S, \quad \|x - x_0\|_A \leq b \quad \text{implies} \quad \|f_{n_0}(x) - f_{n_0}(x_0)\|_A \leq \frac{c}{3} \quad (16)$$

Now, for  $x \in S$  and  $\|x - x_0\|_A \leq b$  we obtain

$$\|f(x) - f(x_0)\|_A \leq \|f(x) - f_{n_0}(x)\|_A + \|f_{n_0}(x) - f_{n_0}(x_0)\|_A + \|f_{n_0}(x_0) - f(x_0)\|_A \quad (17)$$

Then (15) and (16) give

$$\|f(x) - f(x_0)\|_A \leq \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c \quad (18)$$

Hence,  $f$  is continuous at  $x_0$  and the proof is finished.

**Example 44** Let  $C_E(S) = \{f : (S, \|\cdot\|_A) \rightarrow (E, \|\cdot\|_A) : f \text{ is continuous}\}$ , where  $S$  is a compact subset of  $(E, \|\cdot\|_A)$ . Provide  $C_E(S)$  with the cone norm  $\|f\|_c = \sup_{x,y \in S} \|f(x) - f(y)\|_A$ . By the above theorems  $\|f\|_c = \sup_{x,y \in S} \|f(x) - f(y)\|_A$  exists.

To prove that  $(C_E(S), \|\cdot\|_c)$  is complete, by means of Theorem 37, it would be enough to show that  $C_E(S)$  is a (sequential) closed subspace of the cone Banach space  $(E^s, \|\cdot\|_s)$ . But (sequential) closedness directly follows by Theorem 43.

## 4 Conclusion

When  $E = R$ , then  $\|\cdot\|_A = \|\cdot\| = |\cdot|$ ,  $P = [0, \infty)$  and " $\leq$ " is the usual ordering "less than or equal". Hence,  $E^m, E^\infty$ , generalize the normed spaces  $R^m$  and  $l^\infty$  and  $C_E(S)$  generalizes the space of continuous functions  $C[S]$  where  $S = [a, b]$ . Moreover, in this case, the three boundedness definitions given in Definition 15 coincide and both of the  $\|\cdot\|_A$ -convergence and the  $\|\cdot\|$ -convergence gives the usual convergence in  $R$ .

**Problem 45** As it is known that compact subsets of finite dimensional normed spaces are exactly closed and bounded subsets, it is of sense to ask whether it is the case in the  $m$ -Euclidean cone normed space  $E^m$ .

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## STABILITY ANALYSIS OF A CLASS OF GENERALIZED NEUTRAL EQUATIONS

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alisirma01@yahoo.com***Abstract**

In this paper, some sufficient conditions for all solutions of a class of generalized neutral equations to approach zero as  $t \rightarrow \infty$  are presented. Based on Lyapunov's functional approach, some new stability criteria are derived. Our results improve and include some related results existing in the literature.

**1 Introduction.**

In this work, we study two types of neutral differential equations. First, we consider the generalized neutral differential equations of the form

$$\frac{d}{dt}(x(t) + c(t)x(t - \tau)) = -p(t)x(t) - q(t)h(x(t))x(t - \sigma), \quad t \geq t_0, \quad (1.1)$$

where  $\tau$  and  $\sigma$  are positive real numbers,  $c(t), p(t), q(t), h(x(t)) : [t_0, \infty) \rightarrow [0, \infty)$  are continuous functions and  $c(t)$  is differentiable with locally bounded derivative. In the investigating of the stability of the solution of the generalized neutral differential equation (1.1), we follow the way used in the paper of Agarwal and Grace [6]. It should be noted that, in 2000, Agarwal and Grace [6] proved an asymptotic stability result for solutions of the following neutral differential equation:

$$\frac{d}{dt}(x(t) + c(t)x(t - \tau)) = -p(t)x(t) - q(t)x(t - \sigma), \quad t \geq t_0.$$

Hence, it is clear that Equation (1.1) includes the equation investigated by Agarwal and Grace [6]. Namely, when  $h(x(t)) = 1$ , Equation (1.1) reduces to the equation discussed by Agarwal and Grace [6]. They employed Lyapunov's functional approach to verify the results established there. We also utilize the same method.

Second, we investigate the asymptotic behaviors of solutions of the generalized neutral delay equation

$$\frac{d}{dt}(x(t) + px(t - \tau)) = -h(x(t))x(t) + b \tanh x(t - \sigma), \quad t \geq t_0, \quad (1.2)$$

where  $b, \tau$  and  $\sigma$  are positive real numbers,  $|p| < 1$  and  $h(x(t)) : [t_0, \infty) \rightarrow [0, \infty)$  is a continuous function.

It is worth mentioning that, in 2000, 2004 and 2008, El-Morshedy and Gopalsamy [1] and Park [2, 3] discussed the convergence and stability of the solutions of the following neutral differential equation:

$$\frac{d}{dt}(x(t) + px(t - \tau)) = -ax(t) + b \tanh x(t - \sigma), \quad t \geq t_0.$$

Obviously, Equation (1.2) includes the equation investigated El-Morshedy and Gopalsamy [1] and Park [2, 3], since they investigated the case  $h(x(t)) = a$  in Equation (1.2). In the analyzing the stability of the solution of this equation, we follow the procedure introduced in El-Morshedy and Gopalsamy [1] and Park [2, 3].

At the same time, for some papers published on the qualitative behaviors of solutions of various neutral differential equations, we refer the reader to the papers of [1–6] and the references thereof. These types of equations are used for the study of dynamic characteristics of neural networks of Hopfield type (see [4] and references cited therein).

Throughout this article,  $\mathbf{C}([t_0 - H, t_0], \mathbb{R})$  denotes the complete space of continuous functions from  $[t_0 - H, t_0]$  to  $\mathbb{R}$ . With each solution  $x(t)$  of Equations (1.1) and (1.2), we assume the initial condition:

$$x(s) = \phi(s), \quad s \in [t_0 - H, t_0], \quad \text{where } H = \max\{\tau, \sigma\}, \quad \phi \in \mathbf{C}([t_0 - H, t_0], \mathbb{R}).$$

## 2 Main Results

First, we will study the stability of the neural networks of the generalized neutral type differential equation (1.1). We assume that there exist nonnegative constants  $c_1, c_2, p_1, p_2, q_1, q_2, h_1$  and  $h_2$  such that for  $t \geq t_0$ ,

$$p_1 \leq p(t) \leq p_2, \quad q_1 \leq q(t) \leq q_2, \quad c(t) \leq c_1 < 1, \quad |c'(t)| < c_2, \quad \text{and} \quad h_1 \leq h(x(t)) \leq h_2. \quad (2.1)$$

**Theorem 2.1** Assume that condition (2.1) holds. Beside this, if

$$p_1 + q_1 h_1 > (p_2 + q_2 h_2)(c_1 + q_2 h_2 \sigma) \quad (2.2)$$

then every solution  $x(t)$  of Equation (1.1) is asymptotically stable.

**Proof.** Equation (1.1) can be written in the following form

$$\frac{d}{dt} \left( x(t) + c(t)x(t-\tau) - \int_{t-\sigma}^t q(s+\sigma)h(x(s+\sigma))x(s)ds \right) = -p(t)x(t) - q(t+\sigma)h(x(t+\sigma))x(t) \quad (2.3)$$

Define the operator  $D : \mathbf{C}([t_0 - H, t_0], \mathbb{R}) \rightarrow \mathbb{R}$  as

$$D(x_t) = x(t) + c(t)x(t-\tau) - \int_{t-\sigma}^t q(s+\sigma)h(x(s+\sigma))x(s)ds. \quad (2.4)$$

Consider the Lyapunov functional defined by

$$\begin{aligned} V(t) = & D^2(x_t) + \int_{t-\sigma}^t [p(s+\sigma) + q(s+2\sigma)h(s+2\sigma)] \left( \int_s^t q(u+\sigma)h(u+\sigma)x^2(u)du \right) ds \\ & + \int_{t-\tau}^t [p(s+\tau) + q(s+\sigma+\tau)h(s+\sigma+\tau)]c(s+\tau)x^2(s)ds. \end{aligned}$$

The time derivative of  $V$  along solutions of Equation (1.1) is given by

$$\begin{aligned} \frac{dV(t)}{dt} = & 2D(x_t)D'(x_t) \\ & - [p(t) + q(t+\sigma)h(x(t+\sigma))] \int_{t-\sigma}^t q(u+\sigma)h(x(u+\sigma))x^2(u)du \\ & + q(t+\sigma)h(x(t+\sigma))x^2(t) \int_{t-\sigma}^t [p(s+\sigma) + q(s+2\sigma)h(x(s+2\sigma))]ds \\ & + [p(t+\tau) + q(t+\sigma+\tau)h(x(t+\sigma+\tau))]c(t+\tau)x^2(t) - [p(t) + q(t+\sigma)h(x(t+\sigma))]c(t)x^2(t-\tau). \end{aligned}$$

Using the result

$$\begin{aligned} 2D(x_t)D'(x_t) = & -2[p(t) + q(t+\sigma)h(x(t+\sigma))]x^2(t) - 2x(t)x(t-\tau)[p(t)c(t) + c(t)q(t+\sigma)h(x(t+\sigma))] \\ & + 2[p(t) + q(t+\sigma)h(x(t+\sigma))]x(t) \int_{t-\sigma}^t q(s+\sigma)h(x(s+\sigma))x(s)ds \end{aligned}$$

and the inequalities  $-2x(t)x(t-\tau) \leq x^2(t) + x^2(t-\tau)$  and  $2x(t)x(s) \leq x^2(t) + x^2(s)$ , we have

$$\begin{aligned} 2D(x_t)D'(x_t) \leq & [-2(p(t) + q(t+\sigma)h(x(t+\sigma))) + p(t)c(t) + c(t)q(t+\sigma)h(x(t+\sigma))]x^2(t) \\ & + [p(t) + q(t+\sigma)h(x(t+\sigma))]x^2(t) \int_{t-\sigma}^t q(s+\sigma)h(x(s+\sigma))ds \\ & + [p(t) + q(t+\sigma)h(x(t+\sigma))] \int_{t-\sigma}^t q(s+\sigma)h(x(s+\sigma))x^2(s)ds \\ & + [p(t)c(t) + c(t)q(t+\sigma)h(x(t+\sigma))]x^2(t-\tau). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \frac{dV(t)}{dt} &\leq [-2(p(t) + q(t + \sigma)h(x(t + \sigma))) + p(t)c(t) + c(t)q(t + \sigma)h(x(t + \sigma))]x^2(t) \\ &\quad + [p(t) + q(t + \sigma)h(x(t + \sigma))]x^2(t) \int_{t-\sigma}^t q(s + \sigma)h(x(s + \sigma))ds \\ &\quad + q(t + \sigma)h(x(t + \sigma))x^2(t) \int_{t-\sigma}^t [p(s + \sigma) + q(s + 2\sigma)h(x(s + 2\sigma))]ds \\ &\quad + [p(t + \tau) + q(t + \sigma + \tau)h(x(t + \sigma + \tau))]c(t + \tau)x^2(t). \end{aligned}$$

Thereafter, using the inequalities (2.1) and (2.2) we find

$$\frac{dV(t)}{dt} \leq -2[(p_1 + q_1h_1) - (p_2 + q_2h_2)(c_1 + q_2h_2\sigma)]x^2(t) \leq 0, \quad \text{for } t \geq T \geq t_0. \quad (2.5)$$

Integrating inequality (2.5) from  $T$  to  $t$ , we have

$$V(t) + 2[(p_1 + q_1h_1) - (p_2 + q_2h_2)(c_1 + q_2h_2\sigma)] \int_T^t x^2(s)ds \leq V(T) < \infty.$$

Hence, we see that  $V(t)$  is bounded on  $[T, \infty)$  and  $x(t) \in L^2[T, \infty)$ . Since  $V(t)$  is bounded on  $[T, \infty)$ , one can easily see that there exists a constant  $\gamma_1 \geq 0$ ,

$$\begin{aligned} |x(t)| &\leq \gamma_1 + c_1|x(t - \tau)| + \int_{t-\sigma}^t q(s + \sigma)h(x(s + \sigma))x(s)ds \\ &\leq \gamma_1 + c_1|x(t - \tau)| + q_2h_2\sigma \max_{t-H \leq s \leq t} |x(s)| \\ &\leq \gamma_1 + (c_1 + q_2h_2\sigma) \max_{t-H \leq s \leq t} |x(s)| \end{aligned}$$

But the inequality (2.2) yields that  $c_1 + q_2h_2\sigma < \frac{p_1 + q_1h_1}{p_2 + q_2h_2} < 1$ . The rest of the proof is similar to that of Theorem 1 in [6]. ■

In order to illustrate to Theorem 1 proved above, we modified Example 1 in [6] as follows.

**Example 1.** The neutral equation

$$\frac{d}{dt} \left( x(t) + \frac{1}{t}x(t - \sigma) \right) = -x(t) - (1 + e^{2t}) \left( \frac{1}{t} + \frac{1}{t^2} \right) \frac{x^2(t)}{1 + x^2(t)} x(t - \sigma) \quad t \geq 2$$

has a solution  $x(t) = e^{-t} \rightarrow 0$  as  $t \rightarrow \infty$ . All conditions of Theorem 1 are satisfied if we take  $c_1 = \frac{1}{2}$ ,  $c_2 = \frac{1}{4}$ ,  $p_1 = p_2 = 1$ ,  $q_1 = 0$ ,  $q_2 = \frac{3}{4}$ ,  $h_1 = 0$ ,  $h_2 = 1$ , and  $\tau = \sigma < \frac{2}{21}$ . Now, we present the following theorem.

**Theorem 2.2** Assume that in Theorem 2.1, instead of condition (2.2) if we replace

$$p_1 > (p_2 + q_2h_2)c_1 + q_2h_2 \quad (2.6)$$

then the conclusion of Theorem 2.1 still holds.

**Proof.** We use the Lyapunov functional defined by

$$\begin{aligned} V(t) &= (x(t) + c(t)x(t - \tau))^2 + \int_{t-\sigma}^t q(s + \sigma)h(x(s + \sigma))(1 + c(s + \sigma))x^2ds \\ &\quad + \int_{t-\tau}^t [p(s + \tau) + q(s + \tau)h(s + \tau)]c(s + \tau)x^2(s)ds. \end{aligned}$$

Define the operator  $D : \mathbf{C}([t_0 - H, t_0], \mathbb{R}) \rightarrow \mathbb{R}$  as

$$D(x_t) = x(t) + c(t)x(t - \tau).$$



The time derivative of  $V$  along the solution of Equation (1.1) is given by

$$\begin{aligned} \frac{dV(t)}{dt} &= 2D(x_t)D'(x_t) + q(t+\sigma)h(x(t+\sigma))(1+c(t+\sigma))x^2(t) - q(t)h(x(t))(1+c(t))x^2(t-\sigma) \\ &\quad + [p(t+\tau) + q(t+\tau)h(t+\tau)]c(t+\tau)x^2(t) - [p(t) + q(t)h(t)]c(t)x^2(t-\tau). \end{aligned}$$

Since,

$$\begin{aligned} 2D(x_t)D'(x_t) &= -2(x(t) + c(t)x(t-\tau))(p(t)x(t) + q(t)h(x(t))x(t-\sigma)) \\ &= -2p(t)x^2(t) - 2p(t)c(t)x(t-\tau)x(t) - 2q(t)h(x(t))x(t-\sigma)x(t) - 2c(t)q(t)h(x(t))x(t-\sigma)x(t-\tau), \end{aligned}$$

then using the inequality  $-2x(t)y(t) \leq x^2(t) + y^2(t)$ , we obtain

$$\begin{aligned} 2D(x_t)D'(x_t) &\leq -2p(t)x^2(t) + p(t)c(t)x^2(t-\tau) + p(t)c(t)x^2(t) + q(t)h(x(t))x^2(t-\sigma) \\ &\quad + q(t)h(x(t))x^2(t) + c(t)q(t)h(x(t))x^2(t-\sigma) + c(t)q(t)h(x(t))x^2(t-\tau). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{dV(t)}{dt} &= \{-2p(t) + p(t)c(t) + q(t)h(x(t)) + q(t+\sigma)h(x(t+\sigma))(1+c(t+\sigma)) \\ &\quad + [p(t+\tau) + q(t+\tau)h(t+\tau)]c(t+\tau)\}x^2(t) \\ &\leq \{-2p_1 + p_2c_1 + q_2h_2 + q_2h_2(1+c_1) + [p_2 + q_2h_2]c_1\}x^2(t). \end{aligned}$$

So that, we have

$$\frac{dV(t)}{dt} \leq -2[p_1 - (p_2 + q_2h_2)c_1 - q_2h_2]x^2(t) \leq 0, \quad \text{for } t \geq T \geq t_0. \quad (2.7)$$

Integrating inequality (2.7) from  $T$  to  $t$ , we get

$$V(t) + 2[p_1 - (p_2 + q_2h_2)c_1 - q_2h_2] \int_T^t x^2(s)ds \leq V(T) < \infty.$$

The rest of the proof is similar to that of Theorem 1. ■

Now, we are going to investigate the asymptotic stability of generalized neutral differential equation (1.2). In this investigation we will follow the articles [2, 3]. The results we obtained here generalize and include the results in the mentioned articles.  $*$  represents the elements below the main diagonal of a symmetric matrix. The notation  $X > Y$ , where  $X$  and  $Y$  are matrices of same dimensions, means that the matrix  $X - Y$  is positive definite.

Equation (1.2) can be written in the form

$$\frac{d}{dt}[x(t) + px(t-\tau) + b \int_{t-\sigma}^t \tanh x(s)] = -h(x(t))x(t) - b \tanh x(t), \quad t \geq t_0. \quad (2.8)$$

Define the operator  $D : \mathbf{C}([t_0 - H, t_0], \mathbb{R}) \rightarrow \mathbb{R}$  as

$$D(x_t) = x(t) + px(t-\sigma) + b \int_{t-\sigma}^t \tanh x(s)ds. \quad (2.9)$$

Now, we have the following theorem

**Theorem 2.3** Assume that

$$h(x(t)) \geq a > 0.$$

For given  $\sigma > 0$ , every solution  $x(t)$  of Equation (1.2) satisfies  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ , if there exists the positive scalars  $\alpha, \beta$  and  $\gamma$  such that two linear inequalities hold

$$\Omega_1 = \begin{bmatrix} -\gamma a + \alpha & -\gamma p h(x(t)) & \gamma b & -\gamma b h(x(t)) \\ * & -\alpha & \gamma b p & 0 \\ * & * & -\gamma a + \beta \sigma & b^2 \gamma \\ * & * & * & -\frac{\beta}{\sigma} \end{bmatrix} < 0, \quad (2.10)$$

$$\Omega_1 = |p| - 1 + \sigma b < 0. \quad (2.11)$$

**Proof.** Consider the Lyapunov functional defined by

$$V(t) = \gamma D^2(x_t) + \alpha \int_{t-\tau}^t x^2(s) ds + \beta \int_{t-\sigma}^t (\sigma - t + s) \tanh^2 x(s) ds \quad (2.12)$$

where  $\alpha$ ,  $\beta$ , and  $\gamma$  are positive scalars to be chosen later.

The derivative of  $V(t)$  along the solution of Equation (2.8) is given by

$$\begin{aligned} \frac{dV}{dt} &= 2\gamma \left( x(t) + px(t-\tau) + b \int_{t-\sigma}^t \tanh x(s) ds \right) (-h(x(t))x(t) + b \tanh x(t)) + \alpha x^2(t) - \alpha x^2(t-\tau) \\ &\quad + \beta \sigma \tanh^2 x(t) - \beta \int_{t-\sigma}^t \tanh^2 x(s) ds \\ &= -2\gamma x(t)h(x(t))x(t) - 2p\gamma h(x(t))x(t)x(t-\tau) - 2b\gamma h(x(t))x(t) \int_{t-\sigma}^t \tanh x(s) ds + 2b\gamma x(t) \tanh x(t) \\ &\quad + 2bp\gamma x(t-\tau) \tanh x(t) + 2b^2\gamma \tanh x(t) \int_{t-\sigma}^t \tanh x(s) ds + \alpha x^2(t) - \alpha x^2(t-\tau) \\ &\quad + \beta \sigma \tanh^2 x(t) - \beta \int_{t-\sigma}^t \tanh^2 x(s) ds. \\ &\leq -2a\gamma x^2(t) - 2p\gamma h(x(t))x(t)x(t-\tau) - 2bh(x(t))\gamma x(t) \int_{t-\sigma}^t \tanh x(s) ds + 2b\gamma x(t) \tanh x(t) \\ &\quad + 2bp\gamma x(t-\tau) \tanh x(t) + 2b^2\gamma \tanh x(t) \int_{t-\sigma}^t \tanh x(s) ds + \alpha x^2(t) - \alpha x^2(t-\tau) \\ &\quad + \beta \sigma \tanh^2 x(t) - \beta \int_{t-\sigma}^t \tanh^2 x(s) ds. \end{aligned}$$

Here, for vector function  $y$ , using the well-known inequality

$$\left[ \int_{t-\sigma}^t y(s) ds \right]^T \left[ \int_{t-\sigma}^t y(s) ds \right] \leq \sigma \int_{t-\sigma}^t y^T(s)y(s) ds$$

we have

$$-\beta \int_{t-\sigma}^t \tanh^2 x(s) ds \leq -\frac{\beta}{\sigma} \left[ \int_{t-\sigma}^t \tanh x(s) ds \right]^2. \quad (2.13)$$

Also, by utilizing the relation  $\tanh^2 x(t) \leq x^2(t)$ , we have

$$-a\gamma x^2(t) \leq -a\gamma \tanh^2 x(t). \quad (2.14)$$

Substituting the relations (2.13) and (2.14) into the preceding inequality gives that

$$\frac{dV}{dt} \leq \chi^T(t) \Omega_1 \chi(t), \quad (2.15)$$

where

$$\chi = \begin{bmatrix} x(t) \\ x(t-\tau) \\ \tanh x(t) \\ \int_{t-\sigma}^t \tanh x(s) ds \end{bmatrix}. \quad (2.16)$$

Then, the rest of the proof is similar to that of Theorem 1 in [2]. ■

**Remark 1.** In Theorem 2.1 and Theorem 2.2, if we take  $h(x(t)) = 1$ , these theorems reduce to Theorem 1 and 2 in [6], respectively.

**Remark 2.** In Theorem 2.3, if we take  $h(x(t)) = a > 0$ , this theorem reduce to Theorem 1 in [2].

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# An improvement to a $C^2$ quintic spline interpolation scheme on triangulations

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**Abstract.** Wang in [1] solved an open problem described by Farin in [2], he constructed a  $C^2$  quintic spline interpolation scheme on a refined triangulation. In this paper, we would like to improve Wang's study by enforcing some additional smoothness conditions across the interior edges of Wang's refinement of triangulation  $\Delta$ . The results imply that our improvement interpolation scheme is more effective than other counterparts.

*Keywords:* Quintic spline; Improvement; B-coefficients; Spline space; Interpolation scheme.

*MCS:* 65D07; 41A15; 41A63

## 1 Introduction

Given a regular triangulation  $\Delta$  of a connected polygonal domain  $\Omega$ , we denote the set of vertices in  $\Delta$  by  $\mathcal{V}$ , the set of edges by  $\mathcal{E}$  and the set of triangles by  $\mathcal{N}$  in  $\Delta$ . Let

$$S_d^r(\Delta) = \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_d, \forall T \in \mathcal{N}\} \quad (1)$$

be the spline function space of smoothness order  $r$  and degree  $d$ , where  $\mathcal{P}_d$  denotes the space of bivariate polynomials of total degree being at most  $d$ .

Bivariate splines are very flexible for approximating known or unknown functions or any given data sets, they have been playing an important role in surface fitting and computer aided geometric design (CAGD). Many bivariate  $C^1$  and  $C^2$  spline interpolation schemes have

been constructed and used in applications. It is well-known that the dimensions of space  $S_d^r(\Delta)$  with  $d < 3r + 2$  on an arbitrary triangulation are still open. Since the bivariate spline with lower degrees versus smoothness orders are very important and favorable because of the simplicity and efficiency in calculation, many bivariate spline interpolation schemes with lower degrees were constructed on some kinds of special and refined triangulations. For example, related to  $C^2$  quintic spline interpolation scheme, Alfeld in [3] used Clough-Tocher's refinement twice to construct a  $C^2$  quintic spline interpolation scheme on  $\Delta_A$ , where  $\Delta_A$  denotes Alfeld's refinement of  $\Delta$  by using double Clough-Tocher's refinement. Sablonnière in [4], Lai in [5, 6] and Alfeld in [7] used Powell-Sabin's refinement to construct a bivariate  $C^2$  quintic interpolating spline function in  $S_5^2(\Delta_{PS})$  or subspace of  $S_5^2(\Delta_{PS})$ , where  $\Delta_{PS}$  denotes Powell-Sabin's refinement of  $\Delta$ . Wang in [1] subdivided each triangle of  $\Delta$  into 7 subtriangles and constructed a bivariate  $C^2$  quintic interpolating spline function in  $S_5^2(\Delta_W)$ , where  $\Delta_W$  denotes Wang's refinement of  $\Delta$ .

Among these  $C^2$  quintic spline interpolation schemes, Alfeld's and Wang's schemes leads to more basis functions, this may be a disadvantage for certain applications from the point of view of efficiency. In this paper, we would like to improve Wang's study [1] by enforcing some additional smoothness conditions across the interior edges of Wang's refinement. The results imply that our improvement scheme is more advantageous.

## 2 Notations and Preliminaries

Throughout the paper, let  $T$  be a triangle, we denote the three vertices of  $T$  in counter clockwise direction by  $v_0$ ,  $v_1$  and  $v_2$ , then every polynomial  $s \in \mathcal{P}_d$  associated with  $T$  can be written uniquely in the Bernstein-Be zier form

$$s = \sum_{i+j+k=d} c_{ijk}^T \mathcal{B}_{ijk}^d, \quad (2)$$

where  $\{\mathcal{B}_{ijk}^d\}_{i+j+k=d}$  is the Bernstein basis polynomials of degree  $d$  on the triangle  $T$ , and  $c_{ijk}^T$  are called the B-coefficients of  $s$  associated with the domain points  $\xi_{ijk}^T = (iv_0 + jv_1 + kv_2)/d$ ,  $i + j + k = d$ .

For each vertex  $v \in \mathcal{V}$ , we define the usual rings and disks of domain points

$$R_n(v) := \{\xi_{ijk}^T : i = d - n\},$$

$$D_n(v) := \{\xi_{ijk}^T : i \geq d - n\}.$$

Let  $S_d^0(\Delta)$  be the space of continuous splines of  $d$  on the triangulation  $\Delta$ , and let  $\mathcal{D}_{d,\Delta}$  be the union of the sets of domain points associated with each triangle of  $\Delta$ . Then each spline in  $S_d^0(\Delta)$  is uniquely determined by its set of B-coefficients  $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\Delta}}$ , and the B-coefficients

of the polynomial  $s|_T$  are precisely  $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\Delta} \cup T}$ . We recall (cf.[8]) that a determining set for a spline space  $S \subseteq S_d^0(\Delta)$  is a subset  $\mathcal{M}$  of domain points  $\mathcal{D}_{d,\Delta}$  such that if  $s \in S$  and  $c_\xi = 0$  for all  $\xi \in \mathcal{M}$ , then  $c_\eta = 0$  for all  $\eta \in \mathcal{D}_{d,\Delta}$ , i.e.,  $s \equiv 0$ .  $\mathcal{M}$  is called a minimal determining set (MDS) for  $S$  if there is no smaller determining set. It is known that  $\mathcal{M}$  is a MDS for  $S$  if and only if every spline  $s \in S$  is uniquely determined by its set of B-coefficients  $\{c_\xi\}_{\xi \in \mathcal{M}}$ , and  $\dim S = M$ , where  $M$  denotes the cardinality of  $\mathcal{M}$ . Moreover, for each  $\xi \in \mathcal{M}$ , let  $B_\xi$  be the unique spline in  $S$  satisfying

$$\lambda_\eta B_\xi = \delta_{\xi,\eta}, \quad \text{for all } \eta \in \mathcal{M}, \quad (3)$$

where  $\lambda_\eta$  is the linear functional which picks off the B-coefficient  $c_\eta$ . Then the set  $\{B_\xi\}_{\xi \in \mathcal{M}}$  is a basis for  $S$ , we call it the dual basis corresponding to  $\mathcal{M}$ .

Let  $T := \langle v_0, v_1, v_2 \rangle$  and  $\tilde{T} := \langle v_3, v_2, v_1 \rangle$  be two adjacent triangles, they share the common edge  $e := \langle v_1, v_2 \rangle$ , and let  $c_{ijk}$  and  $\tilde{c}_{ijk}$  be the B-coefficients of the B-form of  $s_T$  and  $s_{\tilde{T}}$ , respectively. Following [9], for any  $0 \leq n \leq m \leq d$ , let  $\tau_{e,m}^n$  be the linear functional defined on  $S_d^0(\Delta)$  by

$$\tau_{e,m}^n s := \tilde{c}_{n,m-n,d-m} - \sum_{i+j+k=n} c_{i,j+d-m,k+m-n} \mathcal{B}_{ijk}^n(v_3), \quad (4)$$

then the condition that  $s$  is  $C^r$  smooth across the edge  $e$  is equivalent to

$$\tau_{e,m}^n s = 0, \quad n \leq m \leq d, \quad 0 \leq n \leq r. \quad (5)$$

The following lemma [8] shows how this works for computing coefficients on the ring  $R_m^T(v_2) \cup R_m^{\tilde{T}}(v_2)$ .

**Lemma** Suppose  $T$  and  $\tilde{T}$  are two triangles defined as above, and that all B-coefficients  $c_{ijk}$  and  $\tilde{c}_{ijk}$  of the B-form  $s_T$  and  $s_{\tilde{T}}$  are known except for

$$\begin{aligned} c_\nu &:= c_{\nu,d-m,m-\nu}, \quad \nu = l+1, \dots, q, \\ \tilde{c}_\nu &:= c_{\nu,m-\nu,d-m}, \quad \nu = l+1, \dots, \tilde{q}, \end{aligned} \quad (6)$$

for some  $l, m, q, \tilde{q}$  with  $0 \leq q, \tilde{q}$ ,  $-1 \leq l \leq q, \tilde{q}$ , and  $q + \tilde{q} - l \leq m \leq d$ . Then these B-coefficients are uniquely determined by the smoothness conditions (5).

To be more precise about what we are going to study in this paper, following [1], we introduce some notations and definitions about Wang's refinement of triangle  $T$ .

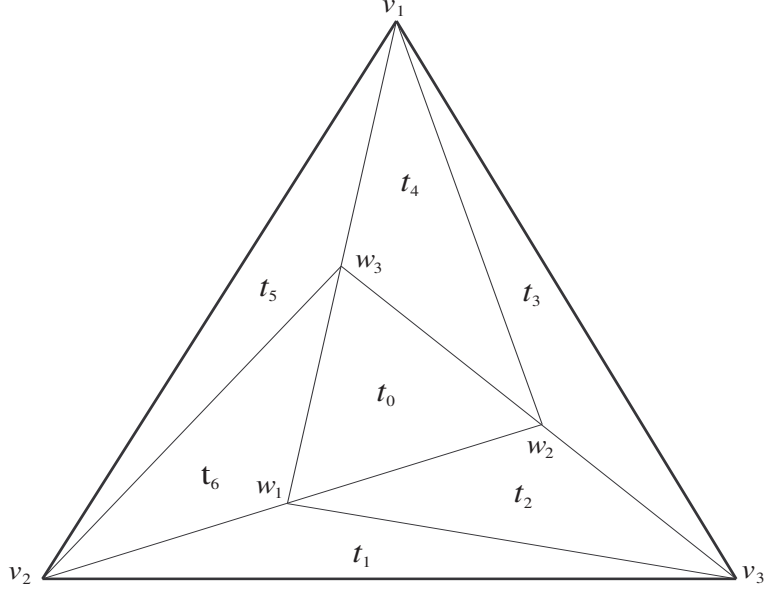
(1) Take three interior points  $w_1, w_2$  and  $w_3$  in triangle  $T$ , where  $w_j = \frac{1}{7}v_j + \frac{4}{7}v_{j+1} + \frac{2}{7}v_{j+2}$  and  $v_{j+3} = v_j$ ,  $j = 1, 2, 3$ ;

(2) Join  $v_j$  to  $w_j$  and  $v_j$  to  $w_{j+1}$ , where  $w_{j+3} = w_j$ ,  $j = 1, 2, 3$ .

By means of the above process, we can get Wang's refinement  $T_W$  (Fig.1) of triangle  $T$ . Now we define subtriangles  $t_k$  ( $k = 0, 1, \dots, 6$ ) in  $T_W$ , suppose

$$t_0 := \langle w_1, w_2, w_3 \rangle, \quad t_{2i-1} := \langle v_{i+1}, v_{i+2}, w_i \rangle, \quad t_{2i} := \langle v_{i+2}, w_{i+1}, w_i \rangle,$$

where  $i = 1, 2, 3$  and  $v_{i+3} = v_i$ ,  $w_{i+3} = w_i$ .

Figure 1: Wang's refinement  $T_W$ .

### 3 The spline space $\hat{S}_5^2(T_W)$

Given Wang's refinement  $T_W$  of triangle  $T$ , we denote the edge  $\langle w_i, w_{i+1} \rangle$  by  $e_i$  for  $i = 1, 2, 3$ , where  $w_4 = w_1$ .

**Theorem 1.** Let  $\hat{S}_5^2(T_W)$  be the subspace of  $S_5^2(T_W)$  satisfying the following set of additional smoothness conditions:

$$\tau_{e_i,3}^3 s = 0, \quad i = 1, 2, 3, \quad (7)$$

then  $\dim \hat{S}_5^2(T_W) = 30$ , and the set  $\mathcal{M}_T$  of the following domain points is a MDS for  $\hat{S}_5^2(T_W)$ .

- 1)  $\xi_{ijk}^{t_{2l}}, i = 3, 4, 5; i + j + k = 5; l = 1, 2, 3$ ,
- 2)  $\xi_{122}^{t_{2l-1}}, \xi_{221}^{t_{2l-1}}, \xi_{212}^{t_{2l-1}}, l = 1, 2, 3$ .
- 3)  $\xi_{221}^{t_0}, \xi_{122}^{t_0}, \xi_{212}^{t_0}$ .

These domain points are marked by  $\bullet$  in Fig.2.

**Proof:** Firstly, we show that  $\mathcal{M}_T$  is a determining set for  $\hat{S}_5^2(T_W)$ . Suppose  $s$  is a spline in  $\hat{S}_5^2(T_W)$  whose B-coefficients corresponding to points in  $\mathcal{M}_T$  are set to prescribed values, then we show that all of its remaining B-coefficients associated with domain points  $T_w$  are uniquely determined.

① The B-coefficients corresponding to points which marked by  $\circ$  in  $D_2(v_i)$  ( $i = 1, 2, 3$ ) can be uniquely computed from those corresponding to domain points in item 1) in Theorem 1 by Lemma.

② Item 2) in Theorem 1 and  $C^2$  smoothness condition across edge  $\langle v_i, w_{i+1} \rangle$  and  $\langle$

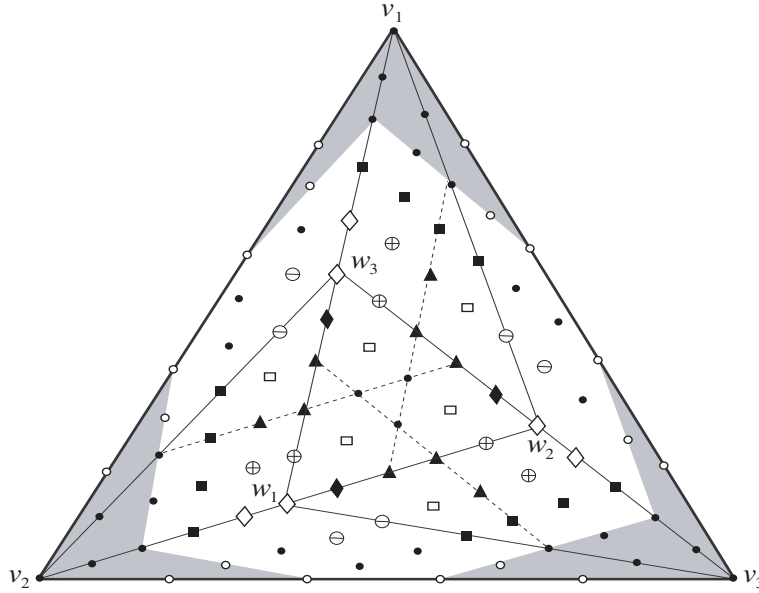


Figure 2: The MDS for spline space  $\hat{S}_5^2(T_W)$  in Theorem 1.

$v_i, w_{i+2} >$  make the remain B-coefficients associated with 4 domain points which marked by  $\blacksquare$  in  $R_3(v_i)$  to be uniquely computed, where  $i = 1, 2, 3$  and  $w_4 = w_1, w_5 = w_2$ . This computation involves solving a non-singular linear system equation.

③ We now compute the B-coefficients associated with 3 domain points which marked by  $\blacktriangle$  in  $t_6$  and  $t_0$ , by using  $C^2$  smoothness condition across the common edge  $e_3 := \langle w_3, w_1 \rangle$  and the condition  $\tau_{e_3,3}^3 s = 0$ , we can get

$$\begin{cases} c_{023}^{t_0} = -\frac{1}{4}c_{221}^{t_0} + c_{122}^{t_0} + \frac{1}{4}c_{221}^{t_6}, \\ c_{122}^{t_6} = -\frac{1}{2}c_{211}^{t_0} + c_{122}^{t_0} + \frac{1}{2}c_{221}^{t_6}, \\ c_{320}^{t_0} = -c_{302}^{t_6} - 8c_{221}^{t_0} - 4c_{122}^{t_0} + 2c_{212}^{t_6}, \end{cases} \quad (8)$$

then these 3 B-coefficients  $c_{023}^{t_0}$ ,  $c_{122}^{t_6}$  and  $c_{320}^{t_0}$  can be uniquely computed. Similarly, we can compute the remain B-coefficients associated with domain points which marked by  $\blacktriangle$  in  $t_0, t_2$  and  $t_4$ .

④ Next we use Lemma to compute the B-coefficients associated with 2 domain points which marked with  $\square$  in  $t_0$  and  $t_{2i}$  for  $i = 1, 2, 3$ .

⑤ Then use Lemma to compute the B-coefficients associated with 2 domain points which marked by  $\oplus$  in  $t_{2i}$  for  $i = 1, 2, 3$ .

⑥ Use Lemma to compute the B-coefficients associated with 2 domain points which marked by  $\ominus$  in  $t_1, t_3$  and  $t_5$ .

⑦ Use Lemma to compute the B-coefficients associated with a domain points which marked



by  $\blacklozenge$  on  $e_i$  for  $i = 1, 2, 3$ .

⑧ Finally, use Lemma to compute the B-coefficients associated with 6 domain points which marked by  $\blacklozenge$  on edge  $\langle v_1, w_3 \rangle$ ,  $\langle v_2, w_1 \rangle$  and  $\langle v_3, w_2 \rangle$ .

Following above computation, we conclude that  $\mathcal{M}$  is a determining set for  $\hat{S}_5^2(T_W)$ . Since spline space  $\hat{S}_5^2(T_W)$  is the subspace of  $S_5^2(T_W)$  which requires additional 3 conditions, thus

$$\dim S_5^2(T_W) - 3 \leq \dim \hat{S}_5^2(T_W) \leq 30.$$

Noticing that  $\dim S_5^2(T_W) = 33$  [1], we conclude that  $\mathcal{M}_T$  is a MDS for  $\hat{S}_5^2(T_W)$  and  $\dim \hat{S}_5^2(T_W) = 30$ . ■

## 4 The spline space $\hat{S}_5^2(\Delta_W)$

Let  $\Delta_W$  be a Wang's refinement of triangulation  $\Delta$  and  $\hat{S}_5^2(\Delta_W)$  be the subspace of all splines in  $S_5^2(\Delta_W)$  satisfying the additional smoothness condition (7) in each  $T_W$ . In addition, we denote the perpendicular cross-derivative across the edge  $e := \langle v_i, v_j \rangle$  by  $D_e$ , for any  $1 \leq k$ , let

$$\eta_{kl}^e := \frac{(k-l+1)v_i + lv_j}{k+1}, \quad l = 1, \dots, k.$$

Based on above results, we have the following

**Theorem 2.**  $\dim \hat{S}_5^2(\Delta_W) = 6V + 3E + 3N$ , and any element in  $\hat{S}_5^2(\Delta_W)$  is uniquely determined by the following conditions:

- 1) the derivatives  $D_x^\alpha D_y^\beta s(v)$  for  $0 \leq \alpha + \beta \leq 2$  and all  $v \in \mathcal{V}$ ,
- 2) the derivatives  $D_e^k s(\eta_{k1}^e), \dots, D_e^k s(\eta_{kk}^e)$  for  $k = 1, 2$  and all  $e \in \mathcal{E}$ ,
- 3) the derivatives  $D_e s(\eta_{11}^e)$  for all  $e \in \mathcal{E}_0$ ,

where  $\mathcal{E}_0$  denotes the set of edges of each triangle  $t_0$  in  $\Delta_W$  and  $V, E, N$  denote the number of vertices, edges, triangles of  $\Delta$ .

**Proof:** Let  $\mathcal{M}_W$  be the set consisting of the following domain points:

- ① For each vertex  $v$  of  $\Delta$ , choose a triangle  $t$  of  $\Delta_W$  attached to  $v$  and include  $D_2^t(v)$ ;
- ② For each edge  $e = \langle v_1, v_2 \rangle$  of  $\Delta$ , let  $t = \langle v_1, v_2, v_3 \rangle$  be a triangle of  $\Delta_W$  containing the edge  $e$ . Then include the points  $\xi_{221}^t, \xi_{122}^t, \xi_{212}^t$ ;
- ③ For each triangle  $t_0$  in  $\Delta_W$ , include the points  $\xi_{221}^{t_0}, \xi_{122}^{t_0}, \xi_{212}^{t_0}$ .

Clearly, the cardinality of  $\mathcal{M}_W$  is  $6V + 3E + 3N$ . Similar to the proof of Theorem 1 and  $C^2$  smoothness on two arbitrary adjacent triangle of the triangulation  $\Delta_W$ , this theorem can be easily proved. ■

In view of Eq. (3) and Theorem 2, we can form a dual basis for  $\hat{S}_5^2(\Delta_W)$ . For each  $\xi \in \mathcal{M}_W$ , it is easy to see that the basis function  $B_\xi$  has local support.

1) If  $\xi$  is point as in item ① of the proof of Theorem 2, then  $\text{supp}(B_\xi)$  is contained in the union of all triangles of  $\Delta$  sharing the vertex  $v$ .

2) If  $\xi$  is point as in item ② of the proof of Theorem 2, then  $\text{supp}(B_\xi)$  is contained in  $T \cup \tilde{T}$ , where  $T$  and  $\tilde{T}$  are the triangles of  $\Delta$  sharing the edge  $e$ . (If  $e$  is a boundary edge of  $\Delta$ , then there is only one such triangle, and it is the support set).

3) If  $\xi$  is point as in item ③ of the proof of Theorem 2, then  $\text{supp}(B_\xi)$  is only one triangle  $T$ , where  $T$  contains  $\xi$  in  $\Delta$ .

We end this paper with the following remarks.

**Remark 1.** The computation detail of item ① and ② in the proof of Theorem 1 are described in [1].

**Remark 2.** In comparison with Wang's interpolation scheme in [1], our improvement method is more efficient. Firstly, the dimension of  $S_5^2(\Delta_W)$  is bigger than that of  $\hat{S}_5^2(\Delta_W)$ , so our improvement method needs fewer locally supported basis functions. Then, in this paper, no large linear system needs to be solved as in [1].

**Remark 3.** For the purpose of comparison, we also list the dimensions of spline spaces mentioned in §1:

- (1)  $\dim S_5^2(T_A) = 37$ , see [3];
- (2)  $\dim S_5^2(T_{PS}) = 31$ , see [4, 6];
- (3)  $\dim S_5^2(T_{PS}) = 37$ , see [5];
- (4)  $\dim S_5^2(T_{PS}) = 30$ , see [7];
- (5)  $\dim S_5^2(T_W) = 33$ , see [1].

**Remark 4.** In [4, 5, 6, 7], they all produce an interpolant spline function which is in  $C^3$  at all vertices of  $\Delta$ , this may be a disadvantage for certain applications.

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# SOME GEOMETRIC AND TOPOLOGICAL PROPERTIES OF A NEW SEQUENCE SPACE DEFINED BY DE LA VALLÉE-POUSSIN MEAN

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ABSTRACT. The main purpose of this paper is to introduce a new sequence space by using de la Vallée-Poussin mean and investigate both the modular structure with some geometric properties and some topological properties with respect to the Luxemburg norm.

## 1. INTRODUCTION

In summability theory, de la Vallée-Poussin's mean is first used to define the  $(V, \lambda)$ -summability by Leindler [9]. Malkowsky and Savaş [14] introduced and studied some sequence spaces which arise from the notion of generalized de la Vallée-Poussin mean. Also the  $(V, \lambda)$ -summable sequence spaces have been studied by many authors including [6] and [21].

Recently, there has been a lot of interest in investigating geometric properties of several sequence spaces. Some of the recent work on sequence spaces and their geometrical properties is given in the sequel: Shue [22] first defined the Cesàro sequence spaces with a norm. In [11], it is shown that the Cesàro sequence spaces  $ces_p$  ( $1 \leq p < \infty$ ) have Kadec-Klee and Local Uniform Rotundity(LUR) properties. Cui-Hudzik-Pluciennik [4] showed that Banach-Saks of type  $p$  property holds in these spaces. In [15], Mursaleen et al. studied some geometric properties of normed Euler sequence space. Karakaya [7] defined a new sequence space involving lacunary sequence space equipped with the Luxemburg norm and studied Kadec-Klee( $H$ ), rotund( $R$ ) properties of this space. Quite recently, Sanhan and Suantai [19] generalized normed Cesàro sequence spaces to paranormed sequence spaces by making use of Köthe sequence spaces. They also defined and investigated modular structure and some geometrical properties of these generalized sequence spaces. In addition, some related papers on this topic can be found in [1],[2],[5],[16],[17],[20] and [24].

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In this paper, our purpose is to introduce a new sequence space defined by de la Vallée-Poussin's mean and investigate some topological and geometric properties of this space.

The organization of our paper is as follows: In the first section, we introduce some definition and concepts that are used throughout the paper. In the second section, we construct a new paranormed sequence space and investigate some geometrical properties of this space. Finally, in the third section, we construct the modular space  $V_\rho(\lambda; p)$  which is obtained by paranormed space  $V(\lambda; p)$  and we investigate the Kadec-Klee property of this space. We also show that the modular space  $V_\rho(\lambda; p)$  is a Banach space under the Luxemburg norm. Also in this section, we investigate the Banach-Saks of type  $p$  property of the space  $V_p(\lambda)$ .

## 2. PRELIMINARIES, BACKGROUND AND NOTATION

The space of all real sequences  $x = (x(i))_{i=1}^\infty$  is denoted by  $\ell^0$ . Let  $(X, \|\cdot\|)$  (for the brevity  $X = (X, \|\cdot\|)$ ) be a normed linear space and let  $S(X)$  and  $B(X)$  be the unit sphere and unit ball of  $X$ , respectively.

A Banach space  $X$  which is a subspace of  $\ell^0$  is said to be a Köthe sequence space, if (see [10]);

(i) for any  $x \in \ell^0$  and  $y \in X$  such that  $|x(i)| \leq |y(i)|$  for all  $i \in \mathbb{N}$ , we have  $x \in X$  and  $\|x\| \leq \|y\|$ ,

(ii) there is  $x \in X$  with  $x(i) > 0$  for all  $i \in \mathbb{N}$ .

We say that  $x \in X$  is order continuous if for any sequence  $(x_n)$  in  $X$  such that  $x_n(i) \leq |x(i)|$  for each  $i \in \mathbb{N}$  and  $x_n(i) \rightarrow 0$  ( $n \rightarrow \infty$ ),  $\|x_n\| \rightarrow 0$  holds. A Köthe sequence space  $X$  is said to be order continuous if all sequences in  $X$  are order continuous. It is easy to see that  $x \in X$  is order continuous if and only if  $\|(0, 0, \dots, 0, x(n+1), x(n+2), \dots)\| \rightarrow 0$  as  $n \rightarrow \infty$ .

A Banach space  $X$  is said to have the *Kadec-Klee property* (or property  $(H)$ ) if every weakly convergent sequence on the unit sphere with the weak limit in the sphere is convergent in norm.

Let  $1 < p < \infty$ . A Banach space is said to have the *Banach – Saks type  $p$*  or property  $(BS_p)$ , if every weakly null sequence  $(x_k)$  has a subsequence  $(x_{k_l})$  such that for some  $C > 0$ ,

$$\left\| \sum_{l=0}^n x_{k_l} \right\| < C(n+1)^{\frac{1}{p}}$$

for all  $n \in \mathbb{N}$  (see [8]).

For a real vector space  $X$ , a functional  $\rho : X \rightarrow [0, \infty]$  is called a *modular* if it satisfies the following conditions:

- i)  $\rho(x) = 0 \Leftrightarrow x = 0$ ,
- ii)  $\rho(\alpha x) = \rho(x)$  for all  $\alpha \in \mathbb{F}$  with  $|\alpha| = 1$ ,
- iii)  $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$  for all  $x, y \in X$  and all  $\alpha, \beta \geq 0$  with  $\alpha + \beta = 1$ .

Further, the modular  $\rho$  is called *convex* if

iv)  $\rho(\alpha x + \beta y) \leq \alpha \rho(x) + \beta \rho(y)$  holds for all  $x, y \in X$  and all  $\alpha, \beta \geq 0$  with  $\alpha + \beta = 1$ .

If  $\rho$  is a modular in  $X$ , we define

$$X_\rho = \{x \in X : \rho(\lambda x) \rightarrow 0 \text{ as } \lambda \rightarrow 0^+\},$$

$$X_\rho^* = \{x \in X : \rho(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

It is clear that  $X_\rho \subseteq X_\rho^*$ . If  $\rho$  is a convex modular for  $x \in X_\rho$ , we define

$$\|x\|_L = \inf \left\{ \lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1 \right\}$$

and

$$\|x\|_A = \inf_{\lambda > 0} \frac{1}{\lambda} (1 + \rho(\lambda x)).$$

If  $\rho$  is a convex modular on  $X$ , then  $X_\rho = X_\rho^*$  and both  $\|\cdot\|_L$  and  $\|\cdot\|_A$  is a norm on  $X_\rho$  for which  $X_\rho$  is a Banach space.

The norms  $\|\cdot\|_L$  and  $\|\cdot\|_A$  are called the *Luxemburg norm* and the *Amemiya norm (Orlicz norm)*, respectively.

In addition

$$\|x\|_L \leq \|x\|_A \leq 2\|x\|_L$$

for all  $x \in X_\rho$  holds (see [18]).

A sequence  $(x_n)$  of elements of  $X_\rho$  is called *modular convergent* to  $x \in X_\rho$  if there exists a  $\lambda > 0$  such that  $\rho(\lambda(x_n - x)) \rightarrow 0$  as  $n \rightarrow \infty$ .

**Proposition 2.1.** *Let  $(x_n) \subset X_\rho$ . Then  $\|x_n\|_L \rightarrow 0$  (or equivalently  $\|x\|_A \rightarrow 0$ ) if and only if  $\rho(\lambda(x_n)) \rightarrow 0$  as  $n \rightarrow \infty$ , for every  $\lambda > 0$ .*

*Proof.* See [18, p.15, Th.1]. ■

Throughout the paper, the sequence  $p = (p_k)$  is a bounded sequence of positive real numbers with  $p_k > 1$ , also  $H = \sup_k p_k$  and  $M = \max\{1, H\}$ .

Besides, we will need the following inequalities in the sequel;

$$(2.1) \quad |a_k + b_k|^{p_k} \leq K (|a_k|^{p_k} + |b_k|^{p_k})$$

$$(2.2) \quad |a_k + b_k|^{t_k} \leq |a_k|^{t_k} + |b_k|^{t_k}$$

where  $t_k = \frac{p_k}{M} \leq 1$  and  $K = \max\{1, 2^{H-1}\}$  with  $H = \sup_k p_k$ .

Now we begin the construction of a new sequence space.

Let  $\Lambda = (\lambda_k)$  be a nondecreasing sequence of positive real numbers tending to infinity and let  $\lambda_1 = 1$  and  $\lambda_{k+1} \leq \lambda_k + 1$ .

The generalized de la Vallée-Poussin means of a sequence  $x = (x_k)$  are defined as follows:

$$t_k(x) = \frac{1}{\lambda_k} \sum_{j \in I_k} x_j \text{ where } I_k = [k - \lambda_k + 1, k] \quad \text{for } k = 1, 2, \dots$$

We write

$$[V, \lambda]_0 = \left\{ x \in \ell^0 : \lim_{k \rightarrow \infty} \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j| = 0 \right\}$$

$$[V, \lambda] = \{ x \in \ell^0 : x - le \in [V, \lambda]_0, \text{ for some } l \in \mathbb{C} \}$$

and

$$[V, \lambda]_\infty = \left\{ x \in \ell^0 : \sup_k \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j| < \infty \right\}$$

for the sequence spaces that are strongly summable to zero, strongly summable and strongly bounded by the de la Vallée-Poussin method (see [9]). In the special case, if we take  $\lambda_k = k$  for  $k = 1, 2, \dots$  the spaces  $[V, \lambda]_0$ ,  $[V, \lambda]$  and  $[V, \lambda]_\infty$  reduce to the spaces  $w_0$ ,  $w$  and  $w_\infty$  introduced by Maddox [12].

We now define the following new paranormed sequence space:

$$V(\lambda; p) := \left\{ x = (x_j) \in \ell^0 : \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j| \right)^{p_k} < \infty \right\}.$$

The space  $V(\lambda; p)$  is reduced to some special sequence spaces corresponding to special cases of sequence  $(\lambda_k)$  and  $(p_k)$ . For example: If we take  $\lambda_k = k$ , we obtain the space  $ces(p)$  defined by [23]. If we take  $\lambda_k = k$  and  $p_k = p$  for all  $k \in \mathbb{N}$ , the space  $V(\lambda; p)$  reduces to the space  $ces_p$  defined by [22].

### 3. SOME TOPOLOGICAL PROPERTIES OF THE SEQUENCE SPACE $V(\lambda; p)$

In this section, we will give the topological properties of the space  $V(\lambda; p)$ . We begin by obtaining the first main result.

**Theorem 3.1.** *a) The space  $V(\lambda; p)$  is a complete paranormed space with respect to paranorm defined by*

$$(3.1) \quad h(x) := \left( \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j| \right)^{p_k} \right)^{\frac{1}{M}}.$$

*b) if  $p_k = p$ ; the space  $V(\lambda; p)$  reduced to  $V_p(\lambda)$  defined by*

$$V_p(\lambda) := \left\{ x = (x_j) \in \ell^0 : \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j| \right)^p < \infty \right\}.$$

*And the space  $V_p(\lambda)$  is a complete normed space defined by*

$$\|x\|_{V_p(\lambda)} := \left( \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j| \right)^p \right)^{\frac{1}{p}} \quad (1 < p < \infty).$$

*Proof.* a) The linearity of  $V(\lambda; p)$  with respect to coordinatewise addition and scalar multiplication follows from the inequality (2.1). Because, for any  $x, y \in V(\lambda; p)$  the following inequalities are satisfied:

$$(3.2) \quad \left( \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j + y_j| \right)^{p_k} \right)^{\frac{1}{M}} \leq \left( \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j| \right)^{p_k} \right)^{\frac{1}{M}} + \left( \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |y_j| \right)^{p_k} \right)^{\frac{1}{M}}$$

and for any  $\alpha \in \mathbb{R}$  (see [13]) we have

$$(3.3) \quad |\alpha|^{p_k} \leq \max\{1, |\alpha|^M\}.$$

It is clear that  $h(\theta) = 0$  and  $h(x) = h(-x)$  for all  $x \in V(\lambda; p)$ . Again the inequalities (3.2) and (3.3) yield the subadditivity of  $h$  and

$$h(\alpha x) \leq \max\{1, |\alpha|\} h(x).$$

Let  $(x^m)$  be any sequence of points of the space  $V(\lambda; p)$  such that  $h(x^m - x) \rightarrow 0$  and  $(\alpha_n)$  also be any sequence of scalars such that  $\alpha_n \rightarrow \alpha$ . Then, since the inequality

$$h(x^m) \leq h(x) + h(x^m - x)$$

holds by subadditivity of  $h$ , the sequence  $(h(x^m))_{m \in \mathbb{N}}$  is bounded and we thus have

$$\begin{aligned} h(\alpha_m x^m - \alpha x) &= \left( \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |\alpha_m x_j^m - \alpha x_j| \right)^{p_k} \right)^{\frac{1}{M}} \\ &\leq |\alpha_m - \alpha| h(x^m) + |\alpha| h(x^m - x). \end{aligned}$$

The last expression tends to zero as  $m \rightarrow \infty$ , that is, the scalar multiplication is continuous. Hence  $h$  is paranorm on the space  $V(\lambda; p)$ .

It remains to prove the completeness of the space  $V(\lambda; p)$ .

Let  $(x^n)$  be any Cauchy sequence in the space  $V(\lambda; p)$ , where  $x = (x_j^n) = (x_1^n, x_2^n, x_3^n, \dots)$ . Then, for a given  $\varepsilon > 0$ , there exists a positive integer  $n_0(\varepsilon)$  such that

$$h(x^n - x^m) < \frac{\varepsilon}{2}$$

for every  $m, n \geq n_0(\varepsilon)$ . By using the definition of  $h$ , we obtain that

$$\left( \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j^n - x_j^m| \right)^{p_k} \right)^{\frac{1}{M}} < \varepsilon^M$$

for every  $m, n \geq n_0(\varepsilon)$ . Also we get, for fixed  $j \in \mathbb{N}$ ,  $|x_j^n - x_j^m| < \varepsilon$  for every  $m, n \geq n_0(\varepsilon)$ . Hence it is clear that the sequences  $(x_j^n)$  is a Cauchy sequence in  $\mathbb{R}$ .



Since the real numbers set is complete, so we have  $x_j^n \rightarrow x_j$  for every  $n \geq n_0(\varepsilon)$  and as  $m \rightarrow \infty$ . Now we get

$$\left( \sum_{k=1}^r \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j^n - x_j| \right)^{p_k} \right) < \varepsilon^M.$$

If we pass to the limit over the  $r$  to infinity and  $n \geq n_0(\varepsilon)$  we obtained  $h(x^n - x) < \varepsilon$ . So, the sequence  $(x^n)$  is a Cauchy sequence in the space  $V(\lambda; p)$ .

It remains to show that the space  $V(\lambda; p)$  is complete. Since we have  $x = x^n - x^n + x$ , we get

$$\sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j| \right)^{p_k} \leq \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j^n - x_j| \right)^{p_k} + \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j^n| \right)^{p_k}.$$

Consequently, we obtain  $x \in V(\lambda; p)$ . This completes the proof.

b) By taking  $p_k = p$  in (a), it can be easily shown the proof of (b). ■

#### 4. SOME GEOMETRIC PROPERTIES OF THE SPACES $V_\rho(\lambda; p)$ AND $V_p(\lambda)$ .

In this section we construct the modular structure of the space  $V(\lambda; p)$  and since the Luxemburg norm is equivalent to usual norm of the space  $V_p(\lambda)$ , we show that the space  $V_p(\lambda)$  has the Banach-Saks type  $p$ .

Firstly, we will introduce a generalized modular sequence space  $V_\rho(\lambda; p)$  by

$$V_\rho(\lambda; p) := \{x \in \ell^0 : \rho(\lambda x) < \infty, \text{ for some } \lambda > 0\},$$

where

$$\rho(x) = \left( \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j| \right)^{p_k} \right).$$

It can be seen that  $\rho : V_\rho(\lambda; p) \rightarrow [0, \infty]$  is a modular on  $V_\rho(\lambda; p)$ .

Note that the Luxemburg norm on the sequence space  $V_\rho(\lambda; p)$  is defined as follows:

$$\|x\|_L = \inf \left\{ \lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1 \right\}, \quad \text{for all } x \in V_\rho(\lambda; p)$$

or equally

$$\|x\|_L = \inf \left\{ \lambda > 0 : \rho\left(\frac{x}{\lambda}\right) = \left( \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j| \right)^{p_k} \right) \leq 1 \right\}.$$

In the same way we can introduce the Amemiya norm (Orlicz norm) on the sequence space  $V_\rho(\lambda; p)$  as follows:

$$\|x\|_A = \inf_{\lambda > 0} \frac{1}{\lambda} (1 + \rho(\lambda x)) \quad \text{for all } x \in V_\rho(\lambda; p).$$

We now give some basic properties of the modular  $\rho$  on the space  $V_\rho(\lambda; p)$ . Also we will investigate some relationships between the modular  $\rho$  and the Luxemburg norm on  $V_\rho(\lambda; p)$ .

**Proposition 4.1.** *The functional  $\rho$  is a convex modular on  $V_\rho(\lambda; p)$ .*

**Proposition 4.2.** *For any  $x \in V_\rho(\lambda; p)$*

- i) if  $\|x\|_L \leq 1$ , then  $\rho(x) \leq \|x\|_L$ ;*
- ii)  $\|x\|_L = 1$  if and only if  $\rho(x) = 1$ .*

**Proposition 4.3.** *For any  $x \in V_\rho(\lambda; p)$ , we have*

- i) If  $0 < a < 1$  and  $\|x\|_L > a$ , then  $\rho(x) > a^H$ ;*
- ii) if  $a \geq 1$  and  $\|x\|_L < a$ , then  $\rho(x) < a^H$ .*

The proofs of the three propositions given above are proved with standard techniques in a similar way as in [19] and [3].

**Proposition 4.4.** *Let  $(x_n)$  be a sequence in  $V_\rho(\lambda; p)$ . Then:*

- i) if  $\lim_{n \rightarrow \infty} \|x_n\|_L = 1$ , then  $\lim_{n \rightarrow \infty} \rho(x_n) = 1$ ;*
- ii) if  $\lim_{n \rightarrow \infty} \rho(x_n) = 0$ , then  $\lim_{n \rightarrow \infty} \|x_n\|_L = 0$ .*

*Proof.* (i) Suppose that  $\lim_{n \rightarrow \infty} \|x_n\|_L = 1$ . Let  $\varepsilon \in (0, 1)$ . Then there exists  $n_0$  such that  $1 - \varepsilon < \|x_n\|_L < 1 + \varepsilon$  for all  $n \geq n_0$ . Since  $(1 - \varepsilon)^H < \|x_n\|_L < (1 + \varepsilon)^H$  for all  $n \geq n_0$  by the Proposition 4.3 (i) and (ii), we have  $\rho(x_n) \geq (1 - \varepsilon)^H$  and  $\rho(x_n) \leq (1 + \varepsilon)^H$ . Therefore  $\lim_{n \rightarrow \infty} \rho(x_n) = 1$ .

(ii) Suppose that  $\|x_n\|_L \rightarrow 0$ . Then there is an  $\varepsilon \in (0, 1)$  and a subsequence  $(x_{n_k})$  of  $(x_n)$  such that  $\|x_{n_k}\|_L > \varepsilon$  for all  $k \in \mathbb{N}$ . By the Proposition 4.3 (i), we obtain that  $\rho(x_{n_k}) > \varepsilon^H$  for all  $k \in \mathbb{N}$ . This implies that  $\rho(x_{n_k}) \not\rightarrow 0$  as  $n \rightarrow \infty$ . Hence  $\rho(x_n) \not\rightarrow 0$ . ■

**Theorem 4.5.** *The space  $V_\rho(\lambda; p)$  is a Banach space with respect to Luxemburg norm defined by*

$$\|x\|_L = \inf \left\{ \lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1 \right\}.$$

*Proof.* We show that every Cauchy sequence in  $V_\rho(\lambda; p)$  is convergent according to the Luxemburg norm.

Let  $(x^n(j))$  be any Cauchy sequence in  $V_\rho(\lambda; p)$  and  $\varepsilon \in (0, 1)$ . Thus, there exists  $n_0$  such that  $\|x_n - x_m\|_L < \varepsilon^M$  for all  $m, n \geq n_0$ . By the Proposition 3.2 (i), we obtain

$$(4.1) \quad \rho(x^n - x^m) < \|x^n - x^m\|_L < \varepsilon^M,$$

for all  $n, m \geq n_0$ , that is;

$$\sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x^n(j) - x^m(j)| \right)^{p_k} < \varepsilon$$

for all  $m, n \geq n_0$ . For fixed  $j \in \mathbb{N}$ , the last inequality gives that

$$|x^n(j) - x^m(j)| < \varepsilon$$

for all  $m, n \geq n_0$ . Hence we obtain that the sequence  $(x^n(j))$  is a Cauchy sequence in  $\mathbb{R}$ . Since  $\mathbb{R}$  is complete,  $x^m(j) \rightarrow x(j)$  as  $m \rightarrow \infty$ . Therefore, we have

$$\sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x^n(j) - x(j)| \right)^{p_k} < \varepsilon$$

for all  $n \geq n_0$ .

It remains to show that the sequence  $(x(j))$  is an element of  $V_\rho(\lambda; p)$ . From the inequality (4.1), we can write

$$\sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x^n(j) - x^m(j)| \right)^{p_k} < \varepsilon$$

for all  $m, n \geq n_0$ . For every  $j \in \mathbb{N}$ , we have  $x^m(j) \rightarrow x(j)$ , so we obtain that

$$\rho(x^n - x^m) \rightarrow \rho(x^n - x)$$

as  $m \rightarrow \infty$ . Since for all  $n \geq n_0$ ,

$$\sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x^n(j) - x^m(j)| \right)^{p_k} \rightarrow \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x^n(j) - x(j)| \right)^{p_k}$$

as  $m \rightarrow \infty$ , then by (4.1) we have  $\rho(x^n - x) \leq \|x^n - x\|_L < \varepsilon$  for all  $n \geq n_0$ . This means that  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . So, we have  $(x_{n_0} - x) \in V_\rho(\lambda; p)$ . Since  $V_\rho(\lambda; p)$  is a linear space, we have  $x = x_{n_0} - (x_{n_0} - x) \in V_\rho(\lambda; p)$ . Therefore the sequence space  $V_\rho(\lambda; p)$  is a Banach space with respect to Luxemburg norm. This completes the proof. ■

Next, we will show that the space  $V_\rho(\lambda; p)$  has Kadec-Klee property. To do this, we need the following Proposition.

**Proposition 4.6.** *Let  $x \in V_\rho(\lambda; p)$  and  $(x_n) \subseteq V_\rho(\lambda; p)$ . If  $\rho(x_n) \rightarrow \rho(x)$  as  $n \rightarrow \infty$  and  $x_n(j) \rightarrow x(j)$  as  $n \rightarrow \infty$  for all  $j \in \mathbb{N}$ , then  $x_n \rightarrow x$  as  $n \rightarrow \infty$ .*

*Proof.* Let  $\varepsilon > 0$ . Since  $\rho(x) = \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x(j)| \right)^{p_k} < \infty$ , there exists  $j \in \mathbb{N}$  such that

$$(4.2) \quad \sum_{k=n_0+1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x(j)| \right)^{p_k} < \frac{\varepsilon}{6K}$$

where  $K = \max\{1, 2^{H-1}\}$ .

Since  $\rho(x_n) - \sum_{k=1}^{n_0} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_n(j)| \right)^{p_k} \rightarrow \rho(x) - \sum_{k=1}^{n_0} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x(j)| \right)^{p_k}$  as  $n \rightarrow \infty$  and  $x_n(j) \rightarrow x(j)$  as  $n \rightarrow \infty$  for all  $j \in \mathbb{N}$ , there exists  $n_0 \in \mathbb{N}$  such that

$$(4.3) \quad \left| \sum_{k=n_0+1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_n(j)| \right)^{p_k} - \sum_{k=n_0+1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x(j)| \right)^{p_k} \right| < \frac{\varepsilon}{3K}$$

for all  $n \geq n_0$ . Also, since  $x_n(j) \rightarrow x(j)$  for all  $j \in \mathbb{N}$ , we have  $\rho(x_n) \rightarrow \rho(x)$  as  $n \rightarrow \infty$ . Hence for all  $n \geq n_0$ , we have  $|x_n(j) - x(j)| < \varepsilon$ . As a result, for all  $n \geq n_0$ , we have

$$(4.4) \quad \sum_{k=1}^{n_0} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_n(j) - x(j)| \right)^{p_k} < \frac{\varepsilon}{3}.$$

Then from (4.2), (4.3) and (4.4) it follows that for  $n \geq n_0$ ,

$$\begin{aligned} \rho(x_n - x) &= \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_n(j) - x(j)| \right)^{p_k} \\ &= \sum_{k=1}^{n_0} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_n(j) - x(j)| \right)^{p_k} + \sum_{k=n_0+1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_n(j) - x(j)| \right)^{p_k} \\ &< \frac{\varepsilon}{3} + K \left[ \sum_{k=n_0+1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_n(j)| \right)^{p_k} + \sum_{k=n_0+1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x(j)| \right)^{p_k} \right] \\ &= \frac{\varepsilon}{3} + K \left[ \rho(x_n) - \sum_{k=1}^{n_0} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_n(j)| \right)^{p_k} + \sum_{k=n_0+1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x(j)| \right)^{p_k} \right] \\ &< \frac{\varepsilon}{3} + K \left[ \rho(x) - \sum_{k=1}^{n_0} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_n(j)| \right)^{p_k} + \frac{\varepsilon}{3K} + \sum_{k=n_0+1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x(j)| \right)^{p_k} \right] \\ &= \frac{\varepsilon}{3} + K \left[ \sum_{k=n_0+1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x(j)| \right)^{p_k} + \frac{\varepsilon}{3K} + \sum_{k=n_0+1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x(j)| \right)^{p_k} \right] \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This shows that  $\rho(x_n - x) \rightarrow 0$  as  $n \rightarrow \infty$ . Hence by Proposition 4.4 (ii), we have  $\|x_n - x\|_L \rightarrow 0$  as  $n \rightarrow \infty$ . ■

Now, we give one of the main result of this paper involving geometric properties of the space  $V_\rho(\lambda; p)$ .

**Theorem 4.7.** *The space  $V_\rho(\lambda; p)$  has the Kadec-Klee property.*

*Proof.* Let  $x \in S(V_\rho(\lambda; p))$  and  $(x_n) \subseteq B(V_\rho(\lambda; p))$  such that  $\|x_n\|_L \rightarrow 1$  and  $x_n \xrightarrow{w} x$  as  $n \rightarrow \infty$ . From Proposition 4.2 (ii), we have  $\rho(x) = 1$ , so it follows

from Proposition 4.4 (i) that  $\rho(x_n) \rightarrow \rho(x)$  as  $n \rightarrow \infty$ . Since  $x_n \xrightarrow{w} x$  and the  $i^{th}$ -coordinate mapping  $\pi_j : V_p(\lambda; p) \rightarrow \mathbb{R}$  defined by  $\pi_j(x) = x(j)$  is continuous linear function on  $V_p(\lambda; p)$ , it follows that  $x_n(j) \rightarrow x(j)$  as  $n \rightarrow \infty$  for all  $j \in \mathbb{N}$ . Thus, by Proposition 4.6 that  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . ■

We prove the following theorem regarding the Banach-Saks of type  $p$  property.

**Theorem 4.8.** *The space  $V_p(\lambda)$  has the Banach-Saks of type  $p$ .*

*Proof.* From the Theorem 3.1 b), it is known that the space  $V_p(\lambda)$  is a Banach space with respect to the norm  $\|x\|_{V_p(\lambda)}$ .

Let  $(\varepsilon_n)$  be a sequence of positive numbers for which  $\sum_{n=1}^{\infty} \varepsilon_n \leq \frac{1}{2}$ . Let  $(x_n)$  be a weakly null sequence in  $B(V_p(\lambda))$ . Set  $b_0 = x_0 = 0$  and  $b_1 = x_{n_1} = x_1$ . Then there exists  $m_1 \in \mathbb{N}$  such that

$$\left\| \sum_{i=m_1+1}^{\infty} b_1(i)e^{(i)} \right\|_{V_p(\lambda)} < \varepsilon_1.$$

Since  $(x_n)$  is a weakly null sequence implies  $x_n \rightarrow 0$  (coordinatewise), there is an  $n_2 \in \mathbb{N}$  such that

$$\left\| \sum_{i=0}^{m_1} x_n(i)e^{(i)} \right\|_{V_p(\lambda)} < \varepsilon_1,$$

where  $n \geq n_2$ . Set  $b_2 = x_{n_2}$ . Then there exists an  $m_2 > m_1$  such that

$$\left\| \sum_{i=m_2+1}^{\infty} b_2(i)e^{(i)} \right\|_{V_p(\lambda)} < \varepsilon_2.$$

By using the fact that  $x_n \rightarrow 0$  (coordinatewise), there exists an  $n_3 > n_2$  such that

$$\left\| \sum_{i=0}^{m_2} x_n(i)e^{(i)} \right\|_{V_p(\lambda)} < \varepsilon_2,$$

where  $n \geq n_3$ .

If we continue this process, we can find two increasing subsequences  $(m_i)$  and  $(n_i)$  such that

$$\left\| \sum_{i=0}^{m_j} x_n(i)e^{(i)} \right\|_{V_p(\lambda)} < \varepsilon_j,$$

for each  $n \geq n_{j+1}$  and

$$\left\| \sum_{i=m_j+1}^{\infty} b_j(i)e^{(i)} \right\|_{V_p(\lambda)} < \varepsilon_j,$$

where  $b_j = x_{n_j}$ . Hence,

$$\left\| \sum_{j=0}^n b_j \right\|_{V_p(\lambda)} = \left\| \sum_{j=0}^n \left( \sum_{i=0}^{m_{j-1}} b_j(i)e^{(i)} + \sum_{i=m_{j-1}+1}^{m_j} b_j(i)e^{(i)} + \sum_{i=m_j+1}^{\infty} b_j(i)e^{(i)} \right) \right\|_{V_p(\lambda)}$$

$$\begin{aligned}
&\leq \left\| \sum_{j=0}^n \left( \sum_{i=m_{j-1}+1}^{m_j} b_j(i) e^{(i)} \right) \right\|_{V_p(\lambda)} + \left\| \sum_{j=0}^n \left( \sum_{i=0}^{m_{j-1}} b_j(i) e^{(i)} \right) \right\|_{V_p(\lambda)} + \left\| \sum_{j=0}^n \left( \sum_{i=m_j+1}^{\infty} b_j(i) e^{(i)} \right) \right\|_{V_p(\lambda)} \\
&\leq \left\| \sum_{j=0}^n \left( \sum_{i=m_{j-1}+1}^{m_j} b_j(i) e^{(i)} \right) \right\|_{V_p(\lambda)} + 2 \sum_{j=0}^n \varepsilon_j.
\end{aligned}$$

On the other hand, since

$$\|x_n\|_{V_p(\lambda)} = \left( \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_{n_k}(j)| \right)^p \right)^{\frac{1}{p}}, \text{ it can be seen that } \|x_n\|_{V_p(\lambda)} < 1.$$

Therefore  $\|x_n\|_{V_p(\lambda)}^p < 1$ . We have

$$\begin{aligned}
\left\| \sum_{j=0}^n \left( \sum_{i=m_{j-1}+1}^{m_j} b_j(i) e^{(i)} \right) \right\|_{V_p(\lambda)}^p &= \sum_{j=0}^n \sum_{i=m_{j-1}+1}^{m_j} \left( \frac{1}{\lambda_i} \sum_{v \in I_i} |b_j(v)| \right)^p \\
&\leq \sum_{j=0}^n \sum_{i=0}^{\infty} \left( \frac{1}{\lambda_i} \sum_{v \in I_i} |b_j(v)| \right)^p \\
&\leq (n+1).
\end{aligned}$$

Hence we obtain,

$$\left\| \sum_{j=0}^n \left( \sum_{i=m_{j-1}+1}^{m_j} b_j(i) e^{(i)} \right) \right\|_{V_p(\lambda)} \leq (n+1)^{\frac{1}{p}}.$$

By using the fact  $1 \leq (n+1)^{\frac{1}{p}}$  for all  $n \in \mathbb{N}$  and  $1 \leq p < \infty$ , we have

$$\left\| \sum_{j=0}^n b_j \right\|_{V_p(\lambda)} \leq (n+1)^{\frac{1}{p}} + 1 \leq 2(n+1)^{\frac{1}{p}}.$$

Hence  $V_p(\lambda)$  has the Banach-Saks type  $p$ . This completes the proof of the theorem. ■

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# Composition operators on logarithmic $\alpha$ -Bloch spaces

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## Abstract

In this paper we discuss the composition operator on the logarithmic  $\alpha$ -Bloch space  $LB_\alpha$  on the unit disk. The main results are as follows: (i) the sufficient and necessary conditions of  $C_\varphi$  being bounded on the  $LB_\alpha$  and  $LB_{\alpha,0}$ ; (ii) the sufficient and necessary conditions of  $C_\varphi$  being compacted on the  $LB_\alpha$  and  $LB_{\alpha,0}$ .

**Keywords:** Composition operator; Bloch space; Boundedness; Compactness

**MSC 2000:** Primary 47B38, secondary 30D05, 30H05

## 1 Introduction

Let  $D = \{z : |z| < 1\}$  be the open unit disk in the complex plane  $\mathbf{C}$ , and  $H(D)$  denote the set of all analytic functions on  $D$ . For  $\alpha > 0$ , a function  $f \in H(D)$  is said to belong to the logarithmic  $\alpha$ -Bloch space  $LB_\alpha$  if

$$\|f\|_{LB_\alpha} = \sup\{(1 - |z|^2)^\alpha \ln\left(\frac{2}{1 - |z|^2}\right)|f'(z)| : z \in D\} < +\infty$$

and to the little logarithmic  $\alpha$ -Bloch space  $LB_{\alpha,0}$  if

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha \ln\left(\frac{2}{1 - |z|^2}\right)|f'(z)| = 0.$$

It is easily proved that  $LB_\alpha$  is a Banach space under the norm  $\|f\|_\alpha = |f(0)| + \|f\|_{LB_\alpha}$  and that  $LB_{\alpha,0}$  is a closed subspace of  $LB_\alpha$ . when  $\alpha = 1$ , the  $LB_\alpha$  space is called the logarithmic Bloch space  $LB_1$ . Some sources for results and references about the logarithmic Bloch functions are the papers of Yoneda [10], Stević [4, 5], and the author [6, 7, 8, 9].

Let  $\varphi$  be a holomorphic self-map of  $D$ . The composition operator  $C_\varphi$  is defined by

$$C_\varphi(f) = f \circ \varphi, \quad f \in H(D).$$

It is easy to see that an operator defined in this manner is linear. It is interesting to provide a function theoretic characterization when  $\varphi$  induces a bounded or compact operator on various spaces (see [1, 2, 3, 11] for more information). In the logarithmic Bloch space  $LB_1$ , the author [8] has characterized the pointwise multiplier operator and R. Yoneda [10] studied the composition operator respectively. We shall study the conditions for which  $C_\varphi$  is a bounded operator or a compact operator on the logarithmic  $\alpha$ -Bloch  $LB_\alpha$  and the little logarithmic  $LB_{\alpha,0}$  spaces. In this paper,  $C$  denotes the constant depending only on the index  $\alpha$ ; the  $C$  may differ at different places.

## 2 The boundedness of $C_\varphi$

**Lemma 2.1** *Let  $\alpha > 0$  and  $f(z) = \frac{(1 - |z|)^\alpha \ln \frac{2}{1 - |z|}}{|1 - z|^\alpha \ln \frac{4}{|1 - z|}}$ ,  $z \in D$ . Then  $|f(z)| \leq \max(1, \frac{1}{\alpha \ln 2})$ .*

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**Proof** Since  $r(x) = x^\alpha \ln \frac{2}{x}$  is increasing on  $(0, 2e^{-\frac{1}{\alpha}}]$ , decreasing on  $[2e^{-\frac{1}{\alpha}}, 2]$  and  $r(2e^{-\frac{1}{\alpha}}) = \frac{2^\alpha}{\alpha e}$ , we have  $|f(z)| \leq 1$  where  $z \in D_1 = \{z \in D : |1 - z| < 2e^{-\frac{1}{\alpha}}\}$ .

On the other hand, for  $z \in D \setminus D_1$ ,

$$|f(z)| \leq \frac{(1 - |z|)^\alpha \ln \frac{2}{1 - |z|}}{(2e^{-\frac{1}{\alpha}})^\alpha \ln 2} \leq \frac{\frac{2^\alpha}{\alpha e}}{(2e^{-\frac{1}{\alpha}})^\alpha \ln 2} = \frac{1}{\alpha \ln 2},$$

hence  $|f(z)| \leq \max(1, \frac{1}{\alpha \ln 2})$ .

**Lemma 2.2** Let  $\alpha > 0$  and  $g(x) = (1 - x)^\alpha \ln \frac{2}{1 - x}$ ,  $x \in [0, 1)$ . Then  $\frac{g(x)}{g(tx)} \leq 1 + \frac{1}{\alpha e \ln 2}$  for each  $t \in [0, 1]$ .

**Proof** Since  $x^\alpha \ln \frac{1}{x} \leq \frac{1}{\alpha e}$  for each  $x \in (0, 1]$ , we have

$$\begin{aligned} \frac{g(x)}{g(tx)} &= \left(\frac{1 - x}{1 - tx}\right)^\alpha \left(\frac{\ln \frac{2}{1 - x}}{\ln \frac{2}{1 - tx}} - 1\right) + \left(\frac{1 - x}{1 - tx}\right)^\alpha \\ &\leq \left(\frac{1 - x}{1 - tx}\right)^\alpha \ln \frac{1 - tx}{1 - x} \frac{1}{\ln \frac{2}{1 - tx}} + 1 \leq \frac{1}{\alpha e \ln 2} + 1. \end{aligned}$$

**Lemma 2.3** Let  $\alpha > 0$  and  $f \in LB_\alpha$ , then exists constant  $C$  such that  $\|f_t\|_\alpha \leq C\|f\|_\alpha$ ,  $0 < t < 1$ , where  $f_t(z) = f(tz)$ .

The result can be easily proved by lemma 2.2.

**Theorem 2.1** Let  $\alpha > 0$ , then  $C_\varphi$  is a bounded operator on  $LB_\alpha$  if and only if

$$\sup\left\{\frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2)^\alpha \ln \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| : z \in D\right\} < +\infty. \quad (1)$$

**Proof** Suppose that (1) holds. For  $\forall f \in LB_\alpha$ , we have

$$\begin{aligned} &\sup_{z \in D} (1 - |z|^2)^\alpha \ln \left(\frac{2}{1 - |z|^2}\right) |(C_\varphi(f))'(z)| \\ &= \sup_{z \in D} (1 - |z|^2)^\alpha \ln \left(\frac{2}{1 - |z|^2}\right) |f'(\varphi(z))| |\varphi'(z)| \\ &\leq \sup_{z \in D} |f'(\varphi(z))| (1 - |\varphi(z)|^2)^\alpha \ln \left(\frac{2}{1 - |\varphi(z)|^2}\right) \times \sup_{z \in D} \frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2)^\alpha \ln \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \\ &\leq C\|f\|_{LB_\alpha}. \end{aligned}$$

This shows that  $C_\varphi$  is bounded.

Conversely, suppose that  $C_\varphi$  is a bounded operator on  $LB_\alpha$ . Then  $\|C_\varphi(f)\|_\alpha \leq \|C_\varphi\| \|f\|_\alpha$  for all  $f \in LB_\alpha$ . On the other hand, we take the test function  $f(z) = z$ , which shows  $\varphi \in LB_\alpha$ . For  $\forall 0 \neq w \in D$ , let

$$f_w(z) = \int_0^z (1 - \frac{\bar{w}^2}{|w|^2} z^2)^{-\alpha} \left(\ln \frac{4}{1 - \frac{\bar{w}^2}{|w|^2} z^2}\right)^{-1} dz.$$

By Lemma 2.1, we have

$$\sup_{z_1 \in D} (1 - |z_1|^2)^\alpha \left(\ln \frac{2}{1 - |z_1|^2}\right) |1 - z_1^2|^{-\alpha} \left|\ln \frac{4}{1 - z_1^2}\right|^{-1} \leq C < +\infty.$$

Applying to  $z_1 = \frac{\bar{w}}{|w|}z$ , we have

$$\sup_{z \in D} (1 - |z|^2)^\alpha \left( \ln \frac{2}{1 - |z|^2} \right) \left| 1 - \frac{\bar{w}^2}{|w|^2} z^2 \right|^{-\alpha} \left| \ln \frac{4}{1 - \frac{\bar{w}^2}{|w|^2} z^2} \right|^{-1} \leq C < +\infty.$$

Hence we have  $f_w \in LB_\alpha$  for  $w \neq 0$ . Then for  $w \neq 0$  we get

$$\|C_\varphi(f_w)\|_{LB_\alpha} \leq \|C_\varphi(f_w)\|_\alpha \leq \|C_\varphi\| \|f_w\|_\alpha = \|C_\varphi\| \|f_w\|_{LB_\alpha} = C < +\infty.$$

So

$$\sup_{z \in D} (1 - |z|^2)^\alpha \ln \left( \frac{2}{1 - |z|^2} \right) |f'_w(\varphi(z))| |\varphi'(z)| \leq C < +\infty. \quad (2)$$

For  $\forall z \in D$  with  $\varphi(z) \neq 0$ , applying  $w = \varphi(z)$  in (2), we have

$$\frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2)^\alpha \ln \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \leq C < +\infty.$$

For  $\forall z \in D$  with  $\varphi(z) = 0$ , since  $\varphi \in LB_\alpha$ , we have

$$\frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2)^\alpha \ln \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \leq \frac{1}{\ln 2} \|\varphi\|_{LB_\alpha} < +\infty.$$

Hence (1) holds. This completes the proof of this theorem.

**Remark 1** *There is a problem in the proof of the Theorem 1 of R. Yonda's 2002 Arch. Math. paper [10]. The reason is that the test function*

$$f_w(z) = \int_0^z \left(1 - \frac{\bar{w}^2}{|w|^2} z^2\right)^{-1} \left(\ln \frac{2}{1 - \frac{\bar{w}^2}{|w|^2} z^2}\right)^{-1} dz.$$

in R. Yonda's paper does not belong to  $LB_1$ . In fact, let  $w = \frac{1}{2}$  and  $z = ir$ , then

$$(1 - |z|^2) \ln \frac{2}{1 - |z|^2} |f'_w(z)| = \frac{(1 - r^2) \ln \frac{2}{1 - r^2}}{(1 + r^2) \ln \frac{2}{1 + r^2}} \longrightarrow \infty$$

as  $r \longrightarrow 1$ .

**Theorem 2.2** *Let  $\alpha > 0$ , then  $C_\varphi$  is a bounded operator on  $LB_{\alpha,0}$  if and only if  $\varphi \in LB_{\alpha,0}$  and*

$$\sup \left\{ \frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2)^\alpha \ln \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| : z \in D \right\} < +\infty. \quad (3)$$

**Proof** Suppose  $\varphi \in LB_{\alpha,0}$  and (3) holds. Let

$$M = \sup \left\{ \frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2)^\alpha \ln \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| : z \in D \right\}.$$

Assume that  $f \in LB_{\alpha,0}$ . Then given  $\epsilon > 0$  there exists  $0 < r < 1$  such that

$$(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2} |f'(z)| < \frac{\epsilon}{M}$$

whenever  $|z| > r$ .

On the other hand, since  $\varphi \in LB_{\alpha,0}$ , there exists  $0 < R < 1$  such that

$$(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2} |\varphi'(z)| < \frac{(1 - r^2)^\alpha \ln \frac{2}{1 - r^2}}{\|f\|_{LB_\alpha}} \epsilon$$

whenever  $|z| > R$ .

Then, for  $|z| > R$  such that  $|\varphi(z)| > r$ , we have

$$\begin{aligned} & (1 - |z|^2)^\alpha \ln \left( \frac{2}{1 - |z|^2} \right) |(C_\varphi(f))'(z)| \\ &= (1 - |\varphi(z)|^2)^\alpha \ln \left( \frac{2}{1 - |\varphi(z)|^2} \right) |f'(\varphi(z))| \times \frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2)^\alpha \ln \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \\ &\leq \frac{\epsilon}{M} M = \epsilon. \end{aligned}$$

For  $|z| > R$  such that  $|\varphi(z)| \leq r$ , we have

$$\begin{aligned} & (1 - |z|^2)^\alpha \ln \left( \frac{2}{1 - |z|^2} \right) |(C_\varphi(f))'(z)| \\ &= (1 - |\varphi(z)|^2)^\alpha \ln \left( \frac{2}{1 - |\varphi(z)|^2} \right) |f'(\varphi(z))| \times \frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2)^\alpha \ln \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \\ &\leq \|f\|_{LB_\alpha} \frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2}}{(1 - r^2)^\alpha \ln \frac{2}{1 - r^2}} |\varphi'(z)| < \epsilon. \end{aligned}$$

Hence we have  $(1 - |z|^2)^\alpha \ln \left( \frac{2}{1 - |z|^2} \right) |(C_\varphi(f))'(z)| < \epsilon$ , which show  $C_\varphi(f) \in LB_{\alpha,0}$ .

Conversely, suppose that  $C_\varphi$  is bounded in  $LB_{\alpha,0}$ . First we take test function  $f(z) = z$ , then  $\varphi \in LB_{\alpha,0}$ .

Next we have  $\|C_\varphi(g)\|_\alpha \leq \|C_\varphi\| \|g\|_\alpha$  for all  $g \in LB_{\alpha,0}$ . For any  $f \in LB_\alpha$  and every  $0 < t < 1$ , we have  $f_t = f(tz) \in LB_{\alpha,0}$ . Then by Lemma 2.3, we get

$$\|C_\varphi(f_t)\|_\alpha \leq \|C_\varphi\| \|f_t\|_\alpha \leq C \|C_\varphi\| \|f\|_\alpha.$$

Let  $t \rightarrow 1$ , we have  $\|C_\varphi(f)\|_\alpha \leq C \|C_\varphi\| \|f\|_\alpha$ . So  $C_\varphi$  is bounded in  $LB_\alpha$ . By Theorem 2.1, (3) holds. This proof is completed.

### 3 The compactness of $C_\varphi$

**Lemma 3.1** Suppose  $\alpha > 0$  and  $f \in LB_\alpha$ , then

- (1)  $|f(z)| \leq C \|f\|_\alpha$ , where  $\alpha < 1$ ;
- (2)  $|f(z)| \leq (1 + C \ln(\ln \frac{2}{1 - |z|})) \|f\|_\alpha$ , where  $\alpha = 1$ ;
- (3)  $|f(z)| \leq (1 + \frac{C}{(1 - |z|)^{\alpha-1}}) \|f\|_\alpha$ , where  $\alpha > 1$ .

The proof follows from the same method as the one for Lemma 2.1 in [4]. We omit the details.

**Lemma 3.2** Let  $C_\varphi$  be a bounded operator on  $LB_\alpha$ , then  $C_\varphi$  is compact if and only if for any bounded sequence  $\{f_n\}$  in  $LB_\alpha$  which converges to 0 uniformly on compact subsets of  $D$ , we have  $\|C_\varphi(f_n)\|_\alpha \rightarrow 0$  as  $n \rightarrow \infty$ .

The result can be proved by using Montel theorem, Lemma 2.3 and 3.1; the details are omitted here.

**Lemma 3.3** Suppose  $\alpha > 0$ , then a closed set  $U$  in  $LB_{\alpha,0}$  is compact if and only if it is bounded and satisfies

$$\lim_{|z| \rightarrow 1} \sup_{f \in U} (1 - |z|^2)^\alpha \ln \left( \frac{2}{1 - |z|^2} \right) |f'(z)| = 0. \quad (4)$$

The proof is similar to that of [1, Lemma 1].

**Theorem 3.1** *Let  $\alpha > 0$ , then  $C_\varphi$  is compact on  $LB_{\alpha,0}$  if and only if*

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2)^\alpha \ln \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| = 0.$$

**Proof** By Lemma 3.3,  $C_\varphi$  is compact if and only if

$$\lim_{|z| \rightarrow 1} \sup_{\|f\|_\alpha \leq 1} (1 - |z|^2)^\alpha \ln \left( \frac{2}{1 - |z|^2} \right) |(C_\varphi(f))'(z)| = 0.$$

On the other hand, we have

$$\begin{aligned} & (1 - |z|^2)^\alpha \ln \left( \frac{2}{1 - |z|^2} \right) |(C_\varphi(f))'(z)| \\ &= (1 - |\varphi(z)|^2)^\alpha \ln \left( \frac{2}{1 - |\varphi(z)|^2} \right) |f'(\varphi(z))| \times \frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2)^\alpha \ln \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)|, \end{aligned}$$

and

$$\sup_{\|f\|_\alpha \leq 1} (1 - |z|^2)^\alpha \ln \left( \frac{2}{1 - |z|^2} \right) |f'(z)| = 1$$

Hence  $C_\varphi$  is compact on  $LB_{\alpha,0}$  if and only if

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2)^\alpha \ln \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| = 0.$$

**Theorem 3.2** *Let  $\alpha > 0$ , then  $C_\varphi$  is a compact operator on  $LB_\alpha$  if and only if for every  $\epsilon > 0$ , there exists  $0 < r < 1$ , such that*

$$\frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2)^\alpha \ln \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| < \epsilon. \quad (5)$$

whenever  $|\varphi(z)| > r$ .

**Proof** Assume that (5) holds. Then it clearly shows that  $\varphi \in LB_\alpha$  and  $C_\varphi$  is bounded by Theorem 2.1. Let  $\{f_n\}$  be a bounded sequence in  $LB_\alpha$  which converges to 0 uniformly on compact subsets of  $D$ . We only need to prove  $\lim_{n \rightarrow \infty} \|C_\varphi(f_n)\|_\alpha = 0$  by Lemma 3.2. Let  $M = \sup_n \|f_n\|_{LB_\alpha} < +\infty$ . Given  $\epsilon > 0$ , there exists  $0 < r < 1$  such that

$$\frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2)^\alpha \ln \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| < \frac{\epsilon}{M},$$

whenever  $|\varphi(z)| > r$ .

Then, for  $|\varphi(z)| > r$ , we have

$$\begin{aligned} \|C_\varphi(f_n)\|_{LB_\alpha} &= (1 - |z|^2)^\alpha \ln \left( \frac{2}{1 - |z|^2} \right) |(C_\varphi(f_n))'(z)| \\ &= (1 - |\varphi(z)|^2)^\alpha \ln \left( \frac{2}{1 - |\varphi(z)|^2} \right) |f'_n(\varphi(z))| \times \frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2)^\alpha \ln \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \\ &\leq M \frac{\epsilon}{M} = \epsilon. \end{aligned}$$

Let

$$M_1 = \sup_{|\varphi(z)| \leq r} \frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2)^\alpha \ln \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \leq \frac{\|\varphi\|_{LB_\alpha}}{(1 - r^2) \ln 2} < +\infty.$$

On the other hand, because  $C_\varphi(f_n)(0)$  and  $(1 - |w|^2)^\alpha \ln \frac{2}{1 - |w|^2} |f'_n(w)|$  converge to 0 uniformly on  $|w| \leq r$  as  $n \rightarrow \infty$ , we have, for large enough  $n$ ,  $C_\varphi(f_n)(0) < \epsilon$  and  $(1 - |w|^2)^\alpha \ln \frac{2}{1 - |w|^2} |f'_n(w)| < \frac{\epsilon}{M_1}$  whenever  $|w| \leq r$ . Then for large enough  $n$  we have

$$\begin{aligned} \|C_\varphi(f_n)\|_\alpha &= C_\varphi(f_n)(0) + \sup_z (1 - |z|^2)^\alpha \ln \left( \frac{2}{1 - |z|^2} \right) |(C_\varphi(f_n))'(z)| \\ &\leq \epsilon + \sup_{|\varphi(z)| > r} (1 - |z|^2)^\alpha \ln \left( \frac{2}{1 - |z|^2} \right) |(C_\varphi(f_n))'(z)| \\ &\quad + \sup_{|\varphi(z)| \leq r} (1 - |z|^2)^\alpha \ln \left( \frac{2}{1 - |z|^2} \right) |(C_\varphi(f_n))'(z)| \\ &\leq 2\epsilon + \sup_{|\varphi(z)| \leq r} (1 - |\varphi(z)|^2)^\alpha \ln \left( \frac{2}{1 - |\varphi(z)|^2} \right) |f'_n(\varphi(z))| \frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2)^\alpha \ln \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \\ &\leq 2\epsilon + \frac{\epsilon}{M_1} M_1 = 3\epsilon. \end{aligned}$$

This shows  $\lim_{n \rightarrow \infty} \|C_\varphi(f_n)\|_\alpha = 0$ .

Conversely, suppose  $C_\varphi$  is compact on  $LB_\alpha$ . Assume that (5) fails. Then there exists a sequence  $\{z_n\} \subset D$  and an  $\epsilon_0 > 0$  such that  $|z_n| \rightarrow 1$  ( $n \rightarrow \infty$ ) and

$$\frac{(1 - |z|^2)^\alpha \ln \frac{2}{1 - |z|^2}}{(1 - |\varphi(z)|^2)^\alpha \ln \frac{2}{1 - |\varphi(z)|^2}} |\varphi'(z)| \geq \epsilon_0.$$

Let  $\varphi(z_n) = r_n e^{i\theta_n}$ , we take

$$f_n(z) = \int_0^z \left( \frac{r_n}{1 - e^{-i\theta_n} w r_n} - \frac{r_n^2}{1 - r_n^2 e^{-i\theta_n} w} \right)^\alpha \left( \ln \frac{4}{1 - r_n^2 e^{-i\theta_n} w} \right)^{-1} dw.$$

We get  $\sup_n \|f_n\|_\alpha < \infty$  and  $|f_n(z)| \leq \left( \frac{1 - r_n}{(1 - |z|^2)^2} \right)^\alpha (\ln 2)^{-1}$  by Lemma 2.1 and 2.2. Then  $\{f_n\}$  is a bounded sequence on  $LB_\alpha$  which converges to 0 uniformly on compact subsets of  $D$ . On the other hand, for enough large  $n$ , it follows that

$$\begin{aligned} \|C_\varphi(f_n)\|_\alpha &\geq (1 - |z_n|^2)^\alpha \ln \frac{2}{1 - |z_n|^2} |f'_n(\varphi(z_n))| |\varphi'(z_n)| \\ &= (1 - |z_n|^2)^\alpha \ln \frac{2}{1 - |z_n|^2} \left( \frac{1}{1 - r_n} - \frac{r_n}{1 - r_n^2} \right)^\alpha \left( \ln \frac{4}{1 - r_n^2} \right)^{-1} |\varphi'(z_n)| \\ &= \left( \frac{r_n}{1 + r_n + r_n^2} \right)^\alpha \frac{(1 - |z_n|^2)^\alpha \ln \frac{2}{1 - |z_n|^2}}{(1 - |\varphi(z_n)|^2)^\alpha \ln \frac{4}{1 - |\varphi(z_n)|^2}} |\varphi'(z_n)| \\ &\geq \frac{1}{3^\alpha} \frac{\epsilon_0}{2}. \end{aligned}$$

This contradicts the compactness of  $C_\varphi$  by Lemma 3.2. The proof is completed.

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# Statistical convergence in fuzzy 2-normed space

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**Abstract.** Motivated by the notion of 2-norm due to Gähler [S. Gähler, 2-metrische Räume und ihre topologische Struktur, Math. Nachr. 26 (1963) 115-148], in this paper we define and study the concept of statistical convergence and statistically Cauchy sequence in fuzzy 2-normed space which provide better tool to study a more general class of sequences. We also introduce here statistical limit point and statistical cluster point in fuzzy 2-normed space.

*Keywords and phrases:* Fuzzy 2-normed space; statistical convergence; statistically Cauchy sequence; statistical limit point; statistical cluster point; 2-normed space.

## 1. Introduction and preliminaries

By modifying own studies on fuzzy topological vector spaces, Katsaras [13] first introduced the notion of fuzzy seminorm and norm on a vector space and later on Felbin [6] gave the concept of a fuzzy normed space (for short, FNS) by applying the notion fuzzy distance of Kaleva and Seikala [12] on vector spaces. Further, Xiao and Zhu [23] improved a bit the Felbin's definition of fuzzy norm of a linear operator between FNSs. Recently, Bag and Samanta [2] has given another notion of boundedness in FNS and introduced another type of boundedness of operators. With the novelty of their approach they can introduce the fuzzy dual spaces and some important analogues of fundamental theorems in classical functional analysis [3]. Certainly there are some situations where the ordinary norm does not work and the concept of fuzzy norm seems to be more suitable in such cases, that is, we can deal with such situations by modelling the inexactness of the norm in some situations.

The idea of statistical convergence was introduced by Fast [5] and Steinhaus [22] independently in the same year 1951 and later on studied by various authors. Active researches on this topic were started after the papers of Šalát [19] and Fridy [8]. Recently, fuzzy version of this concept were discussed in [15,16,20,21].

The concept of 2-normed spaces was initially introduced by Gähler [10] in the 1960s. Since then, this concept has been studied by many authors, see for instance [11,17,18].

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Our aim for this paper is to generalize the definition of Felbin's FNS into fuzzy 2-normed space using the idea of Gähler. We define and study the concept of statistical convergence and statistical Cauchy in fuzzy 2-normed space. Further, we introduce the concept of statistical limit point and statistical cluster point in fuzzy 2-normed spaces.

Firstly, we recall some notations and basic definitions which we will use throughout the paper.

According to Mizumoto and Tanaka [14], a fuzzy number is a mapping  $x : \mathbb{R} \rightarrow [0, 1]$  over the set  $\mathbb{R}$  of all real numbers. A fuzzy number  $x$  is convex if  $x(t) \geq \min\{x(s), x(r)\}$  where  $s \leq t \leq r$ . If there exists a  $t_0 \in \mathbb{R}$  such that  $x(t_0) = 1$ , then  $x$  is called normal. For  $0 < \alpha \leq 1$ ,  $\alpha$ -level set of an upper semi continuous convex normal fuzzy number  $\eta$  (denoted by  $[\eta]_\alpha$ ) is a closed interval  $[a_\alpha, b_\alpha]$ , where  $a_\alpha = -\infty$  and  $b_\alpha = +\infty$  admissible. When  $a_\alpha = -\infty$ , for instance, then  $[a_\alpha, b_\alpha]$  means the interval  $(-\infty, b_\alpha]$ . Similar is the case when  $b_\alpha = +\infty$ . A fuzzy number  $x$  is called non-negative if  $x(t) = 0$ , for all  $t < 0$ . We denote the set of all convex, normal, upper semicontinuous fuzzy real numbers by  $L(\mathbb{R})$  and the set of all non-negative, convex, normal, upper semicontinuous fuzzy real numbers by  $L(\mathbb{R}^*)$ . Given a number  $r \in \mathbb{R}$ , we define a corresponding fuzzy number  $\tilde{r}$  by

$$\tilde{r}(t) = \begin{cases} 1 & \text{if } t = r, \\ 0 & \text{otherwise.} \end{cases}$$

As  $\alpha$ -level sets of a convex fuzzy number is an interval, there is a debate in the nomenclature of fuzzy numbers/fuzzy real numbers. In [4], Dubois and Prade suggested to call this as fuzzy interval.

A partial ordering  $\preceq$  on  $L(\mathbb{R})$  is defined by  $u \preceq v$  if and only if  $u_\alpha^- \leq v_\alpha^-$  and  $u_\alpha^+ \leq v_\alpha^+$  for all  $\alpha \in [0, 1]$ , where  $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$  and  $[v]_\alpha = [v_\alpha^-, v_\alpha^+]$ . The strict inequality in  $L(\mathbb{R})$  is defined by  $u \prec v$  if and only if  $u_\alpha^- < v_\alpha^-$  and  $u_\alpha^+ < v_\alpha^+$  for all  $\alpha \in [0, 1]$ . For  $k > 0$ ,  $ku$  is defined as  $ku(t) = u(t/k)$  and  $(0u)(t)$  is defined to be  $\tilde{0}(t)$ .

According to Mizumoto and Tanaka [14], the arithmetic operations  $\oplus, \ominus, \otimes$  on  $L(\mathbb{R}) \times L(\mathbb{R})$  are defined by

$$(x \oplus y)(t) = \sup_{s \in \mathbb{R}} \min\{x(s), y(t-s)\}, (x \ominus y)(t) = \sup_{s \in \mathbb{R}} \min\{x(s), y(s-t)\} \text{ and}$$

$$(x \otimes y)(t) = \sup_{s \in \mathbb{R}, s \neq 0} \min\{x(s), y(t/s)\},$$

for all  $t \in \mathbb{R}$ .

Let  $u, v \in L(\mathbb{R})$ . Define

$$D(u, v) = \sup_{\alpha \in [0, 1]} \max\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\},$$

then  $D$  is called the supremum metric on  $L(\mathbb{R})$ . Let  $(u_n) \subset L(\mathbb{R})$  and  $u \in L(\mathbb{R})$ . We say that a sequence  $(u_n)$  converges to  $u$  in the metric  $D$  (for short,  $D$ -converges to  $u$ ), written as  $u_n \xrightarrow{D} u$  or  $(D)\text{-}\lim_{n \rightarrow \infty} u_n = u$  if  $\lim_{n \rightarrow \infty} D(u_n, u) = 0$ .



## 2. Fuzzy 2-normed spaces

In this section, we generalize the definition of Felbin's FNS into fuzzy 2-normed space using the idea of Gähler [10]. Here, we also define the convergence of a sequence in fuzzy 2-normed space.

Let  $X$  be a real vector space of dimension  $d$ , where  $2 \leq d \leq \infty$ . A 2-norm on  $X$  is a function  $\|\cdot, \cdot\| : X \times X \rightarrow \mathbb{R}$  which satisfies (i)  $\|x, y\| = 0$  if and only if  $x$  and  $y$  are linearly dependent, (ii)  $\|x, y\| = \|y, x\|$  for all  $x, y \in X$ , (iii)  $\|\alpha x, y\| = |\alpha| \|x, y\|$ , whenever  $x, y \in X$  and  $\alpha \in \mathbb{R}$ , (iv)  $\|x + y, z\| = \|x, z\| + \|y, z\|$  for all  $x, y, z \in X$ .

The pair  $(X, \|\cdot, \cdot\|)$  is then called a 2-normed space.

As an example of a 2-normed space take  $X = \mathbb{R}^2$  being equipped with the 2-norm  $\|x, y\| :=$  the area of the parallelogram spanned by the vectors  $x$  and  $y$ , which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|, \quad \text{where } x = (x_1, x_2), y = (y_1, y_2).$$

We define the following:

**Definition 2.1.** The quadruple  $(X, \|\cdot, \cdot\|^\sim, L, R)$  is said to be fuzzy 2-normed space (for short FTNS) if  $X$  is a vector space over  $\mathbb{R}$ ,  $\|\cdot, \cdot\|^\sim : X \times X \rightarrow L^*(\mathbb{R})$ ,  $L, R : [0, 1] \times [0, 1] \rightarrow [0, 1]$  be symmetric, non-decreasing in both arguments such that  $L(0, 0) = 0$  and  $R(1, 1) = 1$  satisfying the following conditions for every  $x, y, z \in X$  and  $s, t \in \mathbb{R}$ :

- (i)  $\|x, y\|^\sim = \tilde{0}$  if and only if  $x$  and  $y$  are linearly dependent,
- (ii)  $\|x, y\|^\sim = \|y, x\|^\sim$ ,
- (iii)  $\|\alpha x, y\|^\sim = |\alpha| \|x, y\|^\sim$ ,  $\alpha \in \mathbb{R}$ ,
- (iv)  $\|x + y, z\|^\sim(s + t) \geq L(\|x, z\|^\sim(s), \|y, z\|^\sim(t))$  whenever  $s \leq \|x, z\|_1^-, t \leq \|y, z\|_1^-$  and  $s + t \leq \|x + y, z\|_1^-$ ,
- (v)  $\|x + y, z\|^\sim(s + t) \leq R(\|x, z\|^\sim(s), \|y, z\|^\sim(t))$  whenever  $s \geq \|x, z\|_1^-, t \geq \|y, z\|_1^-$  and  $s + t \geq \|x + y, z\|_1^-$ ,

where  $[\|x, z\|^\sim]_\alpha = [\|x, z\|_\alpha^-, \|x, z\|_\alpha^+]$  for  $x, z \in X$ ,  $0 \leq \alpha \leq 1$  and  $\inf_{\alpha \in [0, 1]} \|x, z\|_\alpha^- > 0$ .

In this case  $\|\cdot, \cdot\|^\sim$  is called a fuzzy 2-norm.

let us consider the topological structure of a FTNS  $(X, \|\cdot, \cdot\|^\sim, L, R)$ . For any  $\epsilon > 0, \alpha \in [0, 1]$  and  $x \in X$ , the  $(\epsilon, \alpha)$ -neighborhood of  $x$  is the set

$$\mathcal{N}_x(\epsilon, \alpha) := \{y \in X : \|x - y, z\|_\alpha^+ < \epsilon\},$$

for each nonzero  $z \in X$ .

**Definition 2.2.** Let  $(X, \|\cdot, \cdot\|^\sim, L, R)$  be a FTNS. Then a sequence  $(x_n)$  is said to be *convergent* to  $x \in X$  with respect to the fuzzy 2-norm on  $X$  if for every  $\epsilon > 0$  and every nonzero  $z \in X$  there exists a number  $N = N(\epsilon, z)$  such that

$$D(\|x_n - x, z\|^\sim, \tilde{0}) < \epsilon \text{ for all } n \geq N,$$

or equivalently

$$(D)\text{-}\lim_{n \rightarrow \infty} \|x_n - x, z\|^\sim = \tilde{0}.$$

In this case we write  $(x_n, z) \xrightarrow{FTN} (x, z)$  for every nonzero  $z \in X$ . This means that for every  $\epsilon > 0$  there exists a number  $N = N(\epsilon, z)$  such that

$$\sup_{\alpha \in [0,1]} \|x_n - x, z\|_\alpha^+ = \|x_n - x, z\|_0^+ < \epsilon$$

for all  $n \geq N$ . In terms of neighborhoods, we have  $(x_n, z) \xrightarrow{FTN} (x, z)$  provided that for any  $\epsilon > 0$  there exists a number  $N = N(\epsilon, z)$  such that  $x_n \in \mathcal{N}_x(\epsilon, 0)$  whenever  $n \geq N$ .

### 3. Statistical convergence and statistically Cauchy in FTNS

In this section, we define the notion of statistical convergence and statistically Cauchy sequences in fuzzy 2-normed space. Before proceeding further, we should recall some of the basic concepts on statistical convergence.

Let  $K$  be a subset of  $\mathbb{N}$ , the set of natural numbers. Then the *asymptotic density* of  $K$  denoted by  $\delta(K)$ , is defined as

$$\delta(K) = \lim_n \frac{1}{n} |\{k \leq n : k \in K\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

A number sequence  $x = (x_k)$  is said to be *statistically convergent* to the number  $L$  if for each  $\epsilon > 0$ , the set  $K(\epsilon) = \{k \leq n : |x_k - L| > \epsilon\}$  has asymptotic density zero, i.e.

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case we write  $st\text{-}\lim x = L$  (see [8,22]).

Note that every convergent sequence is statistically convergent to the same limit, but converse need not be true.

Statistical convergence in 2-normed space has been studied by Gürdal and Pehlivan [11].

Let  $(x_n)$  be a sequence in 2-norm space  $(X, \|\cdot, \cdot\|)$ . Then, a sequence  $(x_n)$  is said to be *statistically convergent* to  $x$  if for every  $\epsilon > 0$ , the set

$$\{n \in \mathbb{N} : \|x_n - x, z\| \geq \epsilon\}$$

has natural density zero for each nonzero  $z \in X$ , in other words  $(x_n)$  is *statistically convergent* to  $x$  in 2-norm space  $(X, \|\cdot, \cdot\|)$  if

$$\lim_{m \rightarrow \infty} \frac{1}{m} |\{n \leq m : \|x_n - x, z\| \geq \epsilon\}| = 0$$

for each nonzero  $z \in X$ . It means that for every  $z \in X$ ,

$$\|x_n - x, z\| < \epsilon \quad a.a.n.$$

for almost all  $n$  (for short, *a.a.n.*). In this case we write  $st\text{-}\lim \|x_n - x, z\| = \|x, z\|$ .

Now, we define the statistical convergence in fuzzy 2-normed space.

**Definition 3.1.** Let  $(X, \|\cdot, \cdot\|^\sim, L, R)$  be a FTNS. We say that a sequence  $(x_n)$  is said to be *statistically convergent* to  $x \in X$  with respect to the fuzzy 2-norm on  $X$  if for every  $\epsilon > 0$  and every nonzero  $z$  in  $X$ , we have

$$\delta(\{n \in \mathbb{N} : D(\|x_n - x, z\|^\sim, \tilde{0}) \geq \epsilon\}) = 0, \quad (1)$$

or equivalently

$$\delta(\{n \in \mathbb{N} : D(\|x_n - x, z\|^\sim, \tilde{0}) < \epsilon\}) = 1. \quad (1)'$$

This implies that for each  $\epsilon > 0$  and  $z$  in  $X$ , the set

$$K(\epsilon) := \{n \in \mathbb{N} : \|x_n - x, z\|_0^+ \geq \epsilon\}$$

has natural density zero; namely, for each  $\epsilon > 0$ ,  $\|x_n - x, z\|_0^+ < \epsilon$  for *a.a.n.* In this case we write  $st(FTN)\text{-}\lim \|x_n - x, z\|^\sim = \tilde{0}$  either  $st\text{-}\lim \|x_n - x, z\|^\sim = \tilde{0}$  or  $(x_n, z) \xrightarrow{st(FTN)} (x, z)$ .

In terms of neighborhoods, we have  $(x_n, z) \xrightarrow{st(FTN)} (x, z)$  if for every  $\epsilon > 0$ ,

$$\delta(\{n \in \mathbb{N} : x_n \notin \mathcal{N}_x(\epsilon, 0)\}) = 0,$$

i.e., for each  $\epsilon > 0$ ,  $(x_n) \in \mathcal{N}_x(\epsilon, 0)$  for *a.a.n.*

A useful interpretation of the above definition is the following:

$$(x_n, z) \xrightarrow{st(FTN)} (x, z) \quad \text{iff} \quad st\text{-}\lim \|x_n - x, z\|_0^+ = 0$$

Note that  $st\text{-}\lim \|x_n - x, z\|_0^+ = 0$  implies that

$$st\text{-}\lim \|x_n - x, z\|_\alpha^- = st\text{-}\lim \|x_n - x, z\|_\alpha^+ = 0$$

for each  $\alpha \in [0, 1]$  since

$$0 \leq \|x_n - x, z\|_\alpha^- \leq \|x_n - x, z\|_\alpha^+ \leq \|x_n - x, z\|_0^+$$

holds for every  $n \in \mathbb{N}$  and for each  $\alpha \in [0, 1]$ . Hence the result.

**Remark 3.1.** If a sequence  $(x_n)$  in a fuzzy 2-normed space  $(X, \|\cdot, \cdot\|^\sim, L, R)$  is convergent then it is also statistically convergent but converse need not be true, which can be seen by the following example.

**Example 3.1.** Define a sequence  $(x_n)$  in fuzzy 2-normed space  $(X, \|\cdot, \cdot\|^\sim, L, R)$  by

$$x_n = \begin{cases} (1, n) & ; \text{ if } n = m^2, m \in \mathbb{N} \\ (1, \frac{n-1}{n}) & ; \text{ otherwise.} \end{cases}$$

Let  $L = (1, 1)$  and  $z = (z_1, z_2)$ . If  $z_1 = 0$ . Then for every  $\epsilon > 0$  and  $z \in X$ , the set

$$K(\epsilon) := \{n \in \mathbb{N} : \|x_n - x, z\|_0^+ \geq \epsilon\}$$

has natural density zero. Therefore we have  $z_1 \neq 0$ . For each  $\epsilon > 0$  and  $z \in X$ , the set

$$\left\{n \in \mathbb{N} : n \neq m^2, m \leq \frac{|z_1|}{\epsilon}\right\}$$

is finite. Thus

$$\begin{aligned} & \{n \in \mathbb{N} : \|x_n - x, z\|_0^+ \geq \epsilon\} \\ &= \left\{n \in \mathbb{N} : n = m^2, m \geq \sqrt{\frac{\epsilon}{|z_1|} + 1}\right\} \cup \left\{n \in \mathbb{N} : n \neq m^2, m \leq \frac{|z_1|}{\epsilon}\right\}. \end{aligned}$$

Therefore,

$$\frac{1}{m} |\{n \leq m : \|x_n - x, z\|_0^+ \geq \epsilon\}| = \frac{1}{m} \left| \left\{n \leq m : n = m^2, m \geq \sqrt{\frac{\epsilon}{|z_1|} + 1}\right\} \cup \frac{1}{m} 0(1) \right|$$

for each  $z \in X$ . Hence

$$\delta(\{n \in \mathbb{N} : \|x_n - x, z\|_0^+ \geq \epsilon\}) = 0$$

implies that  $(x_n, z) \xrightarrow{st(FTN)} (x, z)$ , while it is obvious that  $(x_n, z) \not\xrightarrow{FTN} (x, z)$ .

**Definition 3.2.** Let  $(X, \|\cdot, \cdot\|^\sim, L, R)$  be a FTNS. Then a sequence  $(x_n)$  is said to be *statistically Cauchy* with respect to the fuzzy 2-norm on  $X$  if for every  $\epsilon > 0$ , there exists a number  $N = N(\epsilon, z)$  such that

$$\delta(\{n \in \mathbb{N} : \|x_n - x_{N(\epsilon, z)}, z\|_0^+ \geq \epsilon\}) = 0.$$

**Theorem 3.1.** Let  $(x_n)$  and  $(y_n)$  be a sequences in a FTNS  $(X, \|\cdot, \cdot\|^\sim, L, R)$  such that  $(x_n, z) \xrightarrow{st(FTN)} (x, z)$  and  $(y_n, z) \xrightarrow{st(FTN)} (x, z)$ , for all  $x, y \in X$  and nonzero  $z \in X$ . Then we have the following:

$$(i) \quad (x_n + y_n, z) \xrightarrow{st(FTN)} (x + y, z),$$

$$(ii) (\alpha x_n, z) \xrightarrow{st(FTN)} (\alpha x, z), \alpha \in \mathbb{R},$$

$$(iii) st(FTN)-\|x, z\|^\sim = \|x, z\|^\sim.$$

**Proof.** (i) Suppose that  $(x_n, z) \xrightarrow{st(FTN)} (x, z)$  and  $(y_n, z) \xrightarrow{st(FTN)} (x, z)$ . Since  $\|\cdot, \cdot\|_0^+$  is a 2-norm in the usual sense, we get

$$\|(x_n + y_n) - (x + y), z\|_0^+ \leq \|x_n - x, z\|_0^+ + \|y_n - y, z\|_0^+ \quad (3.1.1)$$

for all  $n \in \mathbb{N}$  and every nonzero  $z \in X$ . Write

$$K(\epsilon) := \{n \in \mathbb{N} : \|(x_n + y_n) - (x + y), z\|_0^+ \geq \epsilon\},$$

$$K_1(\epsilon) := \{n \in \mathbb{N} : \|x_n - x, z\|_0^+ \geq \epsilon/2\},$$

$$K_2(\epsilon) = \{n \in \mathbb{N} : \|y_n - y, z\|_0^+ \geq \epsilon/2\}.$$

From (3.1.1), we have  $K(\epsilon) \subseteq K_1(\epsilon) \cup K_2(\epsilon)$ . Now by assumption we have  $\delta(K_1(\epsilon)) = \delta(K_2(\epsilon)) = 0$ . This yields  $\delta(K(\epsilon)) = 0$ , i.e., (i) holds.

(ii) Easy to proof.

(iii) Since  $\|\cdot, \cdot\|_\alpha^-$  and  $\|\cdot, \cdot\|_\alpha^+$  are 2-norms in the usual sense, we have

$$0 \leq \|x_n, z\|_\alpha^- - \|x, z\|_\alpha^- \leq \|x_n - x, z\|_\alpha^-$$

and

$$0 \leq \|x_n, z\|_\alpha^+ - \|x, z\|_\alpha^+ \leq \|x_n - x, z\|_\alpha^+$$

for all  $\alpha \in [0, 1]$ . Therefore

$$0 \leq \max\{|\|x_n, z\|_\alpha^- - \|x, z\|_\alpha^-|, |\|x_n, z\|_\alpha^+ - \|x, z\|_\alpha^+|\} \leq \|x_n - x, z\|_\alpha^+$$

for all  $\alpha \in [0, 1]$ . Taking supremum over  $\alpha \in [0, 1]$ , we get

$$0 \leq D(\|x_n, z\|^\sim, \|x, z\|^\sim) \leq \|x_n - x, z\|_0^+.$$

Hence  $st(FTN)-\|x_n, z\|^\sim = \|x, z\|^\sim$  by Definition 5 in [20].

**Lemma 3.1** [7]. Let  $\{A_i : i \in I\}$  be a countable collection of subset of  $\mathbb{N}$  such that  $\delta(A_i) = 1$  for each  $i \in I$ . Then there is a set  $A \subset \mathbb{N}$  such that  $\delta(A) = 1$  and  $|A \setminus A_i| < \infty$  for all  $i \in I$ .

**Theorem 3.2.** Let  $(X, \|\cdot, \cdot\|^\sim, L, R)$  be a FTNS. Then a sequence  $(x_n)$  is a statistically convergent to  $x$  with respect to the fuzzy 2-norm on  $X$  if and only if  $(x_n)$  is a sequence for which there is a sequence  $(y_n)$  that is convergent such that  $x_n = y_n$  for *a.a.n.*

**Proof.** Suppose that  $(x_n, z) \xrightarrow{st(FTN)} (x, z)$ . For each  $i \in \mathbb{N}$ , let

$$A_i = \{n \in \mathbb{N} : \|x_n - x, z\|_0^+ \leq 1/i\}$$

and  $\delta(A_i) = 1$  for each  $i$ , since  $(x_n)$  is statistically convergent. Let  $A$  be as given in Lemma 3.1. For every  $\epsilon > 0$  there exists a number  $N = N(\epsilon, z)$  such that  $n \geq N$  and  $n \in A$  imply  $\|x_n - x, z\|_0^+ < \epsilon$ . Define a sequence  $(y_n)$  as:

$$y_n = \begin{cases} x_n & ; \text{ for each } n \in A \\ x & ; \text{ for } n \notin A. \end{cases}$$

This shows that the sequence  $(y_n)$  is convergent to  $x$  with respect to the fuzzy 2-norm on  $X$  such that  $y_n = x_n$  for *a.a.n.*

Conversely, suppose that  $x_n = y_n$  for *a.a.n.* and  $(y_n, z) \xrightarrow{FTN} (x, z)$ . Let  $\epsilon > 0$  be given. Then, for each  $m$ , define the following set as:

$$\{n \leq m : \|x_n - x, z\|_0^+ \geq \epsilon\} \subseteq \{n \leq m : x_n \neq y_n\} \cup \{n \leq m : \|y_n - x, z\|_0^+ \geq \epsilon\}. \quad (3.2.1)$$

Since  $(y_n, z) \xrightarrow{FTN} (x, z)$ , the second set on the right hand side of (3.2.1) contains a finite number of elements, say  $p = p(\epsilon, z)$ . Therefore

$$\lim_{m \rightarrow \infty} \frac{1}{m} |\{n \leq m : \|x_n - x, z\|_0^+ \geq \epsilon\}| \leq \lim_{m \rightarrow \infty} \frac{1}{m} |\{n \leq m : x_n \neq y_n\}| + \lim_{m \rightarrow \infty} \frac{p}{m} = 0,$$

since  $x_n = y_n$  for *a.a.n.* Hence  $\|x_n - x, z\|_0^+ < \epsilon$  for *a.a.n.* Hence  $(x_n)$  is statistically convergent with respect to the fuzzy 2-norm on  $X$ .

**Theorem 3.3.** Let  $(X, \|\cdot, \cdot\|^\sim, L, R)$  be a FTNS. Then every statistically convergent sequence  $(x_n)$  is statistically Cauchy sequence with respect to the fuzzy 2-norm on  $X$ .

**Proof.** Assume that  $(x_n, z) \xrightarrow{st(FTN)} (x, z)$ . Then, for given  $\epsilon > 0$  we have  $\|x_n - x, z\|_0^+ < \epsilon/2$  for *a.a.n.* Choose  $N = N(\epsilon, z) \in \mathbb{N}$  such that  $\|x_{N(\epsilon, z)} - x, z\|_0^+ < \epsilon/2$ . Now  $\|\cdot, \cdot\|_0^+$  being a 2-norm in the usual sense, we get

$$\begin{aligned} \|x_n - x_{N(\epsilon, z)}, z\|_0^+ &= \|(x_n - x) + (x - x_{N(\epsilon, z)}), z\|_0^+ \\ &\leq \|x_n - x, z\|_0^+ + \|x_{N(\epsilon, z)} - x, z\|_0^+ < \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

for *a.a.n.* Hence  $(x_n)$  is statistically Cauchy sequence with respect to the fuzzy 2-norm on  $X$ .

**Theorem 3.4.** Let  $(x_n)$  be a sequence in FTNS  $(X, \|\cdot, \cdot\|^\sim, L, R)$  and denote  $E_{N(\epsilon, z)} := \{n \in \mathbb{N} : \|x_n - x_{N(\epsilon, z)}, z\|_0^+ \geq \epsilon\}$ . If  $(x_n)$  is statistically Cauchy, then for every  $\epsilon > 0$  there exists  $A \subset \mathbb{N}$  with  $\delta(A) = 0$  such that  $\|x_m - x_n, z\|_0^+ < \epsilon$  for all  $m, n \notin A$ .

**Proof.** For a given  $\epsilon > 0$ , write  $A = E_{N(\epsilon/2, z)}$ . Since  $(x_n)$  is statistically Cauchy, we can write  $\delta(A) = 0$ . Then, for any  $m, n \notin A$ , we have  $\|x_n - x_{N(\epsilon, z)}, z\|_0^+ < \epsilon/2$  and  $\|x_m - x_{N(\epsilon, z)}, z\|_0^+ < \epsilon/2$ . Hence  $\|x_m - x_n, z\|_0^+ < \epsilon$  for all  $m, n \notin A$ .

**Definition 3.3.** A fuzzy 2-norm  $|||.,.|||^\sim$  on a vector  $X$  is called fuzzy equivalent to a fuzzy 2-norm  $||.,.||^\sim$ , written as  $|||.,.|||^\sim \sim ||.,.||^\sim$ , on  $X$  if there exist  $\mu, \nu \in L(\mathbb{R})$  and  $\mu, \nu \succ \tilde{0}$  such that for all  $x \in X$ , and every nonzero  $z \in X$

$$\mu \otimes \|x, z\|^\sim \preceq |||x, z|||^\sim \preceq \nu \otimes \|x, z\|^\sim.$$

**Theorem 3.5.** Let  $X$  be a vector space over  $\mathbb{R}$  and let  $||.,.||^\sim$  and  $|||.,.|||^\sim$  be fuzzy equivalent fuzzy 2-norms on  $X$ . Let  $(x_n)$  be a sequence in  $X$ . Then

- (i)  $(x_n)$  is statistically convergent to  $x$  in  $(X, ||.,.||^\sim, L, R)$  iff  $(x_n)$  is statistically convergent to  $x$  in  $(X, |||.,.|||^\sim, L, R)$ .
- (ii)  $(x_n)$  is statistically Cauchy in  $(X, ||.,.||^\sim, L, R)$  iff  $(x_n)$  is statistically Cauchy in  $(X, |||.,.|||^\sim, L, R)$ .

**Proof.** (i) Let  $(x_n)$  be statistically convergent to  $x$  in  $(X, ||.,.||^\sim, L, R)$ . Since  $(X, ||.,.||^\sim, L, R)$  and  $(X, |||.,.|||^\sim, L, R)$  are fuzzy equivalent, there exist  $\mu, \nu \in L(\mathbb{R})$  and  $\mu, \nu \succ \tilde{0}$  such that

$$\mu \otimes \|x_n - x, z\|^\sim \preceq |||x_n - x, z|||^\sim \preceq \nu \otimes \|x_n - x, z\|^\sim$$

for all  $x_n, x \in X$  and  $z \in X$ . Thus

$$\mu_0^+ \|x_n - x, z\|_0^+ \leq |||x_n - x, z|||_0^+ \leq \nu_0^+ \|x_n - x, z\|_0^+$$

for all  $n \in \mathbb{N}$ . By assumption, we have  $st(FTN)\text{-}\lim \|x_n - x, z\|_0^+ = 0$ . Hence  $st(FTN)\text{-}\lim |||x_n - x, z|||_0^+ = 0$ , i.e.,  $(x_n, z) \xrightarrow{st(FTN)} (x, z)$  in  $(X, |||.,.|||^\sim, L, R)$ . Similarly, if  $(x_n, z) \xrightarrow{st(FTN)} (x, z)$  then  $(x_n, z) \xrightarrow{st(FTN)} (x, z)$  in  $(X, ||.,.||^\sim, L, R)$ .

(ii) Let  $(x_n)$  be statistically Cauchy in  $(X, ||.,.||^\sim, L, R)$ . Since  $(X, ||.,.||^\sim, L, R)$  and  $(X, |||.,.|||^\sim, L, R)$  are fuzzy equivalent, there exist  $\mu, \nu \in L(\mathbb{R})$  and  $\mu, \nu \succ \tilde{0}$  such that

$$\mu_0^+ \|x, z\|_0^+ \leq |||x, z|||_0^+ \leq \nu_0^+ \|x, z\|_0^+$$

for all  $x \in X$  and  $z \in X$ . For any  $\epsilon > 0$ , there exists  $N(\epsilon, z) \in \mathbb{N}$  such that  $\|x_n - x_{N(\epsilon, z)}\|_0^+ < \epsilon/\nu_0^+$  for a.a.n. Hence

$$|||x_n - x_{N(\epsilon, z)}, z|||_0^+ \leq \nu_0^+ \|x_n - x_{N(\epsilon, z)}, z\|_0^+ < \epsilon$$

for a.a.n. Hence  $(x_n)$  is statistically Cauchy in  $(X, |||.,.|||^\sim, L, R)$ . Similarly, if  $(x_n)$  is statistically Cauchy in  $(X, |||.,.|||^\sim, L, R)$  then it is statistically Cauchy in  $(X, ||.,.||^\sim, L, R)$ .

#### 4. Statistical limit point and statistical cluster point in FTNS

Statistical limit point of sequence  $(x_n)$  has been define and studied by Fridy [9]; and for fuzzy number by Aytar [1]. In this section, we define the notions of thin

subsequence, non-thin subsequence, statistical limit point and statistical cluster point in fuzzy 2-normed space.

**Definition 4.1.** Let  $(x_n)$  be a sequence in FTNS  $(X, \|\cdot, \cdot\|^\sim, L, R)$ . An element  $x \in X$  is said to be a *limit point* of the sequence  $(x_n)$  provided that there is a subsequence of  $(x_n)$  that converges to  $x$  with respect to the fuzzy 2-norm on  $X$ . We denote by  $L_{FTN}(x_n)$ , the set of all limit points of the sequence  $(x_n)$ .

**Definition 4.2.** Let  $(x_n)$  be a sequence in FTNS  $(X, \|\cdot, \cdot\|^\sim, L, R)$  and  $(x_{n_j})$  be a subsequence of  $(x_n)$ . Write  $K = \{n_j : j \in \mathbb{N}\}$ . If  $\delta(K) = 0$  then we say that  $(x_{n_j})$  is a *thin subsequence* on  $(x_n)$ . A subsequence  $(x_{n_j})$  is said to be a *non-thin subsequence* provided that  $\delta(k) > 0$  or  $\delta(k)$  does not exist, namely,  $\bar{\delta}(k) > 0$ .

**Definition 4.3.** Let  $(x_n)$  be a sequence in FTNS  $(X, \|\cdot, \cdot\|^\sim, L, R)$ . An element  $x \in X$  is said to be a *statistical limit point* of the sequence  $(x_n)$  provided that there exists a non-thin subsequence of  $(x_n)$  that converges to  $x$  with respect to the fuzzy 2-norm on  $X$ . By  $\Lambda_{FTN}(x_n)$ , we denote the set of all statistical limit points of the sequence  $(x_n)$ .

**Definition 4.4.** Let  $(x_n)$  be a sequence in FTNS  $(X, \|\cdot, \cdot\|^\sim, L, R)$ . We say that an element  $x \in X$  is said to be a *statistical cluster point* of the sequence  $(x_n)$  with respect to the fuzzy 2-norm on  $X$  provided that for every  $\epsilon > 0$  and  $z \in X$

$$\bar{\delta}(\{n \in \mathbb{N} : \|x_n - x, z\|_0^+ < \epsilon\}) > 0.$$

By  $\Gamma_{FTN}(x_n)$ , we denote the set of all statistical limit points of the sequence  $(x_n)$ .

**Remark 4.1.** An element  $x \in \Gamma_{FTN}(x_n)$  implies that

$$\bar{\delta}(\{n \in \mathbb{N} : \|x_n - x, z\|_\alpha^+ < \epsilon\}) > 0.$$

and

$$\bar{\delta}(\{n \in \mathbb{N} : \|x_n - x, z\|_\alpha^- < \epsilon\}) > 0.$$

for all  $\epsilon > 0$ ,  $\alpha \in [0, 1]$  and  $z \in X$ .

**Theorem 4.1.** Let  $(X, \|\cdot, \cdot\|^\sim, L, R)$  be a FTNS. Then for every sequence  $(x_n)$  in  $X$ , we have

$$\Lambda_{FTN}(x_n) \subseteq \Gamma_{FTN}(x_n) \subseteq L_{FTN}(x_n).$$

**Proof.** Let  $x \in \Lambda_{FTN}(x_n)$ . Then there exists a non-thin subsequence  $(x_{n_j})$  of the sequence  $(x_n)$  that converges to  $x$ , namely,  $\bar{\delta}(\{n_j : j \in \mathbb{N}\}) = d > 0$ . Since

$$\{n \in \mathbb{N} : \|x_n - x, z\|_0^+ < \epsilon\} \supseteq \{n \in \mathbb{N} : \|x_{n_j} - x, z\|_0^+ < \epsilon\}$$

for every  $\epsilon > 0$  and so

$$\{n \in \mathbb{N} : \|x_n - x, z\|_0^+ < \epsilon\} \supseteq \{n_j : j \in \mathbb{N}\} \setminus \{n \in \mathbb{N} : \|x_{n_j} - x, z\|_0^+ \geq \epsilon\}.$$



Since  $(x_{n_j}, z) \xrightarrow{FTN} (x, z)$ , the set  $\{n_j \in \mathbb{N} : \|x_{n_j} - x, z\|_0^+ \geq \epsilon\}$  is finite for any  $\epsilon > 0$ . Hence we have

$$\bar{\delta}(\{n \in \mathbb{N} : \|x_n - x, z\|_0^+ < \epsilon\}) \geq \bar{\delta}(\{n_j : j \in \mathbb{N}\}) - \bar{\delta}(\{n_j \in \mathbb{N} : \|x_{n_j} - x, z\|_0^+ \geq \epsilon\}) = d > 0.$$

Thus, for every  $\epsilon > 0$  and  $z \in X$

$$\bar{\delta}(\{n \in \mathbb{N} : \|x_n - x, z\|_0^+ < \epsilon\}) > 0,$$

i.e.,  $x \in \Gamma_{FTN}(x_n)$ .

Let  $x \in \Gamma_{FTN}(x_n)$ . For every  $\epsilon > 0$  and  $z \in X$ , write

$$\bar{\delta}(\{n \in \mathbb{N} : \|x_n - x, z\|_0^+ < \epsilon\}) > 0.$$

This means that there are infinitely many terms of the sequence  $(x_n)$  in every  $(\epsilon, 0)$ -neighborhood of  $x$ , i.e.,  $x \in L_{FTN}(x_n)$ . Hence the result.

**Theorem 4.2.** Let  $(x_n)$  be a sequence in a FTNS  $(X, \|\cdot, \cdot\|^\sim, L, R)$ . Then  $\Lambda_{FTN}(x_n) = \Gamma_{FTN}(x_n) = \{x\}$ , provided  $(x_n, z) \xrightarrow{st(FTN)} (x, z)$ .

**Proof.** Let  $(x_n, z) \xrightarrow{st(FTN)} (x, z)$ . Therefore  $x \in \Gamma_{FTN}(x_n)$ . Now suppose that there exists atleast one  $y \in \Gamma_{FTN}(x_n)$  such that  $y \neq x$ . For every  $\epsilon > 0$  and every nonzero  $z \in X$  such that

$$\{n \in \mathbb{N} : \|x_n - x, z\|_0^+ \geq \epsilon\} \supseteq \{n \in \mathbb{N} : \|x_n - y, z\|_0^+ < \epsilon\}$$

holds. Hence

$$\bar{\delta}(\{n \in \mathbb{N} : \|x_n - x, z\|_0^+ \geq \epsilon\}) \geq \bar{\delta}(\{n \in \mathbb{N} : \|x_n - y, z\|_0^+ < \epsilon\}).$$

Since  $(x_n, z) \xrightarrow{st(FTN)} (x, z)$ , we have  $\delta(\{n \in \mathbb{N} : \|x_n - x, z\|_0^+ \geq \epsilon\}) = 0$ , which implies that

$$\bar{\delta}(\{n \in \mathbb{N} : \|x_n - x, z\|_0^+ \geq \epsilon\}) = 0.$$

Thus

$$\bar{\delta}(\{n \in \mathbb{N} : \|x_n - y, z\|_0^+ < \epsilon\}) = 0,$$

which is a contradiction to  $y \in \Gamma_{FTN}(x_n)$ . Therefore, we have  $\Gamma_{FTN}(x_n) = \{x\}$ .

On the other hand, since  $(x_n, z) \xrightarrow{st(FTN)} (x, z)$ . By Theorem 3.2 and Definition 4.3, we get  $x \in \Lambda_{FTN}(x_n)$ . Now, Theorem 4.1 yields  $\Lambda_{FTN}(x_n) = \Gamma_{FTN}(x_n) = \{x\}$ .

## 5. Conclusion

The concept of fuzzy 2-normed space, which has been introduced here, is not merely a generalization of fuzzy normed space, but it also provides a bigger setting to deal with the uncertainty and vagueness in natural problems arising in many branches of engineering and science. Some basic results of normed linear spaces have been established here which could be very useful functional tools in the development of fuzzy set theory.

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# A Generalization of the EMMML and ISRA Algorithms for Solving Linear Systems

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## Abstract

From an algebraic point of view, the EMMML and ISRA algorithms for Positron Emission Tomography can be considered as iterative procedures for solving a class of linear system of equations. We introduce an algorithm  $A(p)$ ,  $p \in \mathbb{R}$ , such that  $A(1)$  coincides with EMMML and  $A(-1)$  with a version of ISRA. Some examples illustrate the speed of convergence. Applications are indicated to:

- (i) the Bernstein-Bézier representation;
- (ii) the B-spline interpolation;
- (iii) the inverse problem for Markov chains;
- (iv) the problem of finding the stationary distribution of a regular Markov chain.

**Keywords:** Expectation-Maximization Algorithm, Kullback-Leibler distances, log-likelihood functions, least-squares, linear systems.

## 1 Introduction and notation

Using the Expectation-Maximization (EM) algorithm, L. A. Shepp and Y. Vardi in 1982 and independently K Lange and R. Carson in 1984 pioneered an algorithm in order to compute Maximum Likelihood (ML) estimates for the problem of tomography reconstruction; see [9].

W. H. Richardson in 1972 and independently L. B. Lucy in 1974 obtained the same algorithm in the setting of restoration of astronomical images; see [9].

With the terminology of [11] we shall call this algorithm the EMLL algorithm. It can be considered as a numerical procedure for calculating maximum likelihood estimates, or alternatively as an iterative procedure for solving a class of linear systems of equations; see [1], [3], [4], [9], [10], [11].

In the context of the Positron Emission Tomography problem, M. E. Daube-Witherspoon and G. Muehllehner introduced in 1986 the Image Space Reconstruction Algorithm (ISRA) in order to obtain Least-Squares (LS) estimates of the emission densities; see [9]. Alternatively, ISRA can be viewed as a procedure for solving linear systems; see [1], [3], [4], [9], [10], [11]. Both the ML and LS estimates can be considered as minimum distance estimates, but based on different measures of distance: Kullback-Leibler distance for EMLL and least-squares distance for ISRA; see [1], [3], [4], [9], [10], [11], where the relationship between the two algorithms is discussed.

In this paper we introduce an algorithm  $A(p)$ , depending on a real parameter  $p$ , such that:

- (a)  $A(1)$  coincides with EMLL, and  $A(-1)$  with a version of ISRA;
- (b)  $A(p)$  minimizes a suitable generalized Kullback-Leibler distance and solves a specific problem of convex optimization involving generalized log-likelihood functions and least-squares functions;
- (c)  $A(p)$  solves iteratively linear systems from a certain class and assigns gen-

eralized solutions to inconsistent systems.

Let  $N(p)$  be the number of iterations when one uses the algorithm  $A(p)$ . Numerical experiments show that, if the system has dominant diagonal, then  $N(-p) \approx N(p)$  and  $N(p) < N(q)$  for  $0 \leq p < q$ .

Systems for which  $A(p)$  works and which have a (more or less) dominant diagonal appear quite naturally in practical applications involving the Bernstein-Bézier representation of polynomials, the B-spline interpolation, the inverse problem for Markov chains and the problem of finding the stationary distribution of a Markov chain. We describe such applications in the final sections.

Throughout the paper, we consider the integers  $m \geq 1$ ,  $n \geq 1$ , the matrix  $A = (a_{ij})_{i=1, \dots, n; j=1, \dots, m}$  with  $a_{ij} \geq 0$ ,  $\sum_{i=1}^n a_{ij} > 0$ ,  $\sum_{j=1}^m a_{ij} > 0$ ,  $i = 1, \dots, n$ ;  $j = 1, \dots, m$ , and the vector  $b = (b_1, \dots, b_n)^t$  with  $b_i > 0$ ,  $i = 1, \dots, n$ .

We shall be concerned with the (consistent or inconsistent) system of linear equations (S):

$$Ax = b, \quad (1)$$

where  $x = (x_1, \dots, x_m)^t \in \mathbb{R}^m$ .

We shall use the notation

$$\Pi_m := \{x \in \mathbb{R}^m : x_j > 0, j = 1, \dots, m\},$$

$$\Omega_{m,n} := \{x \in \mathbb{R}^m : (Ax)_i > 0, i = 1, \dots, n\}.$$

Obviously  $\Pi_m \subset \Omega_{m,n}$ .

## 2 Generalized Kullback-Leibler distances

Let  $p \in \mathbb{R}$ . For  $u, t \in (0, \infty)$ , define

$$e_p(u, t) := \begin{cases} t \log t - t \log u + u - t, & p=0; \\ u \log u - u \log t + t - u, & p=1; \\ \frac{u^p t^{1-p} - (1-p)t - pu}{p(p-1)}, & p \neq 0, 1. \end{cases} \quad (2)$$

It is easy to verify that for all  $u, t > 0$  one has:  $\lim_{p \rightarrow 0} e_p(u, t) = e_0(u, t)$ ;  $\lim_{p \rightarrow 1} e_p(u, t) = e_1(u, t)$ ;  $e_p(u, u) = 0$ ;  $e_p(u, t) > 0$  for  $u \neq t$ .

For all  $v, w \in \Pi_n$  we define:

$$d_p(v, w) := \sum_{i=1}^n e_p(v_i, w_i). \quad (3)$$

Then  $d_1$  is the well-known Kullback-Leibler distance; see, e.g., [11] and the references therein. For  $p \in \mathbb{R}$ ,  $d_p$  can be considered as a *generalized Kullback-Leibler distance*.

### 3 Generalized log-likelihood functions and least-squares functions

For  $p \in \mathbb{R}$  consider the function  $F_p : (0, \infty) \rightarrow \mathbb{R}$ ,

$$F_p(t) := \begin{cases} t \log t - t, & p=0; \\ \log t, & p=1; \\ t^{1-p}/(1-p), & p \neq 0, 1. \end{cases}$$

Let now  $L_p : \Omega_{m,n} \rightarrow \mathbb{R}$ ,

$$L_p(x) := \begin{cases} \sum_{i=1}^n ((b_i)^p F_p((Ax)_i) - (Ax)_i), & p \neq 0; \\ \sum_{i=1}^n (F_0((Ax)_i) - (Ax)_i \log b_i), & p=0. \end{cases} \quad (4)$$

Then  $L_1$  is basically the log-likelihood function appearing in the EMLL

algorithm; see [9], [4], [11]. On the other hand,

$$L_{-1} = \frac{1}{2} \sum_{i=1}^n \frac{1}{b_i} ((Ax)_i - b_i)^2 - \frac{1}{2} \sum_{i=1}^n b_i$$

is obviously a least-squares cost function.

Thus the family  $(L_p)_{p \in \mathbb{R}}$  extends the notions of log-likelihood functions and least-squares functions.

It is easy to prove

**Theorem 1.** *The families  $(L_p)$  and  $(d_p)$  are related by*

$$L_1(x) = \sum_{i=1}^n b_i (\log b_i - 1) - d_1(b, Ax), \quad (5)$$

$$L_0(x) = - \sum_{i=1}^n b_i + d_0(b, Ax), \quad (6)$$

$$L_p(x) = \frac{p}{1-p} \sum_{i=1}^n b_i - p d_p(b, Ax), \quad p \neq 0, 1. \quad (7)$$

## 4 Minimizing $d_p(b, Ax)$

We are interested in minimizing  $d_p(b, Ax)$  with respect to  $x \in \Omega_{m,n}$ .

I. Let  $p \leq 0$ . According to Eqs. (6) and (7), minimizing  $d_p(b, Ax)$  is equivalent to minimizing the function  $L_p(x)$ . Remark that in this case  $F_p(t)$  is strictly convex on  $(0, \infty)$ , hence  $L_p(x)$  is convex on  $\Omega_{m,n}$ . By examining the behavior of  $L_p(x)$  when  $x$  approaches the infinity or the boundary of  $\Omega_{m,n}$ , we conclude that  $L_p$  has global minimum points in  $\Omega_{m,n}$ .

II. Let  $p > 0$ . Now Eqs. (5) and (7) show that to minimize  $d_p(b, Ax)$ , means to maximize  $L_p(x)$ . Since  $F_p(t)$  is strictly concave, we infer that  $L_p(x)$  is concave and has global maximum points in  $\Omega_{m,n}$ .

## 5 p-generalized solutions of the system (S)

Let  $x \in \Omega_{m,n}$  be a global extremum point of the function  $L_p$ . Then

$$\frac{\partial L_p(x)}{\partial x_r} = 0, \quad r = 1, \dots, m. \quad (8)$$

For  $p \neq 0$ , Eq. (8) is equivalent to  $(S_p)$ :

$$\sum_{i=1}^n a_{ir} b_i^p / (Ax)_i^p = \sum_{i=1}^n a_{ir}, \quad r = 1, \dots, m, \quad (9)$$

while for  $p = 0$  it is equivalent to  $(S_0)$ :

$$\sum_{i=1}^n (b_i / (Ax)_i)^{a_{ir}} = 1, \quad r = 1, \dots, m. \quad (10)$$

**Theorem 2.** (i) For each  $p \in \mathbb{R}$ , the system  $(S_p)$  is consistent, i.e., has solutions in  $\Omega_{m,n}$ .

(ii) If  $x_p$  is a solution of  $(S_p)$ , then  $x_p$  minimizes  $d_p(b, Ax)$ . Moreover,  $x_p$  minimizes (if  $p \leq 0$ ), respectively maximizes (if  $p > 0$ ) the function  $L_p(x)$ .

(iii) If  $(S)$  has a solution  $x \in \Omega_{m,n}$ , then  $x$  is a solution of  $(S_p)$ .

(iv) If  $\text{rank}(A)=n$ , then each solution of  $(S_p)$  is a solution of  $(S)$ .

**Proof.** According to [6], pp. 14-15, the set

$$\begin{aligned} \{x \in \Omega_{m,n} : x \text{ is a global extremum point of } L_p\} = \\ = \{x \in \Omega_{m,n} : x \text{ is a solution of } (S_p)\} \end{aligned}$$

is a nonempty convex subset of  $\Omega_{m,n}$ . This proves statement (i) in the theorem.

Statement (ii) is a consequence of the results presented in Section 4. (iii) is obvious, and (iv) is an easy exercise in algebra.



**Definition 1.** Each solution of  $(S_p)$  will be called a ***p-generalized solution*** of  $(S)$ .

According to Theorem 2, each solution of  $(S)$  is also a  $p$ -generalized solution,  $p \in \mathbb{R}$ . If  $\text{rank}(A) = n$ , the  $p$ -generalized solutions coincide with the solutions of  $(S)$ . For each  $p \in \mathbb{R}$ ,  $(S)$  has a  $p$ -generalized solution even if  $(S)$  itself is inconsistent.

## 6 The $A(p)$ algorithm

In order to find the  $p$ -generalized solutions of  $(S)$ , i.e., the global extremum points of  $L_p$ , we shall apply the Maximization-Minimization (or Minimization-Maximization) Algorithm; see [7], [9].

For a given  $k \in \mathbb{N}$  let  $x^{(k)} \in \Pi_m$  be an arbitrary vector. If  $p \neq 0$ , define

$$l_p(x|x^{(k)}) := \sum_{i=1}^n b_i^p \sum_{j=1}^m \frac{a_{ij}x_j^{(k)}}{(Ax^{(k)})_i} F_p \left( \frac{(Ax^{(k)})_i}{x_j^{(k)}} x_j \right) - \sum_{i=1}^n (Ax)_i.$$

If  $p = 0$ , let

$$l_0(x|x^{(k)}) := \sum_{i=1}^n \sum_{j=1}^m \frac{a_{ij}x_j^{(k)}}{(Ax^{(k)})_i} F_0 \left( \frac{(Ax^{(k)})_i}{x_j^{(k)}} x_j \right) - \sum_{i=1}^n (Ax)_i \log b_i.$$

It is easy to verify that

$$l_p(x^{(k)}|x^{(k)}) = L_p(x^{(k)}), \quad p \in \mathbb{R}. \quad (11)$$

As consequence of Jensen's inequality, we get

$$L_p(x) \leq l_p(x|x^{(k)}), \quad x \in \Pi_m, \quad p \leq 0, \quad (12)$$

$$L_p(x) \geq l_p(x|x^{(k)}), \quad x \in \Pi_m, \quad p > 0. \quad (13)$$

With straightforward computation we find that for each  $p \in \mathbb{R}$  the system

$$\frac{\partial l_p}{\partial x_r}(x|x^{(k)}) = 0, \quad r = 1, \dots, m$$

has a unique solution in  $\Pi_m$ , denoted by  $x^{(k+1)}$  and given by

$$A(p) : \quad x_r^{(k+1)} = x_r^{(k)} \left( \frac{\sum_{i=1}^n a_{ir} (b_i / (Ax^{(k)})_i)^p}{\sum_{i=1}^n a_{ir}} \right)^{1/p} \quad (14)$$

for  $p \neq 0$  and  $r = 1, \dots, m$ ;

$$A(0) : \quad x_r^{(k+1)} = x_r^{(k)} \prod_{i=1}^n \left( \frac{b_i}{(Ax^{(k)})_i} \right)^{a_{ir} / \sum_{s=1}^n a_{sr}} \quad (15)$$

for  $p = 0$  and  $r = 1, \dots, m$ .

I. Let  $p \leq 0$ . Then  $l_p(x|x^{(k)})$  is convex and  $x^{(k+1)}$  is a minimum point of it. According to Eqs. (12) and (11),

$$L_p(x^{(k+1)}) \leq l_p(x^{(k+1)}|x^{(k)}) \leq l_p(x^{(k)}|x^{(k)}) = L_p(x^{(k)}).$$

So, starting from an arbitrary  $x^{(0)} \in \Pi_m$ , the algorithm  $A(p)$  generates a sequence  $x^{(0)}, x^{(1)}, x^{(2)}, \dots$  such that  $L_p(x^{(0)}) \geq L_p(x^{(1)}) \geq L_p(x^{(2)}) \geq \dots$ . The sequence  $(x^{(k)})$  is bounded since  $L_p(x) \rightarrow \infty$  as  $x$  approaches the infinity.

II. Let  $p > 0$ . It can be proved similarly that starting from an arbitrary  $x^{(0)} \in \Pi_m$ , the algorithm  $A(p)$  generates a bounded sequence  $(x^{(k)})$  such that  $L_p(x^{(0)}) \leq L_p(x^{(1)}) \leq L_p(x^{(2)}) \leq \dots$ .

These results, combined with Eqs. (9), (10), (14) and (15), lead to

**Theorem 3.** *If for given  $x^{(0)} \in \Pi_m$  and  $p \in \mathbb{R}$  the sequence  $(x^{(k)})$  generated by  $A(p)$  is convergent and*

$$x^* := \lim_{k \rightarrow \infty} x^{(k)} \in \Pi_m, \quad (16)$$

*then  $x^*$  is a  $p$ -generalized solution of  $(S)$ .*

To conclude this section, let us remark that  $A(1)$  is equivalent to the EMMML Algorithm (see [9], [4], [11]) and, when  $b_1 = \dots = b_n$ ,  $A(-1)$ , is equivalent to the ISRA Algorithm (see [9], [3], [11]). The convergence of the sequence  $(x^{(k)})$  is governed by the general rules of the Expectation-Maximization Algorithm; see [9].

## 7 Examples and applications

For the sake of brevity we shall present only some examples and applications involving systems with exactly one solution; underconstrained and overconstrained systems will be considered in subsequent papers, as well as comparisons with other methods.

### 7.1 Dominant diagonal

**Example 1.**

For illustrative purposes, consider the system for which

$$A = \begin{pmatrix} 100000 & 1000 & 1000 & 1000 & 1000 & 1000 \\ 1 & 100 & 1 & 1 & 1 & 1 \\ 1 & 1 & 100 & 1 & 1 & 1 \\ 1 & 1 & 1 & 100 & 1 & 1 \\ 1 & 1 & 1 & 1 & 100 & 1 \\ 1000 & 1000 & 1000 & 1000 & 1000 & 100000 \end{pmatrix}$$

and  $b = (1050000 \ 1050 \ 1050 \ 1050 \ 1050 \ 1050000)^t$ .

The exact solution is  $(10 \ 10 \ 10 \ 10 \ 10 \ 10)^t$ . Taking as initial solution  $x^{(0)} = (1 \ 1000 \ 1 \ 1000 \ 1 \ 1000)^t$  we get as approximate solution  $(10 \ 10 \ 10 \ 10 \ 10 \ 10)^t$ . The number of iterations, corresponding to  $p \in [-5, 5]$ , can be seen in Fig. 1.

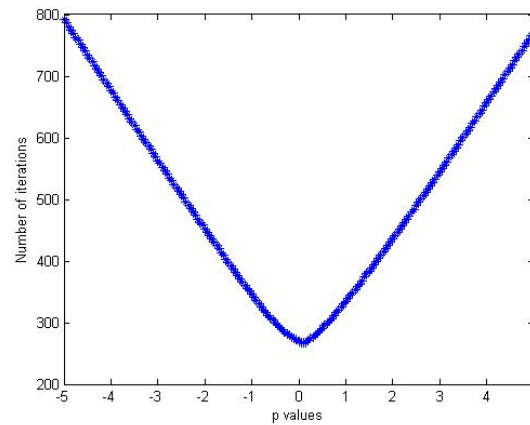


Figure 1: *The number of iterations (Example 1)*

Remark that  $N(0) < N(-1) \approx N(1)$ ; this happened in all our numerical experiments with systems having dominant diagonal. Moreover, in such cases the algorithm exhibits little sensitivity with respect to changing the initial solution.

## 7.2 Non-dominant diagonal

### Example 2.

The dominant diagonal makes the computation easier. The opposite circumstance is illustrated here by the very simple system

$$\begin{pmatrix} 0.9 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 190 \\ 200 \end{pmatrix}.$$

Indeed, we know that solving the system with  $A(p)$  is equivalent to finding the extremum points of the function  $L_p$ . The surfaces representing  $L_p$  for  $p \in \{-1, 0, 1\}$  can be seen in Fig. 2.

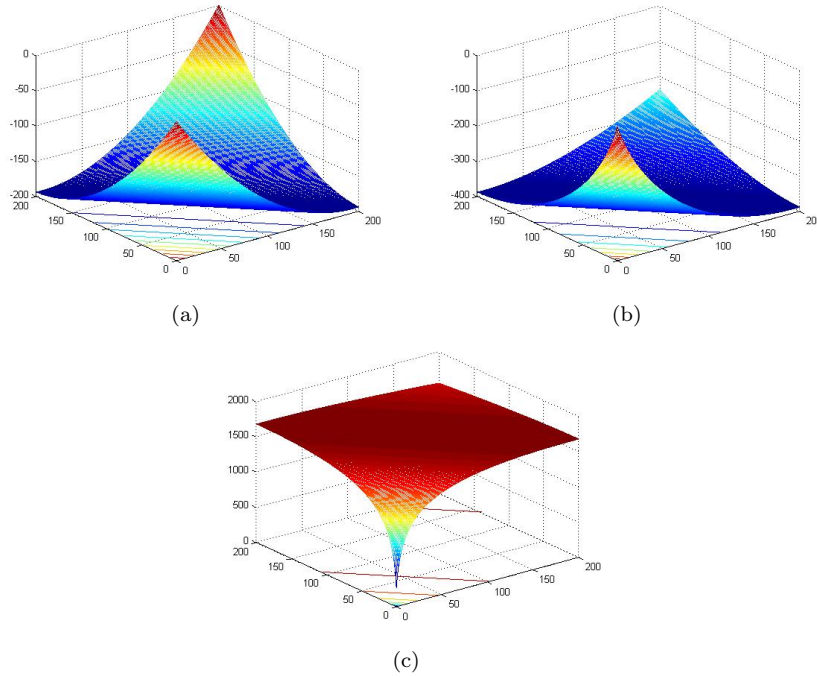


Fig. 2: The surface representation of the function  $L_p$  for (a)  $p=-1$ , (b)  $p=0$  and (c)  $p=1$  (Example 2)

We remark that they are similar to the surface representing the *Rosenbrock*

*function:* "long, narrow, parabolic shaped flat valley" (see [12]).

The difficulties in finding the extremum points for such functions are notorious; they explain the difficulties in solving the corresponding system with  $A(p)$ .

In fact, taking  $x^{(0)} = (50 \ 150)^t$  as initial solution of the above system, we get the approximate solution  $(99.8 \ 100)^t$  after a number of iterations presented in Fig. 3.

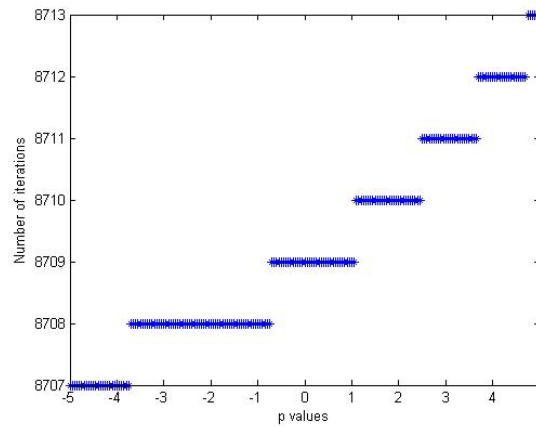


Fig. 3: The number of iterations (Example 2)

### Example 3.

By contrast, consider the system given by

$$A = \begin{pmatrix} 100 & 1 \\ 1 & 100 \end{pmatrix}, \quad b = (10100 \ 10100)^t,$$

and take  $x^{(0)} = (1 \ 199)^t$  as initial solution. We get as approximate solution  $(100 \ 100)^t$  with a number of iterations illustrated in Fig. 4.

In this case the function  $L_p$ ,  $p \in \{-1, 0, 1\}$  looks like in Fig. 5.

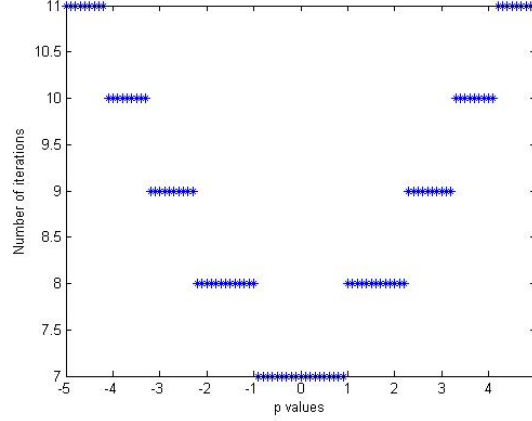


Fig. 4: The number of iterations (Example 3)

### 7.3 Bernstein-Bézier representation

Let  $P_n$  be the space of the real polynomial functions of degree at most  $n$ , defined on  $[0, 1]$ . Let  $b_{nj}(t) := \binom{n}{j} t^j (1-t)^{n-j}$ ,  $t \in [0, 1]$ ,  $j = 0, 1, \dots, n$ . Then  $\{b_{nj} : j = 0, 1, \dots, n\}$  is a basis of  $P_n$ , called the *Bernstein-Bézier basis*; see [5], [8];

Let  $f \in P_n$ ; suppose that the numbers  $f(\frac{i}{n}) > 0$ ,  $i = 0, 1, \dots, n$ , are known. We want to represent  $f$  with respect to the Bernstein-Bézier basis, i.e., to find the coefficients  $c_0, c_1, \dots, c_n$  such that  $f = \sum_{j=0}^n c_j b_{nj}$ . Then we have to solve the system  $Ax = b$ , where  $A = (b_{nj}(\frac{i}{n}))_{i,j=0,\dots,n}$ ,  $x = (c_0, \dots, c_n)^t$ ,  $b = (f(0), f(\frac{1}{n}), \dots, f(1))^t$ .

The algorithm  $A(p)$  works particularly well if one takes  $x^{(0)} := b$ .

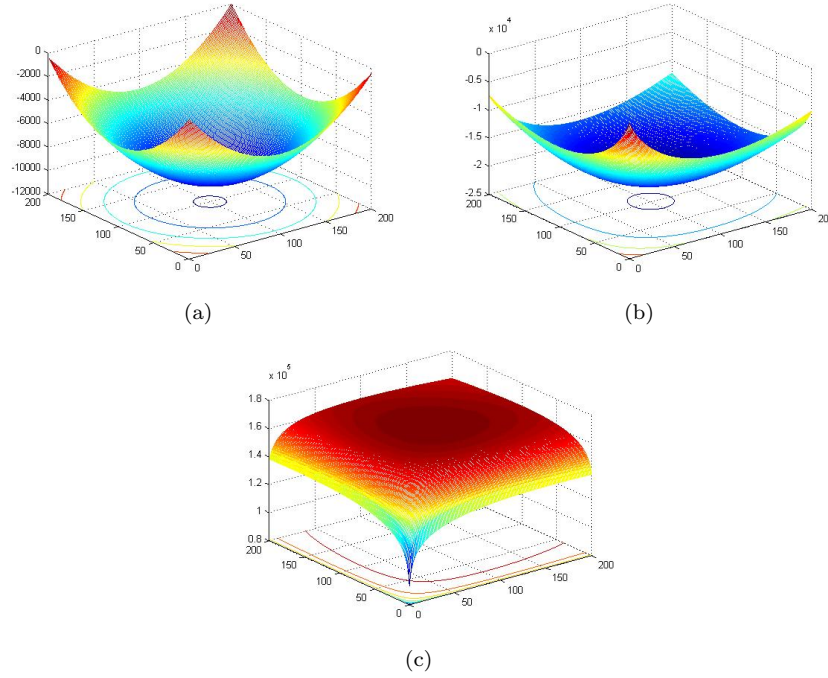


Fig. 5: The surface representation of the function  $L_p$  for (a)  $p=-1$ , (b)  $p=0$  and (c)  $p=1$  (Example 3)

## 7.4 B-spline interpolation

In solving the problem of *B-spline interpolation* (see [5], [8]), one has to consider systems with

$$M = \begin{pmatrix} 1 & 1 & & \\ a_1 & b_1 & c_1 & \\ & \dots & & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & 1 & 1 \end{pmatrix} \quad (17)$$

where  $a_i > 0$ ,  $c_i > 0$ ,  $b_i = 2(a_i + c_i)$ ,  $i = 1, \dots, n-1$ .

Once again, the  $A(p)$  algorithm can be used.



## 7.5 The inverse problem of Markov chains

Let  $(X_k)_{k \geq 0}$  be a homogeneous Markov chain with state space  $\{1, \dots, n\}$  and transition matrix  $T = (p_{ij})_{i,j=1,\dots,n}$ . Let  $b_i := P(X_k = i)$ ,  $i = 1, \dots, n$ , where  $k \geq 1$  is a given integer. Then  $(b_1, \dots, b_n)$  describes the probability distribution at the moment  $k$ . Suppose that we know the vector  $b$  and want to find the probability distribution at the moment  $k-1$ . Then we have to solve the system

$$T^t x = b^t, \quad (18)$$

where  $x_j = P(X_{k-1} = j)$ ,  $j = 1, \dots, n$ , and we can use  $A(p)$ . So we can solve *the inverse problem for the Markov chain*  $(X_k)$  (see [2], p. 304). In particular,

$$P(X_{k-1} = j | X_k = i) = p_{ij} x_j / b_i, \quad i, j = 1, \dots, n.$$

## 7.6 Stationary distribution

Suppose that the above Markov chain is regular. Then it has a stationary distribution (see [2]) given by  $w_i = \lim_{k \rightarrow \infty} P(X_k = i)$ ,  $i = 1, \dots, n$ . The vector  $w$  is the unique eigenvector of the matrix  $T^t$ , associated with the eigenvalue 1 and having positive components with sum equal to 1. It is easy to see that  $w$  is the solution of the system

$$\begin{cases} p_{11}w_1 + (p_{21} + 1)w_2 + \dots + (p_{n1} + 1)w_n = 1 \\ (p_{12} + 1)w_1 + p_{22}w_2 + \dots + (p_{n2} + 1)w_n = 1 \\ \dots \\ (p_{1n} + 1)w_1 + (p_{2n} + 1)w_2 + \dots + p_{nn}w_n = 1. \end{cases}$$

So we can use the algorithm  $A(p)$  in order to find the stationary distribution.

**Example 4.**

If the transition matrix is

$$T = \begin{pmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.1 & 0.1 & 0.1 & 0.3 \\ 0.2 & 0.4 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.3 & 0.4 & 0.1 & 0.1 \\ 0.3 & 0.3 & 0.2 & 0.1 & 0.1 \end{pmatrix}$$

then we have to solve the system for which

$$A = \begin{pmatrix} 0.2 & 1.4 & 1.2 & 1.1 & 1.3 \\ 1.2 & 0.1 & 1.4 & 1.3 & 1.3 \\ 1.2 & 1.1 & 0.2 & 1.4 & 1.2 \\ 1.2 & 1.1 & 1.1 & 0.1 & 1.1 \\ 1.2 & 1.3 & 1.1 & 1.1 & 0.1 \end{pmatrix}$$

and  $b = (1, 1, 1, 1, 1)^t$ .

Taking as initial solution  $x^{(0)} = (0.05 \ 0.2 \ 0.5 \ 0.05 \ 0.2)^t$  we get the approximate solution (stationary distribution)  $(0.2540 \ 0.2455 \ 0.2005 \ 0.1254 \ 0.1745)^t$ .

The number of iterations is presented in Fig. 6.

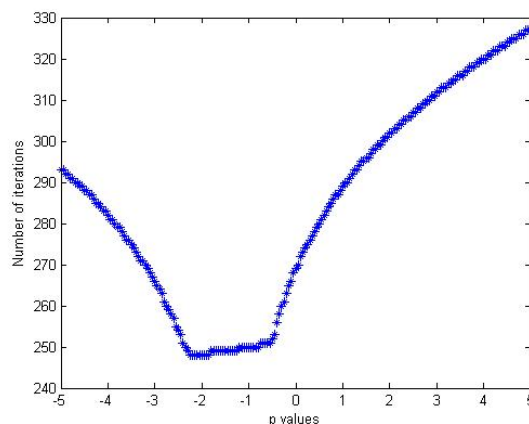


Fig. 6: The number of iterations (Example 4)

## 8 Concluding remarks

In the frame of the EM Algorithm, the  $A(p)$  algorithm generalizes the known algorithms EMLL (i.e.,  $A(1)$ ) and ISRA (i.e.,  $A(-1)$ ). For systems with dominant diagonal  $A(0)$  is better than  $A(1)$  and  $A(-1)$  with respect to the number of iterations.  $A(p)$  can be applied to concrete problems, as shown by the above examples. Subsequent papers will be devoted to applications involving underconstrained and overconstrained systems, as well as to the problem of identifying classes of systems for which a certain  $A(p)$  works better than other algorithms.

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# On Local Properties of Summability of Fourier Series

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## Abstract

In this note we have improved the result of Sulaiman [2] on local property of absolute weighted mean summability with index  $\delta$  and  $k$ , of factored Fourier series by proving under weaker conditions.

## 1 Introduction

Let  $\sum a_n$  be a given series with partial sums  $(s_n)$ , and let  $(p_n)$  be a sequence of positive numbers such that  $P_n = p_0 + p_1 + \dots + p_n \rightarrow \infty$  as  $n \rightarrow \infty$ . The sequence- to- sequence transformation

$$T_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v$$

defines the sequence  $(T_n)$  of the  $(\overline{N}, p_n)$  means of the the sequence  $(s_n)$ , generated the sequence coefficients  $(p_n)$ . The series  $\sum a_n$  is summable the  $|\overline{N}, p_n; \delta|_k$  summability,  $k \geq 1, \delta \geq 0$ , if (see [1] )

$$\sum_{n=1}^{\infty} (P_n / p_n)^{\delta k + k - 1} |T_n - T_{n-1}|^k < \infty. \quad (1)$$

In the special case when  $\delta = 0$ ,  $|\overline{N}, p_n; \delta|_k$  summability is the same as  $|\overline{N}, p_n|_k$  summability. If we take  $p_n = 1/(n+1)$ ,  $k = 1, \delta = 0$ ,  $|\overline{N}, p_n; \delta|_k$  is reduced to  $|R, \log n, 1|$  summability.

Let  $f$  be a function with period  $2\pi$ , integrable  $(L)$  over  $(-\pi, \pi)$ . Without any loss of generality we may assume that the constant term in the Fourier series of  $f$  is zero, so that

$$\int_{-\pi}^{\pi} f(t) dt = 0$$

and

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$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} C_n(t). \quad (2)$$

It is well known [3] that convergence of a Fourier series at any point  $t = x$  is a local property  $f$ , i.e., for arbitrarily small  $\delta > 0$ , the behaviour of  $(s_n(t))$ , the  $n$ -th partial sum of the series (2), depends only the natura of  $f$  in the interval  $(x - \delta, x + \delta)$  and is not affected by the values it takes outside the interval. The local property problem of the factored Fourier series has been by several authors (see [1] for detail).

Quite recently Sulaiman [2] proved the following theorem on local property of  $\left| \overline{N}, p_n; \delta \right|_k$  summability of factored Fourier series.

**Theorem A.** Let  $k \geq 1$  and  $\delta \geq 0$ . If  $(p_n)$  and  $(\lambda_n)$  satisfy the conditions

$$\Delta X_n = O(1/n), \quad X_n = (np_n)^{-1} P_n \quad (3)$$

$$\sum_{n=1}^{\infty} (P_n / p_n)^{\delta k - 1} \left\{ |\lambda_n|^k + |\lambda_{n+1}|^k \right\} X_n^k < \infty, \quad (4)$$

$$\sum_{n=1}^{\infty} (P_n / p_n)^{\delta k} X_{n+1} |\Delta \lambda_n| < \infty, \quad (5)$$

$$\sum_{n=v}^{\infty} (P_n / p_n)^{\delta k - 1} (1 / P_n) = O \left\{ (P_v / p_v)^{\delta v} (1 / P_v) \right\}, \quad (6)$$

then the summability  $\left| \overline{N}, p_n; \delta \right|_k$  of the series  $\sum_{n=1}^{\infty} \lambda_n X_n C_n(t)$  at a point can be ensured by a local property.

The aim of this paper is to establish Theorem A under weaker conditions. Now, we shall prove the following theorem.

**Theorem.** Let  $k \geq 1$  and  $\delta \geq 0$ . If  $(p_n)$  and  $(\lambda_n)$  satisfy the conditions (3), (6),

$$\sum_{n=1}^{\infty} (P_n / p_n)^{\delta k - 1} X_n^k |\lambda_n|^k < \infty, \quad X_n = (np_n)^{-1} P_n \quad (7)$$

and

$$\sum_{n=1}^{\infty} (P_n / p_n)^{\delta} X_{n+1} |\Delta \lambda_n| < \infty, \quad (8)$$

then the summability  $\left| \overline{N}, p_n; \delta \right|_k$  of the series  $\sum_{n=1}^{\infty} \lambda_n X_n C_n(t)$  at a point can be ensured by a local property.

**Remark.** It may be noticed that our result is an improvement of Theorem A in the sense that the conditions (7) and (8) are weaker than (4) and (5).

**2. Proof of the Theorem.** As mentioned in the beginning, the convergence of Fourier series at a point is a local property. Therefore in order to prove the theorem it is sufficient to prove that if  $(s_n)$  is bounded, then under the conditions of our theorem, the series  $\sum X_n \lambda_n a_n$  is summable  $|\overline{N}, p_n; \delta|_k$ . Now, let  $(T_n)$  denote the  $(\overline{N}, p_n)$  means of this series. Then we have

$$T_n - T_{n-1} = (P_n P_{n-1})^{-1} p_n \sum_{v=1}^n P_{v-1} X_v \lambda_v a_v .$$

Applying Abel's transformation to this sum we get

$$\begin{aligned} T_n - T_{n-1} &= (P_n P_{n-1})^{-1} p_n \sum_{v=1}^n s_n \Delta(P_{v-1} X_v \lambda_v) + (P_n P_{n-1})^{-1} p_n s_n P_{n-1} X_n \lambda_n \\ &= (P_n P_{n-1})^{-1} p_n \sum_{v=1}^n s_v \lambda_v \Delta(P_{v-1} X_v) + (P_n P_{n-1})^{-1} p_n \sum_{v=1}^n s_v P_v X_{v+1} \Delta \lambda_v \\ &\quad + (P_n)^{-1} p_n s_n X_n \lambda_n = T_1 + T_2 + T_3, \text{ say.} \end{aligned}$$

For the proof of the lemma, by Minkowski's inequality, it suffices to show that

$$\sum_{n=1}^{\infty} (P_n / p_n)^{\delta_{k+k-1}} |T_r|^k < \infty, \quad r = 1, 2, 3, \dots$$

Now, since  $s_n = O(1)$ , It follows that

$$\sum_{n=2}^{m+1} (P_n / p_n)^{\delta_{k+k-1}} |T_1|^k = O(1) \sum_{n=2}^{m+1} (P_n / p_n)^{\delta_{k-1}} (P_{n-1})^{-k} \left\{ \sum_{v=1}^{n-1} |\lambda_v| |\Delta(P_{v-1} X_v)| \right\}^k .$$

In addition, in view of  $\Delta(P_{v-1} X_v) = -p_v X_v + P_v \Delta X_v = -v^{-1} P_v + P_v \Delta X_v = P_v (-v^{-1} + \Delta X_v)$ , it is clear that the condition  $\Delta X_v = O(1/v)$  is equivalent to  $\Delta(P_{v-1} X_v) = O(v^{-1} P_v)$ . Therefore, applying Hölder's inequality, we get

$$\begin{aligned} \sum_{n=2}^{m+1} (P_n / p_n)^{\delta_{k+k-1}} |T_1|^k &= O(1) \sum_{n=2}^{m+1} (P_n / p_n)^{\delta_{k-1}} (P_{n-1})^{-k} \left\{ \sum_{v=1}^{n-1} |\lambda_v| v^{-1} P_v \right\}^k \\ &= O(1) \sum_{n=2}^{m+1} (P_n / p_n)^{\delta_{k-1}} (P_{n-1})^{-k} \left\{ \sum_{v=1}^{n-1} |\lambda_v| X_v P_v \right\}^k \end{aligned}$$

$$\begin{aligned}
&= O(1) \sum_{n=2}^{m+1} (P_n / p_n)^{\delta k-1} (P_{n-1})^{-k} \left\{ \sum_{v=1}^{n-1} |\lambda_v|^k X_v^k p_v \right\} \left\{ \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
&= O(1) \sum_{v=1}^m |\lambda_v|^k X_v^k p_v \sum_{n=v+1}^{m+1} (P_n / p_n)^{\delta k-1} (P_{n-1})^{-1} \\
&= O(1) \sum_{v=1}^m (P_v / p_v)^{\delta k-1} X_v^k |\lambda_v|^k = O(1) \text{ as } m \rightarrow \infty.
\end{aligned}$$

Again,

$$\begin{aligned}
\sum_{n=2}^{m+1} (P_n / p_n)^{\delta k+k-1} |T_2|^k &= O(1) \sum_{n=2}^{m+1} (P_n / p_n)^{\delta k-1} (P_{n-1})^{-k} \left\{ \sum_{v=1}^{n-1} (P_v / p_v)^{-\delta} P_v (P_v / p_v)^{\delta} X_{v+1} |\Delta \lambda_v| \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} (P_n / p_n)^{\delta k-1} (P_{n-1})^{-k} \sum_{v=1}^{n-1} (P_v / p_v)^{\delta-\delta k} P_v^k X_{v+1} |\Delta \lambda_v| \left\{ \sum_{v=1}^{n-1} (P_v / p_v)^{\delta} X_{v+1} |\Delta \lambda_v| \right\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} (P_n / p_n)^{\delta k-1} (P_{n-1})^{-k} \sum_{v=1}^{n-1} (P_v / p_v)^{\delta-\delta k} P_v^k X_{v+1} |\Delta \lambda_v| \\
&= O(1) \sum_{v=1}^m (P_v / p_v)^{\delta-\delta k} P_v^k X_{v+1} |\Delta \lambda_v| \sum_{n=v+1}^{m+1} (P_n / p_n)^{\delta k-1} (P_{n-1})^{-k} \\
&= O(1) \sum_{v=1}^m (P_v / p_v)^{\delta} X_{v+1} |\Delta \lambda_v| = O(1), \text{ as } m \rightarrow \infty,
\end{aligned}$$

by virtue of (8). Finally, it is clear that  $\sum_{n=1}^{\infty} (P_n / p_n)^{\delta k+k-1} |T_3|^k < \infty$  by virtue of (7). This completes the proof.

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# Higher Order Generalization of $q$ -Bernstein Operators

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## Abstract

We introduce higher order generalization of the  $q$ -Bernstein operators. Then we study approximation properties and a Voronovskaja-type theorem for higher order  $q$ -Bernstein operators.

## 1 Introduction

Nowadays it is known that the theory of  $q$ -calculus plays an important role on analytic number theory and theoretical physics. For example, various applications of this theory have appeared in the study of hypergeometric series [1], in the approximation theory [2], [17], [18] while some other important applications have been related with the quantum theory. In this paper, using the moment estimates from [9] and with the techniques of the works [5], [6], we study the approximation properties of an  $r$ th order generalization of the  $q$ -Bernstein polynomials.

We first recall some basic definitions used in the paper. The  $q$ -Bernstein operators are given by

$$B_{n,q}(f; x) = \sum_{k=0}^n f\left(\frac{[k]}{[n]}\right) p_{n,k}(q; x), \quad n \in \mathbb{N}, \quad 0 \leq x \leq 1,$$

$$p_{n,k}(q; x) = \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1} (1 - q^s x).$$

Recall that  $[n] = [n]_q$  and  $\begin{bmatrix} n \\ k \end{bmatrix}$  denotes the  $q$ -integers and  $q$ -Gaussian binomial, which are defined, respectively by

$$[n] = \begin{cases} (1 - q^n) / (1 - q), & \text{if } q \neq 1 \\ n, & \text{if } q = 1 \end{cases}, \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]! [n-k]!},$$

where  $[n]!$  denotes the  $q$ -factorial given by

$$[n]! = \begin{cases} [n] \dots [2] [1], & \text{if } n \geq 1, \\ 1, & \text{if } n = 0. \end{cases}$$

After  $q$ -Bernstein polynomials were introduced by Phillips [15] they have been the object of several investigations in approximation theory (cf. [3]-[21]). Surveys of results on the  $q$ -Bernstein polynomials together with comprehensive lists of references on the subject are given in [12].

We introduce a new sequence of positive linear operators so-called higher ( $r$ th) order  $q$ -Bernstein operators.

**Definition 1** Let  $r \in \mathbb{N} \cup \{0\}$  be a fixed number. For  $f \in C^r[0, 1]$  and  $n \in \mathbb{N}$  we define the  $r$ th order generalization of the  $q$ -Bernstein operators as follows

$$B_{n,q}^{[r]}(f; x) := \sum_{k=0}^n p_{n,k}(q; x) \sum_{j=0}^r \frac{1}{j!} f^{(j)}\left(\frac{[k]}{[n]}\right) \left(x - \frac{[k]}{[n]}\right)^j.$$

It is clear that for  $r = 0$ ,  $B_{n,q}^{[0]}(f; x)$  becomes the  $q$ -Bernstein operator,  $B_{n,q}(f; x)$ , defined by Phillips. Also it can be easily shown that  $B_{n,q}^{[r]}(f; 1) = f(1)$ . It should be stressed out that this kind of generalization was considered in [5], [10], [14], [6].

In the paper, we investigate the rate of convergence for the sequence  $\{B_{n,q}^{[r]}(f; x)\}$  by the modulus of continuity of the  $r$ th order derivative of  $f$  in the case  $0 < q < 1$ . Moreover we study a Voronovskaja-type theorem for higher order  $q$ -Bernstein operators.

## 2 Auxiliary results

**Theorem 2** [9] *For any  $m \in \mathbb{N}$ ,  $0 < q < 1$ , there exists a constant  $\tilde{K}_m > 0$  such that*

$$|B_{n,q}((t-x)^m, x)| \leq \tilde{K}_m \frac{x(1-x)}{[n]^{\lfloor \frac{m+1}{2} \rfloor}},$$

where  $x \in [0, 1]$  and  $\lfloor a \rfloor$  is the integer part of  $a \geq 0$ .

**Corollary 3** *For any  $m \in \mathbb{N}$ ,  $0 < q < 1$ , there exists a constant  $K_m > 0$  such that*

$$B_{n,q}(|t-x|^m, x) \leq K_m \frac{x(1-x)}{[n]^{m/2}}. \quad (1)$$

**Proof.** Indeed if  $m$  is even

$$B_{n,q}(|t-x|^m, x) = B_{n,q}((t-x)^m, x) \leq \tilde{K}_m \frac{x(1-x)}{[n]^{\lfloor (m+1)/2 \rfloor}} = K_m \frac{x(1-x)}{[n]^{m/2}}.$$

On the other hand if  $m$  is odd, say  $m = 2k + 1$ , we have

$$\begin{aligned} B_{n,q}(|t-x|^{2k+1}, x) &\leq \sqrt{B_{n,q}(|t-x|^{4k}, x)} \sqrt{B_{n,q}(|t-x|^2, x)} \\ &\leq \sqrt{\tilde{K}_{4k} \frac{x(1-x)}{[n]^{\lfloor (4k+1)/2 \rfloor}}} \sqrt{\tilde{K}_2 \frac{x(1-x)}{[n]^{\lfloor 3/2 \rfloor}}} \\ &= \sqrt{\tilde{K}_{4k} \frac{x(1-x)}{[n]^{2k}}} \sqrt{\tilde{K}_2 \frac{x(1-x)}{[n]}} \\ &= K_{2k+1} \frac{x(1-x)}{[n]^{(2k+1)/2}}. \end{aligned}$$

■

**Lemma 4** *Let  $0 < q < 1$ .  $B_{n,q}^{[r]}(f; x)$  is an operator from  $C^r[0, 1]$  into  $C[0, 1]$ . Furthermore there exists a constant  $C(r)$  such that for every  $f \in C^r[0, 1]$  we have*

$$\|B_{n,q}^{[r]}(f)\|_{C[0,1]} \leq C(r) \sum_{j=0}^r \|f^{(j)}\| = C(r) \|f\|_{C^r[0,1]}. \quad (2)$$

**Proof.** It is obvious that  $B_{n,q}^{[r]}(f; x)$  is continuous on  $[0, 1]$ . To show (2) from the definition we deduce that

$$B_{n,q}^{[r]}(f; x) = \sum_{j=0}^r \frac{(-1)^j}{j!} B_{n,q}((t-x)^j f^{(j)}(t); x).$$

Further, by the Corollary 3 we have

$$\left| B_{n,q} \left( (t-x)^j f^{(j)}(t); x \right) \right| \leq \|f^{(j)}\| B_{n,q} \left( |t-x|^j; x \right) \leq K_j \|f^{(j)}\| [n]^{-j/2}.$$

Consequently

$$\left\| B_{n,q}^{[r]}(f) \right\| \leq \sum_{j=0}^r \frac{(-1)^j}{j!} \left\| B_{n,q} \left( (t-x)^j f^{(j)}(t); x \right) \right\| \leq C(r) \sum_{j=0}^r \|f^{(j)}\|.$$

Thus the proof is completed. ■

### 3 Convergence properties of $B_{n,q}^{[r]}$

We use the modulus of continuity of the derivative  $f^{(r)}$  :

$$\omega \left( f^{(r)}; t \right) := \sup \left\{ \left| f^{(r)}(x) - f^{(r)}(y) \right| : |x-y| \leq t, \quad x, y \in [0, 1] \right\}.$$

**Theorem 5** *Let  $0 < q < 1$  and  $r \in N \cup \{0\}$  be a fixed number. Then there exists  $A_r > 0$  such that for every  $f \in C^r[0, 1]$  and  $n \in N$  the following inequality holds*

$$\left\| B_{n,q}^{[r]}(f) - f \right\| \leq A_r \frac{1}{[n]^{\frac{r}{2}}} \omega \left( f^{(r)}; \frac{1}{\sqrt{[n]}} \right). \quad (3)$$

**Proof.** The estimation (3) for  $r = 0$  follows from [19, Theorem 4.1].

Let  $r \in N$ . We apply the following Taylor formula for  $f \in C^r[0, 1]$  at a given point  $t \in [0, 1]$ :

$$f(x) = \sum_{j=0}^r \frac{f^{(j)}(t)}{j!} (x-t)^j + \frac{(x-t)^r}{(r-1)!} \int_0^1 (1-u)^{r-1} \left[ f^{(r)}(t+u(x-t)) - f^{(r)}(t) \right] du.$$

Applying  $B_{n,q}$  we get

$$\begin{aligned} f(x) - B_{n,q}^{[r]}(f; x) &= \sum_{k=0}^n \frac{(x - [k]/[n])^r}{(r-1)!} \\ &\quad \times \int_0^1 (1-u)^{r-1} \left[ f^{(r)} \left( \frac{[k]}{[n]} + u \left( x - \frac{[k]}{[n]} \right) \right) - f^{(r)} \left( \frac{[k]}{[n]} \right) \right] dup_{n,k}(q; x). \end{aligned} \quad (4)$$

The definition and properties of modulus of continuity of function imply that

$$\begin{aligned} \left| f^{(r)} \left( \frac{[k]}{[n]} + u \left( x - \frac{[k]}{[n]} \right) \right) - f^{(r)} \left( \frac{[k]}{[n]} \right) \right| &\leq \omega \left( f^{(r)}; u \left| x - \frac{[k]}{[n]} \right| \right) \\ &\leq \omega \left( f^{(r)}; \left| x - \frac{[k]}{[n]} \right| \right) \leq \left( \sqrt{[n]} \left| x - \frac{[k]}{[n]} \right| + 1 \right) \omega \left( f^{(r)}; \frac{1}{\sqrt{[n]}} \right), \end{aligned} \quad (5)$$

for every  $0 \leq u \leq 1$ ,  $0 \leq x \leq 1$ ,  $k \in N \cup \{0\}$ ,  $n \in N$ . From (4) and (5) we get

$$\begin{aligned} &\left| B_{n,q}^{[r]}(f; x) - f(x) \right| \\ &\leq \frac{1}{r!} \omega \left( f^{(r)}; \frac{1}{\sqrt{[n]}} \right) \sum_{k=0}^n \left| x - \frac{[k]}{[n]} \right|^r \left( \sqrt{[n]} \left| x - \frac{[k]}{[n]} \right| + 1 \right) p_{n,k}(q; x) \\ &= \frac{1}{r!} \omega \left( f^{(r)}; \frac{1}{\sqrt{[n]}} \right) \left( \sqrt{[n]} B_{n,q} \left( |x-t|^{r+1}; x \right) + B_{n,q} \left( |x-t|^r; x \right) \right) \end{aligned} \quad (6)$$

for  $0 \leq x \leq 1$ ,  $n \in N$ . Using (1) in (6), we have

$$\begin{aligned} \left| B_{n,q}^{[r]}(f; x) - f(x) \right| &\leq \frac{1}{r!} (K_{r+1} + K_r) \left( \frac{1}{\sqrt{[n]}} \right)^r \omega \left( f^{(r)}; \frac{1}{\sqrt{[n]}} \right) \\ &= A_r \left( \frac{1}{\sqrt{[n]}} \right)^r \omega \left( f^{(r)}; \frac{1}{\sqrt{[n]}} \right). \end{aligned}$$

■

It is easy to see that if  $q_n \rightarrow 1$  for  $n \rightarrow \infty$  then  $\lim_{n \rightarrow \infty} [n]_{q_n} = +\infty$ . Therefore, the sequence  $B_{n,q_n}^{[r]}(f; x)$  converges uniformly to  $f$  for each continuous function  $f$  if  $\lim_{n \rightarrow \infty} q_n = 1$ . From Theorem 5 we can derive the following two corollaries.

**Corollary 6** Let  $f \in C^r[0, 1]$ ,  $0 < q_n < 1$ ,  $\lim_{n \rightarrow \infty} q_n = 1$  and  $r \in N \cup \{0\}$  be a fixed number. Then

$$\lim_{n \rightarrow \infty} [n]_{q_n}^{r/2} \left\| B_{n,q_n}^{[r]}(f) - f \right\| = 0.$$

We note that a function  $f \in C[0, 1]$  belongs to  $\text{Lip}_M(\alpha)$ ,  $0 < \alpha \leq 1$ , provided

$$|f(x) - f(y)| \leq M |x - y|^\alpha \quad (x, y \in [0, 1] \text{ and } M > 0).$$

**Corollary 7** Let  $0 < q_n < 1$ ,  $\lim_{n \rightarrow \infty} q_n = 1$ ,  $r \in N \cup \{0\}$ ,  $f \in C^r[0, 1]$ . If  $f^{(r)} \in \text{Lip}_M(\alpha)$  then

$$\left\| B_{n,q_n}^{[r]}(f) - f \right\| = O \left( [n]_{q_n}^{-\frac{r+\alpha}{2}} \right).$$

**Proof.** Since  $f^{(r)} \in \text{Lip}_M(\alpha)$  we have immediately

$$\left\| B_{n,q_n}^{[r]}(f) - f \right\| \leq A_r \frac{1}{[n]_{q_n}^{\frac{r}{2}}} \omega \left( f^{(r)}; \frac{1}{\sqrt{[n]_{q_n}}} \right) \leq A_r M \frac{1}{[n]_{q_n}^{\frac{r}{2}}} \frac{1}{[n]_{q_n}^{\frac{\alpha}{2}}}.$$

■

Consequently, according to (3) the sequence  $\{B_{n,q_n}^{[r]}(f)\}$  converges uniformly to any  $f \in C^r[0, 1]$  if  $q_n \rightarrow 1$  for  $n \rightarrow \infty$ .

**Theorem 8** (Voronovskaja) Let  $0 < q < 1$ . Suppose that  $f \in C^{r+2}[0, 1]$  where  $r \in N \cup \{0\}$  is fixed. Then

$$\begin{aligned} &\left| B_{n,q}^{[r]}(f; x) - f(x) - \frac{(-1)^r f^{(r+1)}(x) B_{n,q}((t-x)^{r+1}; x)}{(r+1)!} - \frac{(-1)^r (r+1) f^{(r+2)}(x) B_{n,q}((t-x)^{r+2}; x)}{(r+2)!} \right| \\ &\leq (K_{r+2} + K_{r+4}) \frac{x(1-x)}{[n]^{\frac{r}{2}+1}} \sum_{j=0}^r \frac{1}{j!(r+2-j)!} \omega \left( f^{(r+2-j)}; [n]^{-\frac{1}{2}} \right). \end{aligned}$$

**Proof.** Fix  $0 \leq x \leq 1$ . For  $f \in C^{r+2}[0, 1]$  we have  $f^{(j)} \in C^{r+2-j}[0, 1]$ ,  $0 \leq j \leq r$ , and by the Taylor formula we can write

$$\begin{aligned} f^{(j)}(t) &= \sum_{i=0}^{r+2-j} \frac{f^{(j+i)}(x)}{i!} (t-x)^i + R_{r+2-j}(f; t, x), \\ R_{r+2-j}(f; t, x) &= \frac{f^{(r+2-j)}(\xi_t) - f^{(r+2-j)}(x)}{(r+2-j)!} (t-x)^{r+2-j}, \end{aligned} \tag{7}$$

where  $\xi_t$  is situated between  $x$  and  $t$ , therefore,  $|\xi_t - x| < |t - x|$ . Taking  $t = \tau_{nk} = \frac{[k]}{[n]}$  in (7) and applying this formula to  $B_{n,q}^{[r]}(f)$  we have

$$\begin{aligned} B_{n,q}^{[r]}(f; x) &= \sum_{k=0}^n p_{nk}(q; x) \sum_{j=0}^r \frac{(x - \tau_{nk})^j}{j!} \sum_{i=0}^{r+2-j} \frac{f^{(j+i)}(x)}{i!} (\tau_{nk} - x)^i \\ &\quad + \sum_{k=0}^n p_{nk}(q; x) \sum_{j=0}^r \frac{(x - \tau_{nk})^j}{j!} R_{r+2-j}(f; \tau_{nk}, x) \\ &= I_1 + I_2. \end{aligned} \quad (8)$$

It follows that

$$\begin{aligned} |B_{n,q}^{[r]}(f; x) - I_1| &= |I_2| = \left| \sum_{k=0}^n p_{nk}(q; x) \sum_{j=0}^r \frac{(-1)^j}{j!} \frac{f^{(r+2-j)}(\xi_{\tau_{nk}}) - f^{(r+2-j)}(x)}{(r+2-j)!} (\tau_{nk} - x)^{r+2} \right| \\ &= \left| B_{n,q} \left( \sum_{j=0}^r \frac{(-1)^j}{j!} \frac{f^{(r+2-j)}(\xi_t) - f^{(r+2-j)}(x)}{(r+2-j)!} (t - x)^{r+2}; x \right) \right| \end{aligned}$$

For the estimate of the remainder  $|I_2|$  we shall use the well-known inequality

$$\omega(f, \lambda\delta) \leq (1 + \lambda^2) \omega(f, \delta).$$

We have

$$\begin{aligned} |f^{(r+2-j)}(\xi_t) - f^{(r+2-j)}(x)| &\leq \omega(f^{(r+2-j)}, |\xi_t - x|) \leq \omega(f^{(r+2-j)}, |t - x|) \\ &\leq \omega(f^{(r+2-j)}, [n]^{-\frac{1}{2}}) (1 + [n](t - x)^2). \end{aligned}$$

Hence

$$\begin{aligned} |I_2| &\leq B_{n,q} \left( \sum_{j=0}^r \left| \frac{(-1)^j}{j!} \frac{f^{(r+2-j)}(\xi_t) - f^{(r+2-j)}(x)}{(r+2-j)!} \right| |t - x|^{r+2}; x \right) \\ &\leq B_{n,q} \left( \sum_{j=0}^r \frac{1}{j! (r+2-j)!} \omega(f^{(r+2-j)}, [n]^{-\frac{1}{2}}) (1 + [n](t - x)^2) |t - x|^{r+2}; x \right) \\ &= \sum_{j=0}^r \frac{1}{j! (r+2-j)!} \omega(f^{(r+2-j)}, [n]^{-\frac{1}{2}}) (B_{n,q}(|t - x|^{r+2}; x) + [n] B_{n,q}(|t - x|^{r+4}; x)) \\ &\leq \sum_{j=0}^r \frac{1}{j! (r+2-j)!} \omega(f^{(r+2-j)}, [n]^{-\frac{1}{2}}) \left( K_{r+2} \frac{x(1-x)}{[n]^{\frac{r}{2}+1}} + K_{r+4} \frac{[n]x(1-x)}{[n]^{\frac{r}{2}+2}} \right) \\ &= (K_{r+2} + K_{r+4}) \frac{x(1-x)}{[n]^{\frac{r}{2}+1}} \sum_{j=0}^r \frac{1}{j! (r+2-j)!} \omega(f^{(r+2-j)}, [n]^{-\frac{1}{2}}), \end{aligned}$$

and consequently

$$|B_{n,q}^{[r]}(f; x) - I_1| \leq (K_{r+2} + K_{r+4}) \frac{x(1-x)}{[n]^{\frac{r}{2}+1}} \sum_{j=0}^r \frac{1}{j! (r+2-j)!} \omega(f^{(r+2-j)}, [n]^{-\frac{1}{2}}).$$

In order to end the proof it remains to simplify  $I_1$ . Standard calculations give

$$\begin{aligned}
 I_1 &= \sum_{k=0}^n p_{nk}(q; x) \sum_{j=0}^r \frac{(x - \tau_{nk})^j}{j!} \sum_{l=j}^{r+2} \frac{f^{(l)}(x)}{(l-j)!} (\tau_{nk} - x)^{l-j} \\
 &= \sum_{k=0}^n p_{nk}(q; x) \sum_{j=0}^r \frac{(-1)^j}{j!} \left\{ \sum_{l=j}^r \frac{f^{(l)}(x)}{(l-j)!} (\tau_{nk} - x)^l \right. \\
 &\quad \left. + \frac{f^{(r+1)}(x)}{(r+1-j)!} (\tau_{nk} - x)^{r+1} + \frac{f^{(r+2)}(x)}{(r+2-j)!} (\tau_{nk} - x)^{r+2} \right\} \\
 &= \sum_{k=0}^n p_{nk}(q; x) \sum_{l=0}^r \frac{f^{(l)}(x)}{l!} (\tau_{nk} - x)^l \sum_{j=0}^l \binom{l}{j} (-1)^j \\
 &\quad + \frac{f^{(r+1)}(x)}{(r+1)!} \sum_{k=0}^n p_{nk}(q; x) (\tau_{nk} - x)^{r+1} \sum_{j=0}^r \binom{r+1}{j} (-1)^j \\
 &\quad + \frac{f^{(r+2)}(x)}{(r+2)!} \sum_{k=0}^n p_{nk}(q; x) (\tau_{nk} - x)^{r+2} \sum_{j=0}^r \binom{r+2}{j} (-1)^j
 \end{aligned}$$

for  $n \in N$ . Using the following inequalities

$$\sum_{j=0}^r \binom{r+1}{j} (-1)^j = (-1)^r,$$

$$\sum_{j=0}^r \binom{r+2}{j} (-1)^j = (r+1) (-1)^r, \quad r \in N \cup \{0\},$$

we obtain

$$\begin{aligned}
 I_1 &= f(x) + \frac{(-1)^r f^{(r+1)}(x) B_{n,q}((t-x)^{r+1}; x)}{(r+1)!} \\
 &\quad + \frac{(-1)^r (r+1) f^{(r+2)}(x) B_{n,q}((t-x)^{r+2}; x)}{(r+2)!}, \quad n \in N.
 \end{aligned}$$

Thus the proof is completed. ■

Theorem 8 implies the following Voronovskaja type theorem for operators  $B_{n,q}(f)$  proved by Videnskii [19].

**Corollary 9** *If  $f \in C^2[0, 1]$  and  $q_n \rightarrow 1$  as  $n \rightarrow \infty$ , then*

$$\left| B_{n,q_n}(f; x) - f(x) - \frac{f''(x)}{2} \frac{x(1-x)}{[n]_{q_n}} \right| \leq K \frac{x(1-x)}{[n]_{q_n}} \omega\left(f'', [n]_{q_n}^{-\frac{1}{2}}\right),$$

for every  $x \in [0, 1]$ , where  $K = (K_2 + K_4)/2$ . Moreover

$$\lim_{n \rightarrow \infty} [n]_{q_n} (B_{n,q_n}(f; x) - f(x)) = \frac{x(1-x)}{2} f''(x)$$

uniformly on  $[0, 1]$ .

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# Calculating zeros of the second kind Euler polynomials

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**Abstract :** Many mathematicians have studied the second kind Euler numbers and polynomials in the complex plane. One purpose of this paper is to investigate the zeros of the second kind Euler polynomials  $E_n(x)$ . We also display the shape of the second kind Euler polynomials  $E_n(x)$ .

**Key words :** Bernoulli numbers, Bernoulli polynomials, Euler numbers, Euler polynomials, the second kind Euler numbers and polynomials

## 1. Introduction

Several mathematicians have studied the second kind Euler numbers and polynomials (see [1,2,3,4]). The second kind Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. In this paper, we introduce the second kind Euler numbers  $E_n$  and polynomials  $E_n(x)$ . In order to study the second kind Euler numbers  $E_n$  and polynomials  $E_n(x)$ , we must understand the structure of the second kind Euler numbers  $E_n$  and polynomials  $E_n(x)$ . Therefore, using computer, a realistic study for the second kind Euler numbers  $E_n$  and polynomials  $E_n(x)$  is very interesting. It is the aim of this paper to observe an interesting phenomenon of ‘scattering’ of the zeros of the second kind Euler polynomials  $E_n(x)$  in complex plane. The outline of this paper is as follows. We introduce the second kind Euler numbers  $E_n$  and polynomials  $E_n(x)$ . In Section 2, we describe the beautiful zeros of the second kind Euler polynomials  $E_n(x)$  using a numerical investigation. Finally, we investigate the roots of the second kind Euler polynomials  $E_n(x)$ .

First, we introduce the second kind Euler numbers  $E_n$  and polynomials  $E_n(x)$ . The second kind Euler numbers  $E_n$  are defined by the generating function:

$$F(t) = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad (|t| < \frac{\pi}{2}), \text{ cf. [1, 3]} \quad (1)$$

where we use the technique method notation by replacing  $E^n$  by  $E_n (n \geq 0)$  symbolically. From (1), we have

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} = e^{Et}$$

which yields

$$2 = e^{(E+1)t} + e^{(E-1)t}.$$

Using Taylor expansion of exponential function, we obtain

$$2 = \sum_{n=0}^{\infty} ((E+1)^n + (E-1)^n) \frac{t^n}{n!}.$$

By comparing the coefficients, we have

$$(E+1)^n + (E-1)^n = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases}$$



We obtain the first value of the second kind Euler numbers  $E_n$  :

$$\begin{aligned} E_0 &= 1, & E_1 &= 0, & E_2 &= -1, & E_3 &= 0, & E_4 &= 5, & E_5 &= 0, & E_6 &= -61, & E_7 &= 0, \\ E_8 &= 1385, & E_9 &= 0, & E_{10} &= -50521, & E_{11} &= 0, & E_{12} &= 2702765, & E_{14} &= -199360981, \\ E_{16} &= 19391512145, & E_{18} &= -2404879675441 & E_{20} &= 370371188237525, \dots \end{aligned}$$

In general, it satisfies  $E_3 = E_5 = E_7 = \dots = 0$ . We consider the second kind Euler polynomials  $E_n(x)$  as follows:

$$F(x, t) = \frac{2}{e^t + e^{-t}} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \quad (2)$$

By the above definition, we obtain

$$\begin{aligned} \sum_{l=0}^{\infty} E_l(x) \frac{t^l}{l!} &= \frac{2}{e^t + e^{-t}} e^{xt} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \sum_{m=0}^{\infty} x^m \frac{t^m}{m!} \\ &= \sum_{l=0}^{\infty} \left( \sum_{n=0}^l E_n \frac{t^n}{n!} x^{l-n} \frac{t^{l-n}}{(l-n)!} \right) = \sum_{l=0}^{\infty} \left( \sum_{n=0}^l \binom{l}{n} G_n x^{l-n} \right) \frac{t^l}{l!}. \end{aligned}$$

By using comparing coefficients  $\frac{t^l}{l!}$ , we have

$$E_n(x) = \sum_{k=0}^n \binom{n}{k} E_k x^{n-k}.$$

In the special case  $x = 0$ , we define  $E_n(0) = E_n$ . Let  $m$  be odd. It is easy to see that

$$\begin{aligned} \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} &= \frac{2}{e^t + e^{-t}} e^{xt} = \sum_{a=0}^{m-1} (-1)^a \frac{2e^{mt}}{e^{2mt} + 1} e^{(2a+x+1-m)t} \\ &= \sum_{a=0}^{m-1} (-1)^a \frac{2}{e^{mt} + e^{-mt}} e^{\frac{2a+x+1-m}{m} mt} \\ &= \sum_{a=0}^{m-1} (-1)^a \sum_{n=0}^{\infty} E_n \left( \frac{2a+x+1-m}{m} \right) \frac{(mt)^n}{n!} \\ &= \sum_{n=0}^{\infty} \left( m^n \sum_{a=0}^{m-1} (-1)^a E_n \left( \frac{2a+x+1-m}{m} \right) \right) \frac{t^n}{n!}. \end{aligned}$$

Hence we have the below theorem.

**Theorem 1.** For any positive integer  $m$ (=odd), we obtain

$$E_n(x) = m^n \sum_{i=0}^{m-1} (-1)^i E_n \left( \frac{2i+x+1-m}{m} \right) \text{ for } n \geq 0.$$

Since

$$\begin{aligned} \sum_{l=0}^{\infty} E_l(x+y) \frac{t^l}{l!} &= \frac{2}{e^t + e^{-t}} e^{(x+y)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \sum_{m=0}^{\infty} y^m \frac{t^m}{m!} \\ &= \sum_{l=0}^{\infty} \left( \sum_{n=0}^l E_n(x) \frac{t^n}{n!} y^{l-n} \frac{t^{l-n}}{(l-n)!} \right) \\ &= \sum_{l=0}^{\infty} \left( \sum_{n=0}^l \binom{l}{n} E_n(x) y^{l-n} \right) \frac{t^l}{l!}, \end{aligned}$$

we have the following theorem.

**Theorem 2.** The second kind Euler polynomials  $E_n(x)$  satisfies the following relation:

$$E_l(x+y) = \sum_{n=0}^l \binom{l}{n} E_n(x) y^{l-n}.$$

Because

$$\frac{\partial}{\partial x} F(t, x) = tF(t, x) = \sum_{n=0}^{\infty} \frac{d}{dx} E_n(x) \frac{t^n}{n!},$$

it follows the important relation

$$\frac{d}{dx} E_n(x) = nE_{n-1}(x).$$

We have the integral formula as follows:

$$\int_a^b E_{n-1}(x) dx = \frac{1}{n} (E_n(b) - E_n(a)).$$

Since

$$\begin{aligned} \sum_{n=0}^{\infty} E_n(-x) \frac{(-t)^n}{n!} &= F(-x, -t) = \frac{2}{e^{-t} + e^t} e^{(-x)(-t)} \\ &= \frac{2}{e^t + e^{-t}} e^{xt} = F(x, t) = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \end{aligned}$$

we obtain the following theorem.

**Theorem 3.** For  $n \in \mathbb{N}$ , we have

$$E_n(x) = (-1)^n E_n(-x).$$

By using computer, the second kind Euler polynomials  $E_n(x)$  can be determined explicitly. A few of them are

$$\begin{aligned} E_0(x) &= 1, \\ E_1(x) &= x, \\ E_2(x) &= x^2 - 1, \\ E_3(x) &= x^3 - 3x, \\ E_4(x) &= x^4 - 6x^2 + 5, \\ E_5(x) &= x^5 - 10x^3 + 25x, \\ E_6(x) &= x^6 - 15x^4 + 75x^2 - 61, \\ E_7(x) &= x^7 - 21x^5 + 175x^3 - 427x, \\ E_8(x) &= x^8 - 28x^6 + 350x^4 - 1708x^2 + 1385, \\ E_9(x) &= x^9 - 36x^7 + 630x^5 - 5124x^3 + 12465x, \\ E_{10}(x) &= x^{10} - 45x^8 + 1050x^6 - 12810x^4 + 62325x^2 - 50521, \\ E_{11}(x) &= x^{11} - 55x^9 + 1650x^7 - 28182x^5 + 228525x^3 - 555731x \\ &\dots \end{aligned}$$

## 2. Zeros of the second kind Euler polynomials $E_n(x)$

In this section, we display the shapes of the second kind Euler polynomials  $E_n(x)$  and we investigate the zeros of the second Euler polynomials  $E_n(x)$ . For  $n = 1, \dots, 10$ , we can draw a plot of the second kind Euler polynomials  $E_n(x)$ , respectively. This shows the ten plots combined into one. We display the shape of  $E_n(x)$ ,  $-8 \leq x \leq 8$ . (Figure 1).

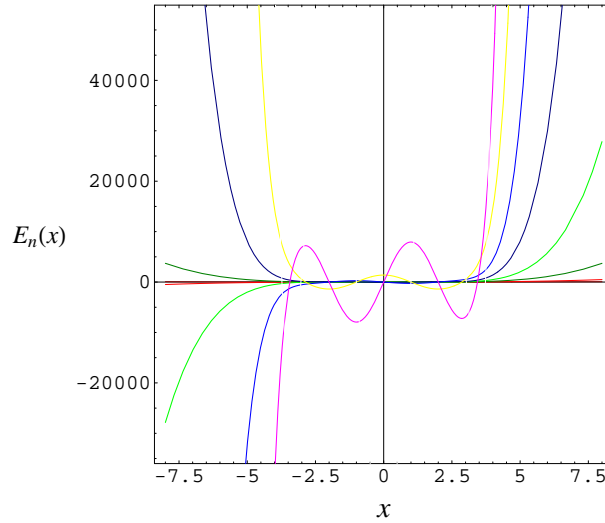


Figure 1: Curve of  $E_n(x)$

We investigate the beautiful zeros of the  $E_n(x)$  by using a computer. We plot the zeros of the second kind Euler polynomials  $E_n(x)$  for  $n = 20, 30, 40, 50$  and  $x \in \mathbb{C}$ . (Figure 2). Stacks of zeros of  $E_n(x)$  for  $1 \leq n \leq 40$  from a 3-D structure are presented. (Figure 3). Our numerical results for approximate solutions of real zeros of  $E_n(x)$  are displayed. (Tables 1, 2).

**Table 1.** Numbers of real and complex zeros of  $E_n(x)$

degree $n$	real zeros	complex zeros
2	2	0
3	3	0
4	4	0
5	5	0
6	2	4
7	3	4
8	3	4
9	5	4
10	5	4
11	3	8

We observe a remarkably regular structure of the complex roots of the Euler polynomials  $E_n(x)$ . We hope to verify a remarkably regular structure of the complex roots of the Euler polynomials  $E_n(x)$ . (Table 1). Next, we calculated an approximate solution satisfying  $E_n(x)$ ,  $x \in \mathbb{R}$ . The results are

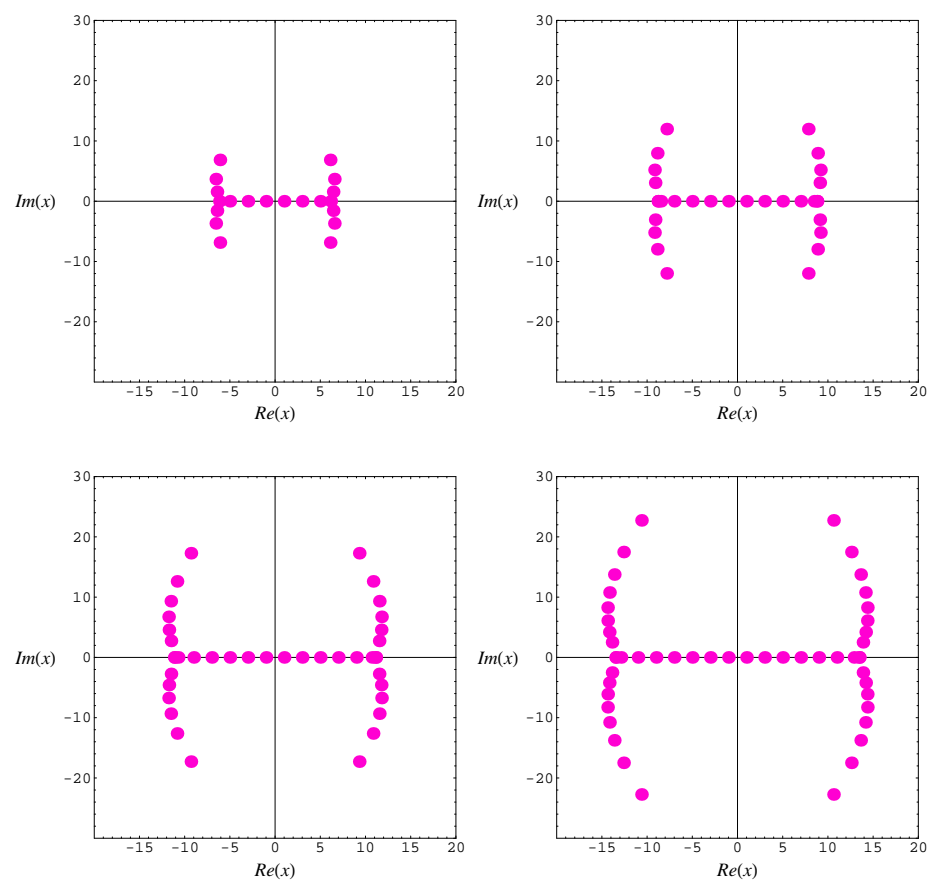
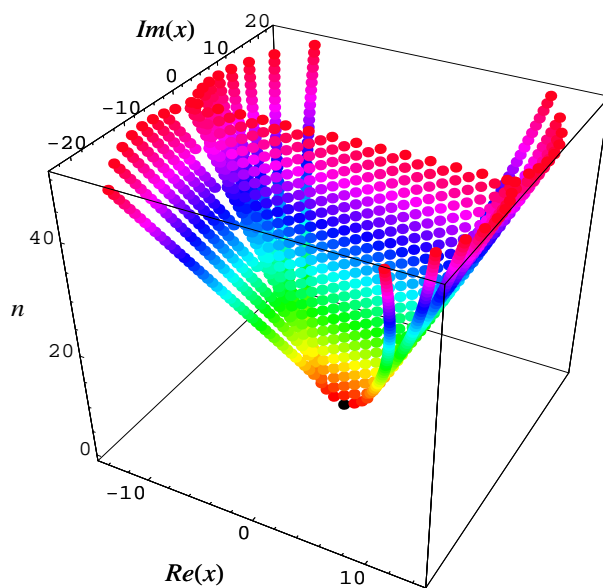


Figure 2: Zeros of  $E_n(x)$  for  $n = 20, 30, 40, 50$

given in Table 2.

**Table 2.** Approximate solutions of  $E_n(x) = 0, x \in \mathbb{R}$

degree $n$	$x$
2	-1.0000, 1.0000
3	-1.7321, 0.0000, 1.7321
4	-2.236, -1.0000, 1.0000, 2.236
5	-2.236, -2.236, 0.0000, 2.236, 2.236
6	-1.000, 1.000
7	-1.995463, 0.0000, 1.995463
8	-2.86466, -1.0000000, 1.0000000, 2.86466
9	-3.4395, -2.00016, 0.0000 2.00016, 3.4395
10	-3.730, -3.030 - 1.000, 1.000, 3.030, 3.730
11	-2.0000, 0.0000, 2.0000

Figure 3: Stacks of zeros of  $E_n(x)$ ,  $1 \leq n \leq 50$ 

### 3. Directions for Further Research

Finally, we shall consider the more general problems. Prove that  $E_n(x) = 0$  has  $n$  distinct solutions. Find the numbers of complex zeros  $C_{E_n(x)}$  of  $E_n(x)$ ,  $Im(x) \neq 0$ . Since  $n$  is the degree of the polynomial  $E_n(x)$ , the number of real zeros  $R_{E_n(x)}$  lying on the real plane  $Im(x) = 0$  is then  $R_{E_n(x)} = n - C_{E_n(x)}$ , where  $C_{E_n(x)}$  denotes complex zeros. See Table 1 for tabulated values of  $R_{E_n(x)}$  and  $C_{E_n(x)}$ . Find the equation of envelope curves bounding the real zeros lying on the plane. We prove that  $E_n(x)$ ,  $x \in \mathbb{C}$ , has  $Re(x) = 0$  reflection symmetry in addition to the usual  $Im(x) = 0$  reflection symmetry analytic complex functions. The author has no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the second kind Euler polynomials  $E_n(x)$  to appear in mathematics and physics. For related topics the interested reader is referred to [1,4].

### Acknowledgment

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# Block incomplete LU factorization for block-tridiagonal $M$ -matrices \*

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**Suggested running head:** Block ILU Factorization

## Abstract

Here we propose a block ILU preconditioner for block-tridiagonal  $M$ -matrices, and some theoretical properties for the block ILU preconditioner are studied. Numerical results of the BICGSTAB using the block ILU and ILU(0) as the preconditioners are compared to see the effective of the block ILU preconditioners.

*Key words:* Block incomplete LU factorization; preconditioner; block-tridiagonal matrix

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## 1 INTRODUCTION

The discretization of partial differential equations in 2D or 3D, by finite difference or finite element approximation, leads often to large sparse block-tridiagonal linear systems. In this paper, we consider the matrix equations

$$Ax = b, \quad x, b \in \mathbb{R}^n, \quad (1)$$

where  $A$  is a block-tridiagonal  $M$ -matrix and have the following form

$$A = \begin{pmatrix} D_1 & E_1 & & & \\ C_2 & D_2 & E_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & E_{n-1} \\ & & & C_n & D_n \end{pmatrix} \in \mathbb{R}^{n \times n}. \quad (2)$$

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Since  $A$  is  $M$ -matrix the diagonal blocks  $D_i$  of  $A$  are square  $M$ -matrices too and  $C_i$  and  $E_i$  are nonpositive matrices. The matrix  $A$  is generally sparse and with a large dimension, thus the direct solvers become prohibitively expensive. Alternatively, the preconditioned conjugate gradient (CG) iterative method is widely used. For the non-symmetric case, the other Krylov subspace methods are widely used, such as the GMRES and BICGSTAB methods. In general, the convergence rate of the above methods may be extremely slow, so the preconditioning matrix or preconditioner will always be needed and will be applied to the iterative solvers to accelerate the convergence rate. If  $K$  is a approximation of matrix  $A$ , then we can perform the left preconditioner  $K^{-1}Ax = K^{-1}b$  or the right preconditioned linear system  $AK^{-1}y = b$ , where  $y = Kx$ . The preconditioner  $K$  should be chosen so that the eigenvalue of  $K^{-1}A$  or  $AK^{-1}$  is a more centralized.

The ultimate goal of the preconditioned iterative methods is to reduce the total execution time, one of the powerful preconditioning methods in terms of reducing the number of iterations and executing time is the ILU factorization method. However, the parallelization of the construction of the ILU preconditioner is one of the problems which will influence the time used to construct the preconditioner, while the efficient of the preconditioner to accelerate the convergence rate of the iterative method such as the Krylov subspace method is another which will reduce the time used to solve the whole linear system.

The purpose of this paper is to propose a kind of block ILU preconditioner which can be computed in parallel for block-tridiagonal  $M$ -matrices. In Section 2, we review some properties of the ILU factorizations for block-tridiagonal matrix. In Section 3, the new block ILU factorization preconditioners for block-tridiagonal  $M$ -matrices are proposed and some of theoretical properties. In Section 4 the construction of a effective block preconditioners for a special type of matrices which arises from five-point discretization of the second-order PDE is given, and the numerical results of the BICGSTAB with the block ILU preconditioners proposed in this paper and the standard ILU(0) preconditioners are compared.

## 2 ILU FACTORIZATION FOR $M$ -MATRICES

J. H. Yun [2] proposed a new block incomplete factorization preconditioner for a symmetric block-tridiagonal  $M$ -matrix which can be computed in parallel. For the linear system of equations (2), where  $A$  is a symmetric block-tridiagonal  $M$ -matrix blocked in the form of

$$A = \begin{pmatrix} B_1 & -C_1 & & & \\ -C_1^T & B_2 & -C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -C_{n-1} \\ & & & -C_{n-1}^T & B_n \end{pmatrix} \in \mathbb{R}^{n \times n}, \quad (3)$$

the following theorem is given.

**Theorem 2.1.** ([2]) Let  $A$  be a symmetric block-tridiagonal  $M$ -matrix of the form (3) and let  $B_i = U_i^T D_i U_i - R_i$  be a regular splitting of  $B_i$  which can be obtained by the IC factorization process for each  $i = 1, 2, \dots, n$ . Suppose that for each  $i = 1, 2, \dots, n-1$ ,  $E_i$  is a matrix which satisfies  $U_i^T D_i E_i \leq C_i \leq E_i$ . Let

$$D = \begin{pmatrix} D_1 & 0 & & & \\ 0 & D_2 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 0 \\ & & & 0 & D_n \end{pmatrix}, \quad U = \begin{pmatrix} U_1 & 0 & & & \\ 0 & U_2 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 0 \\ & & & 0 & U_n \end{pmatrix},$$

$$\bar{U} = \begin{pmatrix} U_1 & -C_1 & & & \\ 0 & U_2 & -C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -C_{n-1} \\ & & & 0 & U_n \end{pmatrix},$$

$$\hat{U} = \begin{pmatrix} U_1 & -E_1 & & & \\ 0 & U_2 & -E_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -E_{n-1} \\ & & & 0 & U_n \end{pmatrix},$$

$$\tilde{U} = \begin{pmatrix} U_1 & -(U_1^T D_1)^{-1} C_1 & & & \\ 0 & U_2 & -(U_2^T D_2)^{-1} C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -(U_{n-1}^T D_{n-1})^{-1} C_{n-1} \\ & & & 0 & U_n \end{pmatrix} \in \mathbb{R}^{n \times n},$$

$M = U^T D U$ ,  $\bar{M} = \bar{U}^T D \bar{U}$ ,  $\hat{M} = \hat{U}^T D \hat{U}$  and  $\tilde{M} = \tilde{U}^T D \tilde{U}$ . Then the following holds:

- (a)  $R = M - A \geq 0$ ,  $\bar{R} = \bar{M} - A \geq 0$ ,  $\hat{R} = \hat{M} - A \geq 0$ ,  $\tilde{R} = \tilde{M} - A \geq 0$ ;
- (b)  $0 \leq U^{-1} \leq \bar{U}^{-1} \leq \hat{U}^{-1} \leq \tilde{U}^{-1}$ ;
- (c)  $0 \leq M^{-1} \leq \bar{M}^{-1} \leq \hat{M}^{-1} \leq \tilde{M}^{-1}$ ;
- (d)  $A = M - R = \bar{M} - \bar{R} = \hat{M} - \hat{R} = \tilde{M} - \tilde{R}$  are regular splittings of  $A$ ;
- (e)  $\rho(\tilde{M}^{-1} R) \leq \rho(\bar{M}^{-1} R) \leq \rho(\hat{M}^{-1} R) \leq \rho(M^{-1} R) \leq 1$ .

The most favorite character of the four types of IC factorization preconditioners presented by [2] is that it can be computed in parallel. Block IC factorization preconditioner of type  $\tilde{M}$  may not be used in practical situations since it requires a lot



of fill-in elements causing too much storage and arithmetic operations. Block preconditioner of type  $M$  has rich parallelism since both the computation of preconditioner and preconditioner solve step of the PCG can be done in parallel, but its effectiveness is much worse than the other types of block preconditioners. So the preconditioner of type  $\hat{M}$  is recommended. Yun [11] have already make it to more generalized case for nonsymmetric block-tridiagonal  $M$ -matrices.

Saad [3] and M. H. Koulai [4] discussed the block ILU factorization method too, since the block-tridiagonal matrix  $A$  has the form

$$A = L + D + U, \quad (4)$$

where  $D$  is the diagonal block part of matrix  $A$ ,  $L$  is the lower block part of the matrix  $A$  and  $U$  is the upper block part. Then a block factorization can be defined as

$$M = (L + S)S^{-1}(S + U), \quad (5)$$

where  $L$  and  $U$  are the same as those above, and  $S$  is a block-diagonal matrix whose blocks  $S_i$  are defined by the recurrence:

$$S_1 = D_1, \quad S_i = D_i - C_i a_{i-1} E_{i-1},$$

in which  $a_i$  is the sparse approximate to  $S_i^{-1}$ . In this case for computing the blocks  $a_i$ , it is natural to keep the blocks  $a_i$  banded, particular keep it tridiagonal. Then the method proposed by [8-10] can be used to compute the approximate inverse of  $S_{i-1}^{-1}$ . Koulai [4] proposed two algorithm for computing the inverse factors directly which is needed for computing  $a_i$ . Using the LU factorization of  $S_i$

$$S_i = L_i U_i,$$

the preconditioner  $M$  can be expressed in terms of lower and upper triangular factors

$$M = \begin{pmatrix} L_1 & & & & & \\ V_2 & L_2 & & & & \\ & \ddots & \ddots & & & \\ & & V_{n-1} & L_{n-1} & & \\ & & & V_n & L_n & \end{pmatrix} \begin{pmatrix} U_1 & W_2 & & & & \\ & U_2 & W_3 & & & \\ & & \ddots & \ddots & & \\ & & & U_{n-1} & W_n & \\ & & & & U_n & \end{pmatrix}, \quad (6)$$

where

$$V_i = C_i U_{i-1}^{-1}, \quad W_i = L_{i-1}^{-1} E_{i-1}.$$

This form is generally more efficient. But since  $S_i$  is computed by the recurrence, the construction of the preconditioner  $M$  is less parallel than Yun's [2].

### 3 BLOCK ILU FACTORIZATION PRECONDITIONERS

We consider the matrix  $A$  which is a large sparse block-tridiagonal  $M$ -matrix blocked in the form (2) too, to compute the  $S_i$  we define the recurrence as:

$$S_1 = D_1, \quad S_i = D_i - C_i D_{i-1}^{-1} E_{i-1},$$

using the ILU factorization  $S_i = L_i U_i$ , the preconditioner  $M$  can be expressed in a similar form as equation(6).

$$M = LU = \begin{pmatrix} L_1 & & & & \\ V_2 & L_2 & & & \\ & \ddots & \ddots & & \\ & & V_{n-1} & L_{n-1} & \\ & & & V_n & L_n \end{pmatrix} \begin{pmatrix} U_1 & W_2 & & & \\ & U_2 & W_3 & & \\ & & \ddots & \ddots & \\ & & & U_{n-1} & W_n \\ & & & & U_n \end{pmatrix},$$

where  $V_i = C_i U_{i-1}^{-1}$  and  $W_i = L_i^{-1} E_{i-1}$  too.

The only different is the construction of the  $S_i$ , we define is as  $S_i = D_i - C_i D_{i-1}^{-1} E_{i-1}$  instead of  $S_i = D_i - C_i S_{i-1}^{-1} E_{i-1}$  used by Saad [3] and M. H. Koulaei [4] or  $S_i = D_i$  used by J. H. Yun [2]. Then we have the following Theorems.

**Lemma 3.1**([2]). *Let  $A$  and  $B$  be square  $M$ -matrices, and let  $A = L_1 U_1 - R_1$  and  $B = L_2 U_2 - R_2$  be ILU factorizations corresponding to the same zero pattern set  $P \in P_n$ . If  $A \leq B$ , then  $L_2^{-1} \leq L_1^{-1}$  and  $U_2^{-1} \leq U_1^{-1}$ .*

**Theorem 3.2.** *Let  $A$  be a block-tridiagonal  $M$ -matrix of the form (2), and let  $M = LU$  and the computation of  $S_i$  defined as above. Then*

$$S_i + C_i S_{i-1}^{-1} E_{i-1} \geq D_i.$$

**Proof.** Since  $A$  is a block-tridiagonal  $M$ -matrix, from the definition of  $S_i$ , we can get  $S_{i-1} \leq D_{i-1}$  and that they are both  $M$ -matrices. So

$$S_{i-1}^{-1} \geq D_{i-1}^{-1},$$

Then

$$S_i + C_i S_{i-1}^{-1} E_{i-1} - D_i = C_i S_{i-1}^{-1} E_{i-1} - C_i D_{i-1}^{-1} E_{i-1} \geq 0.$$

Thus the proof complete.  $\square$

**Theorem 3.3.** *Let  $A$  be a block-tridiagonal  $M$ -matrix of the form (2), and let  $M = LU$  defined as above. Then the following holds:*

(a)  $R = M - A \geq 0$ ;

- (b)  $U^{-1} \geq 0$  and  $L^{-1} \geq 0$ ;  
 (c)  $A = M - R$  is a regular splitting of  $A$ .

**Proof.** By simple calculation, we can obtain

$$R = M - A = \begin{pmatrix} 0 & & & \\ & S_2 + C_2 S_1^{-1} E_1 - D_2 & & \\ & & \ddots & \\ & & & S_n + C_n S_{n-1}^{-1} E_{n-1} - D_n \end{pmatrix}.$$

Then  $R = M - A \geq 0$  can be easily obtained from Theorem 3.2. Since  $L$  and  $U$  are both  $Z$ -matrices with positive diagonal, they are  $M$ -matrices, then we can get  $U^{-1} \geq 0$  and  $L^{-1} \geq 0$ , and

$$M^{-1} = U^{-1} L^{-1} \geq 0.$$

Therefore  $A = M - R$  is a regular splitting of  $A$ .  $\square$

**Theorem 3.4.** Let  $A$  be a block-tridiagonal  $M$ -matrix of the form (2), let  $M = LU$  defined as above, and  $\bar{M} = \bar{L}\bar{U}$ , where  $\bar{L}$  and  $\bar{U}$  are constructed as follow:

$$B_i = \bar{L}_i \bar{U}_i,$$

and

$$\bar{M} = \bar{L}\bar{U} = \begin{pmatrix} \bar{L}_1 & & & \\ \bar{V}_2 & \bar{L}_2 & & \\ & \ddots & \ddots & \\ & & \bar{V}_{n-1} & \bar{L}_{n-1} \\ & & & \bar{V}_n & \bar{L}_n \end{pmatrix} \begin{pmatrix} \bar{U}_1 & \bar{W}_2 & & \\ & \bar{U}_2 & \bar{W}_3 & \\ & & \ddots & \ddots \\ & & & \bar{U}_{n-1} & \bar{W}_n \\ & & & & \bar{U}_n \end{pmatrix},$$

where

$$\bar{V}_i = C_i \bar{U}_{i-1}^{-1}, \quad \bar{W}_i = \bar{L}_{i-1}^{-1} E_{i-1}.$$

Then

$$M^{-1} \geq \bar{M}^{-1} \quad \text{and} \quad \rho(M^{-1}R) \leq \rho(\bar{M}^{-1}\bar{R}).$$

**Proof.** From the definition of  $S_i$ , it is easy to obtain that  $S_i \leq D_i$ . From Lemma 1, we know that

$$L_i \leq \bar{L}_i \quad \text{and} \quad U_i \leq \bar{U}_i.$$

So

$$\begin{aligned} V_i &= C_i U_{i-1}^{-1} \leq \bar{V}_i = C_i \bar{U}_{i-1}^{-1}, \\ W_i &= L_{i-1}^{-1} E_{i-1} \leq \bar{W}_i = \bar{L}_{i-1}^{-1} E_{i-1}. \end{aligned}$$

Then

$$L \leq \bar{L} \quad \text{and} \quad U \leq \bar{U},$$

so

$$M^{-1} = U^{-1}L^{-1} \geq \bar{M}^{-1} = \bar{U}^{-1}\bar{L}^{-1}.$$

Since  $A = M - R$  and  $A = \bar{M} - \bar{R}$  are both regular splittings, and  $M^{-1} \geq \bar{M}^{-1}$ . It is naturally  $\rho(M^{-1}R) \leq \rho(\bar{M}^{-1}\bar{R})$ .  $\square$

For the matrix  $A$  which is a large sparse block-tridiagonal  $M$ -matrix blocked in the form (2), when we perform the block ILU factorization we can also define the recurrence of the computation of  $S_i$  as:

$$S_1 = D_1,$$

for  $i = 3, 5, \dots, 2i + 1, \dots,$

$$S_i = D_i - C_i D_{i-1}^{-1} E_{i-1};$$

For  $i = 2, 4, \dots, 2i, \dots,$

$$S_i = D_i - C_i S_{i-1}^{-1} E_{i-1}.$$

In other words, if we define the recurrence for the first steps as  $S_1 = D_1$ , and the following  $k - 1$  steps as  $S_i = D_i - C_i S_{i-1}^{-1} E_{i-1}$ . At the  $k + 1$  step we restart and define the recurrence as  $S_i = D_i - C_i D_{i-1}^{-1} E_{i-1}$  and the following  $k - 1$  steps we also define it as  $S_i = D_i - C_i S_{i-1}^{-1} E_{i-1}$ , repeat such process until it has been exceeded the matrix dimension. We call such process the  $k$  step restart method. To simplify the  $k$  step restart method we can choose the number  $k$  so that the matrix dimension  $n$  can be divided exactly by  $k$ .

Then for  $k \geq 2$ , the  $k$  step restart method can be expressed as

$$S_1 = D_1,$$

for  $i = 2, 3, \dots, n$ , if  $(i) \bmod(k) = 1$ ,

$$S_i = D_i - C_i D_{i-1}^{-1} E_{i-1},$$

otherwise,

$$S_i = D_i - C_i S_{i-1}^{-1} E_{i-1}.$$

For  $k < 2$ , when  $k = 0$  we just define the  $S_i$  by  $S_i = D_i$  and when  $k = 1$  we define the  $S_i$  by  $S_1 = D_1$  and for  $i = 2, 3, \dots, n$ ,  $S_i = D_i - C_i D_{i-1}^{-1} E_{i-1}$ , for  $k \geq 2$ , it can be performed as we discussed above. It is easy to prove that when we perform the block ILU factorization combining with the  $k$  step restart method to compute the  $S_i$ , then for all  $k = 2, \dots, n$ , Theorem 3.2, Theorem 3.3 and Theorem 3.4 are consistent. And when  $k = n$ , it is just the method Saad [3] and Koulai [4] discussed.

## 4 NUMERICAL RESULTS

The construction of the block ILU factorization preconditioner presented in section 3 will be considered for a special type of matrices described below. The matrix arises from five-point discretization of the second-order self-adjoint elliptic partial differential equation:

$$-au_{xx} - bu_{yy} + cu_x + du_y + fu = g, \quad (7)$$

with  $a(x, y) > 0$ ,  $b(x, y) > 0$ ,  $c(x, y), d(x, y)$  and  $f(x, y)$  defined on the unit square region  $\Omega$ , and with suitable boundary conditions on  $\partial\Omega$ .

Therefore  $C_i$  and  $E_i$  are diagonal and  $D_i$  is tridiagonal. Since  $D_i$  is a tridiagonal matrix, the complete LU factorization of  $D_i$  is equal to ILU(0). Considering the recurrence  $S_i = D_i - C_i D_{i-1}^{-1} E_{i-1}$ , the approximate inverse of  $D_i$  should be simple for the calculation of the  $S_i$  and do better make the  $S_i$  have the same zero pattern as  $D_i$ , by other words we can make the approximation of inverse of  $D_i$  still tridiagonal. Thus we will use the  $L_{D_i}^{-1}$  and  $U_{D_i}^{-1}$  to compute the tridiagonal part of the  $D_i^{-1}$  where  $D_i = L_{D_i} U_{D_i}$ . Since the inverse of the matrix  $D_i$  can be calculated parallel, the calculation of the matrix  $S_i$  can be done respectively too. The inverse of  $L_i$  and  $U_i$ ,  $S_i = L_i U_i$ , are also used to construct the block ILU factorization, so we denote the term  $L_i^{-1j}$ , the lower banded matrices with  $j$  lower diagonal by drop some nonzero elements of  $L_i^{-1}$ , and  $U_i^{-1j}$  respectively. The nonzero structures of  $L_i^{-1j}$  are illustrated in Fig.1.

By  $L^j$  we means the block ILU preconditioner constructed by using  $L_i^{-1j}$  and  $U_i^{-1j}$  to compute the  $S_i$  and  $W_i$  and  $V_i$ . The nonzero structures of  $L^j$  are illustrated in Fig. 2. And the term  $M^j$  denotes  $M^j = L^j U^j$ .

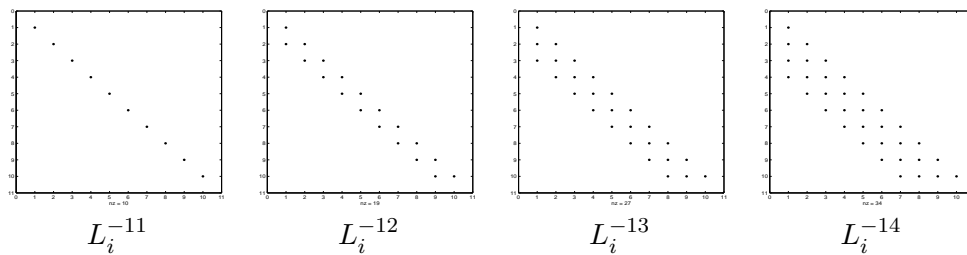
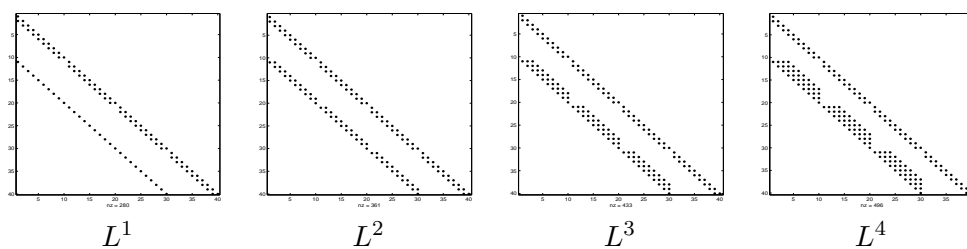
Now we provide numerical results of the BICGSTAB method using different preconditioners  $M^j$  and ILU(0) for linear systems  $Ax = b$ , where  $A$  is the block tridiagonal  $M$ -matrices.

**Example 1.** This example considers Eq. (7) with  $a(x, y) = b(x, y) = 1$ ,  $c(x, y) = -10(x + y)$ ,  $d(x, y) = -10(x - y)$ , and  $f(x, y) = 1$ .

**Example 2.** This example considers Eq. (7) with  $a(x, y) = c(x, y) = 1$ ,  $b(x, y) = -(1 + y^2)$ ,  $d(x, y) = (1 + y^2)$ , and  $f(x, y) = 1$ .

**Example 3.** This example considers Eq. (7) with  $a(x, y) = b(x, y) = 1$ ,  $c(x, y) = \cos(x/6)$ ,  $d(x, y) = \sin(y/6)$ , and  $f(x, y) = 1$ .

**Example 4.** This example considers Eq. (7) with  $a(x, y) = b(x, y) = 1$ ,  $c(x, y) = 10e^{xy}$ ,  $d(x, y) = 10e^{-xy}$ , and  $f(x, y) = 1$ .

Figure 1: Nonzero structures of  $L_i^{-1j,s}$ .Figure 2: Nonzero structures of  $L^j$ 's.

In the examples above, we all used a uniform mesh of  $\Delta x = \Delta y = 1/(m+1)$ , which leads to a matrix of order  $n = m \times m$ , where  $\Delta x$  and  $\Delta y$  refer to the mesh sizes in the  $x$ -direction and  $y$ -direction respectively. The right-hand side  $b$  is created from  $Ae$ , where  $e = (1, \dots, 1)^T \in R^n$ . Therefore, the right-hand side function  $g(x, y)$  in (7) is not relevant. We will show the results of the BICGSTAB with ILU(0) and with our Preconditioners when  $m = 32$  in Table 1, Table 2 and Table 3 and  $m = 48$  in Table 4, Table 5 and Table 6. The  $L^j$  and  $U^j$  are constructed as above.

Table 1: The number of iterations of BICGSTAB with ILU(0) and with our Preconditioners when  $m = 32$  with 1 step restart.

	example 1	example 2	example 3	example 4
ILU(0)	16	20	17	17
$M^1$	15	18	16	16
$M^2$	12	13	12	11
$M^3$	11	11	10	9
$M^4$	9	11	10	9

Table 2: The number of iterations of BICGSTAB with ILU(0) and with our Preconditioners when  $m = 32$  with 2 step restart.

	example 1	example 2	example 3	example 4
ILU(0)	16	20	17	17
$M^1$	14	20	16	16
$M^2$	10	12	12	10
$M^3$	9	11	10	9
$M^4$	9	10	9	8

Table 3: The number of iterations of BICGSTAB with ILU(0) and with our Preconditioners when  $m = 32$  with 3 step restart.

	example 1	example 2	example 3	example 4
ILU(0)	16	20	17	17
$M^1$	15	18	16	16
$M^2$	10	12	12	10
$M^3$	9	10	10	9
$M^4$	9	10	9	8

Table 4: The number of iterations of BICGSTAB with ILU(0) and with our Preconditioners when  $m = 48$  with 1 step restart.

	example 1	example 2	example 3	example 4
ILU(0)	25	29	23	23
$M^1$	25	27	23	21
$M^2$	17	21	16	16
$M^3$	15	19	14	13
$M^4$	14	19	14	12

Table 5: The number of iterations of BICGSTAB with ILU(0) and with our Preconditioners when  $m = 48$  with 2 step restart.

	example 1	example 2	example 3	example 4
ILU(0)	25	29	23	23
$M^1$	21	28	22	22
$M^2$	15	18	16	14
$M^3$	13	16	14	13
$M^4$	14	16	13	12

Table 6: The number of iterations of BICGSTAB with ILU(0) and with our Preconditioners when  $m = 48$  with 3 step restart.

	example 1	example 2	example 3	example 4
ILU(0)	25	29	23	23
$M^1$	21	26	24	21
$M^2$	15	16	15	15
$M^3$	14	16	14	13
$M^4$	12	15	12	11

## 5 CONCLUSIONS

We presented in this paper a block ILU factorization method. Notice that the construction of  $M^j$  requires more storage and arithmetic operations when  $j$  becomes large, it is not recommended to use a large value of  $j$  and  $j = 2, 3$  is better enough from our experiments. Theoretically speaking, the block ILU factorization method combining with the  $k$  step restart method will performance better when the number  $k$  is bigger, especially for the block ILU preconditioner to reduce the iteration counts of the iterative method. But from our experiments it is not always right, when  $k$  is large enough the iteration counts of the BICGSTAB will not reduce as it is expected. And with a small  $k$  it means the parallelization of the calculation of the block ILU preconditioner will be enhanced, thus a less time will be used to construct the block ILU preconditioner, of course a less efficient of the block ILU preconditioner will be encountered. A pity is that the parrel computation of the block ILU preconditioner is beyond our ability now, we can not do much more compare with the time used to construct the preconditioner, so only the iteration counts are compared to see the efficient of the proposed block ILU preconditioners.



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# A numerical investigation on the structure of the roots of ( $h, q$ )-extension of $q$ -analogue of Genocchi polynomials

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**Abstract :** In this paper, we construct a new  $(h, q)$ -extension of  $q$ -analogue of Genocchi polynomials  $G_{n,q}^{(h)}(x)$ . We consider the behavior of real roots of the  $(h, q)$ -extension of  $q$ -analogue of Genocchi polynomials  $G_{n,q}^{(h)}(x)$ , using numerical investigation. By numerical experiments, we demonstrate a remarkably regular structure of the complex roots of the  $G_{n,q}^{(h)}(x)$ . Finally, we give a table for the solutions of the  $(h, q)$ -extension of  $q$ -analogue of Genocchi polynomials  $G_{n,q}^{(h)}(x)$ .

**Key words :** Euler numbers, Euler polynomials, Genocchi numbers, Genocchi polynomials,  $q$ -analogue of Genocchi numbers,  $q$ -analogue of Genocchi polynomials,  $(h, q)$ -extension of  $q$ -analogue of Genocchi polynomials

**2000 Mathematics Subject Classification :** 11S80, 11B68

## 1. Introduction

The  $q$ -analogue of Genocchi numbers  $G_{n,q}$  and polynomials  $G_{n,q}(x)$  using generating functions was constructed by T. Kim [1]. The significance of the  $q$ -analogue of Genocchi numbers  $G_{n,q}$  and polynomials  $G_{n,q}(x)$  was already pointed out in [1,2]. In this paper, we construct a new  $(h, q)$ -extension of  $q$ -analogue of Genocchi polynomials,  $G_{n,q}^{(h)}(x)$ . In order to study  $(h, q)$ -extension of  $q$ -analogue of Genocchi polynomials,  $G_{n,q}^{(h)}(x)$ , we must understand the structure of the  $(h, q)$ -extension of  $q$ -analogue of Genocchi polynomials,  $G_{n,q}^{(h)}(x)$ . Therefore, using computer, a realistic study for the  $(h, q)$ -extension of  $q$ -analogue of Genocchi polynomials,  $G_{n,q}^{(h)}(x)$  is very interesting. The main purpose of this paper is to describe the distribution and structure of the zeros of the  $(h, q)$ -extension of  $q$ -analogue of Genocchi polynomials,  $G_{n,q}^{(h)}(x)$  for values of the index  $n$  by using computer. The outline of this paper is as follows. We introduce the  $(h, q)$ -extension of  $q$ -analogue of Genocchi polynomials,  $G_{n,q}^{(h)}(x)$ . In Section 2, we describe the beautiful zeros of the  $(h, q)$ -extension of  $q$ -analogue of Genocchi polynomials,  $G_{n,q}^{(h)}(x)$  using a numerical investigation. Finally, we also investigate the roots of the  $(h, q)$ -extension of  $q$ -analogue of Genocchi polynomials,  $G_{n,q}^{(h)}(x)$ .

First, we introduce the Genocchi numbers and Genocchi polynomials. The Genocchi numbers  $G_n$  are defined by the generating function:

$$G(t) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, (|t| < \pi), \text{ cf. [1, 2, 3]}, \quad (1)$$

where we use the technique method notation by replacing  $G^n$  by  $G_n (n \geq 0)$  symbolically. For  $x \in \mathbb{R}$

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(= the field of real numbers), we consider the Genocchi polynomials  $G_n(x)$  as follows:

$$G(x, t) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}. \quad (2)$$

Note that  $G_n(x) = \sum_{k=0}^n \binom{n}{k} G_k x^{n-k}$ . In the special case  $x = 0$ , we define  $G_n(0) = G_n$ .

The Euler numbers are defined by the following generating function

$$F(t) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, (|t| < \pi), \text{ cf. [2, 3]}, \quad (3)$$

where we use the technique method notation by replacing  $E^n$  by  $E_n$  symbolically. For  $x \in \mathbb{R}$  (= the field of real numbers), we consider the Euler polynomials  $E_n(x)$  as follows:

$$F(x, t) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \quad (4)$$

## 2. $(h, q)$ -extension of $q$ -analogue of Genocchi polynomials

Let  $q$  be a complex number with  $|q| < 1$ . The  $q$ -analogue of  $n$  is denoted by

$$[n]_q = \frac{1 - q^n}{1 - q}.$$

In [1], Kim introduced  $q$ -analogue of Euler numbers and polynomials. We consider the following generating functions:

$$F_q(t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]_q t} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}, \quad (5)$$

and

$$F_q(x, t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n+x]_q t} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \quad (6)$$

We see that

$$\lim_{q \rightarrow 1} F_q(x, t) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad \lim_{q \rightarrow 1} F_q(t) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

T. Kim constructed  $q$ -analogue of Genocchi numbers and polynomials(see [1]). We introduce the following generating functions:

$$G_q(t) = [2]_q t e^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \frac{(-1)^n}{1 + q^{n+1}} \left( \frac{1}{1 - q} \right)^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}, \quad (7)$$

and

$$G_q(x, t) = [2]_q q^x t e^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \frac{(-1)^n}{1 + q^{n+1}} q^{nx} \left( \frac{1}{1 - q} \right)^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}, \quad (8)$$

Note that

$$\lim_{q \rightarrow 1} G_q(x, t) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad \lim_{q \rightarrow 1} G_q(t) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}.$$

We now construct a new  $(h, q)$ -extension of  $q$ -analogue of Genocchi polynomials  $G_{n,q}^{(h)}(x)$ .  $(h, q)$ -extension of  $q$ -analogue of Genocchi polynomials  $G_{n,q}^{(h)}(x)$  are defined by means of the generation function

$$G_q^{(h)}(t) = [2]_q t \sum_{n=0}^{\infty} (-1)^n q^{nh} e^{[n]_q t} = \sum_{n=0}^{\infty} G_{n,q}^{(h)} \frac{t^n}{n!}, \quad (9)$$

and

$$G_q^{(h)}(x, t) = [2]_q t \sum_{n=0}^{\infty} (-1)^n q^{(n+x)h} e^{[n+x]_q t} = \sum_{n=0}^{\infty} G_{n,q}^{(h)}(x) \frac{t^n}{n!}. \quad (10)$$

Hence, we have the following remark.

**Remark.** For  $n \geq 0$ , we have

- (1)  $\lim_{q \rightarrow 1} G_q^{(h)}(x, t) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}.$
- (2)  $\lim_{q \rightarrow 1} G_q^{(h)}(t) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}.$
- (3) If  $h = 1$ , then  $G_q^{(h)}(x, t) = G_q(x, t)$  and  $G_q^{(h)}(t) = G_q(t).$
- (4)  $G_{n,q}^{(h)}(0) = G_{n,q}^{(h)}.$
- (5)  $\lim_{q \rightarrow 1} G_{n,q}^{(h)}(x) = G_n(x)$  and  $\lim_{q \rightarrow 1} G_{n,q}^{(h)} = G_n.$
- (6) If  $h = 1$ , then  $G_{n,q}^{(h)}(x) = G_{n,q}(x)$  and  $G_{n,q}^{(h)} = G_{n,q}.$

By (10), we obtain

$$\begin{aligned} G_q^{(h)}(x, t) &= [2]_q t \sum_{m=0}^{\infty} (-1)^m q^{(m+x)h} e^{[m+x]_q t} \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m q^{(m+x)h} \sum_{n=0}^{\infty} n[m+x]_q^{n-1} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left( [2]_q n \left( \frac{1}{1-q} \right)^{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{(-1)^l}{1+q^{l+h}} q^{(l+h)x} \right) \frac{t^n}{n!}. \end{aligned}$$

Hence, we have the following theorem.

**Theorem 1.** For  $n \geq 0$ , we have

$$G_{n,q}^{(h)}(x) = [2]_q n \left( \frac{1}{1-q} \right)^{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{(-1)^l}{1+q^{l+h}} q^{(l+h)x}.$$

By definition of  $(h, q)$ -extension of  $q$ -analogue of Genocchi polynomials  $G_{n,q}^{(h)}(x)$ , we obtain the following corollary.

**Corollary 2.** For  $n \geq 0$ , we have

$$G_{n,q}^{(h)} = [2]_q n \left( \frac{1}{1-q} \right)^{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{(-1)^l}{1+q^{l+h}}.$$

We obtain the first value of the  $(h, q)$ -extension of  $q$ -analogue of Genocchi numbers  $G_{n,q}^{(h)}$  :

$$\begin{aligned} G_{0,q}^{(h)} &= 0, \quad G_{1,q}^{(h)} = \frac{1+q}{1+q^h}, \\ G_{2,q}^{(h)} &= -\frac{2q^h(1+q)}{(1+q^h)(1+q^{1+h})}, \\ G_{3,q}^{(h)} &= \frac{3q^h(1+q)(-1+q^{1+h})}{(1+q^h)(1+q^{1+h})(1+q^{2+h})}, \\ G_{4,q}^{(h)} &= -\frac{4q^h(1+q)(1-2q^{1+h}-2q^{2+h}+q^{3+2h})}{(1+q^h)(1+q^{1+h})(1+q^{2+h})(1+q^{3+h})} \cdots \end{aligned}$$

In [5], the  $(h, q)$ -Euler polynomials  $E_{n,q}^{(h)}(x)$  are defined by means of the generating function

$$F_q^{(h)}(x, t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{[x+n]_q t}. \quad (11)$$

Since

$$[2]_q \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{q^x [n]_q t} = \sum_{n=0}^{\infty} q^{nx} \frac{G_{n+1,q}^{(h)}}{n+1} \frac{t^n}{n!},$$

we obtain

$$\begin{aligned} F_q^{(h)}(x, t) &= [2]_q \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{[n+x]_q t} = e^{[x]_q t} [2]_q \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{q^x [n]_q t} \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \binom{n}{k} [x]_q^{n-k} q^{nx} \frac{G_{n+1,q}^{(h)}}{n+1} \right) \frac{t^n}{n!}. \end{aligned}$$

By using comparing coefficients  $\frac{t^n}{n!}$ , we have the following theorem.

**Theorem 3.** For any positive integer  $n$ , we have

$$E_{n,q}^{(h)}(x) = \sum_{k=0}^n \binom{n}{k} [x]_q^{n-k} q^{nx} \frac{G_{n+1,q}^{(h)}}{n+1}.$$

By (10), we obtain

$$\sum_{n=0}^{\infty} G_{n,q}^{(h)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \frac{[2]_q}{[2]_q^m} [m]_q^{n-1} \sum_{a=0}^{m-1} (-1)^a G_{n,q^m}^{(h)} \left( \frac{x+a}{m} \right) \right) \frac{t^n}{n!} \text{ for } m \text{ odd}.$$

Thus we have the following theorem.

**Theorem 4.** Let  $m \in \mathbb{N}$  and  $m$  odd. Then we have

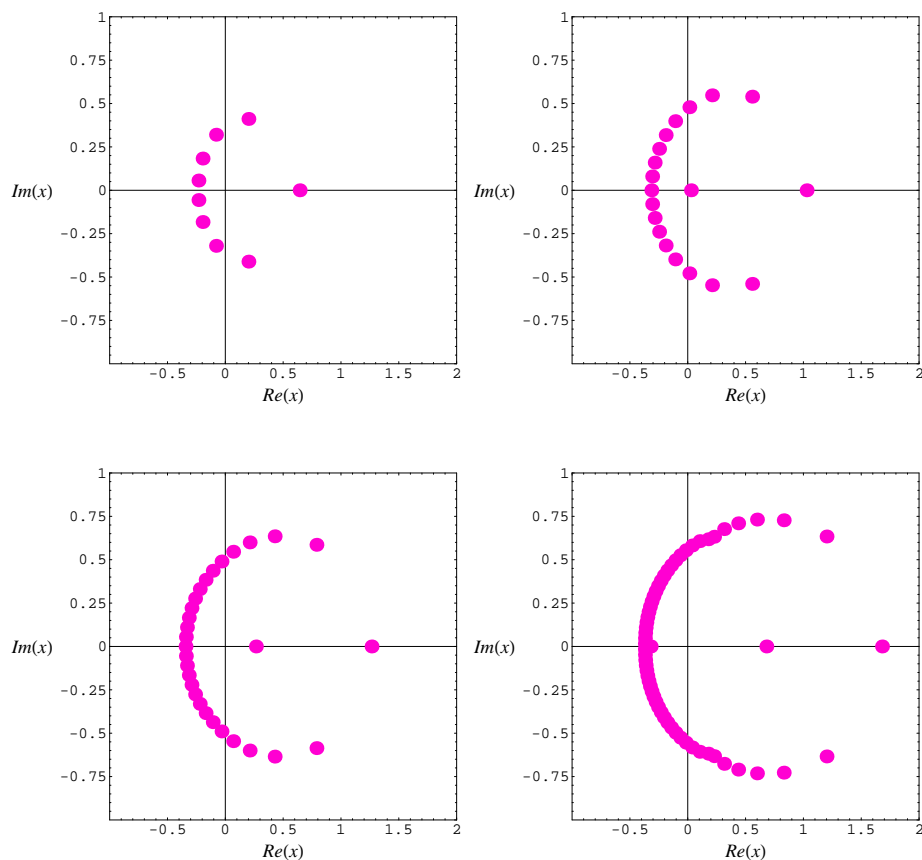
$$G_{n,q}^{(h)}(x) = \frac{[2]_q}{[2]_q^m} [m]_q^{n-1} \sum_{a=0}^{m-1} (-1)^a G_{n,q^m}^{(h)} \left( \frac{x+a}{m} \right) = \sum_{k=0}^n \binom{n}{k} q^{xk} q^{(h-1)x} G_{n,q}^{(h)} [x]_q^{n-k}.$$

### 3. Distribution and Structure of the Zeros

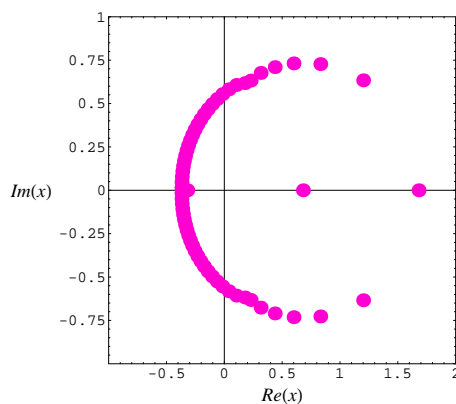
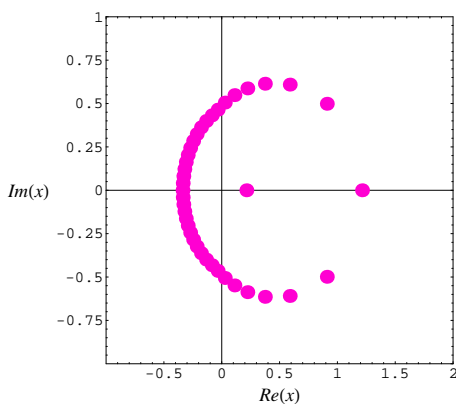
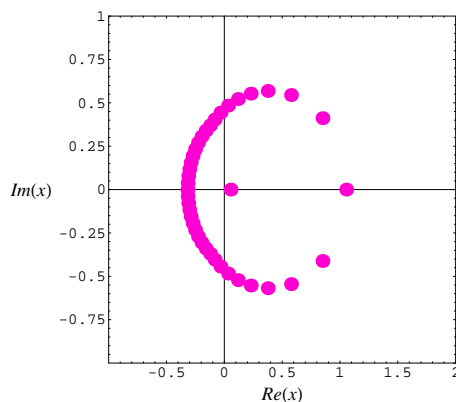
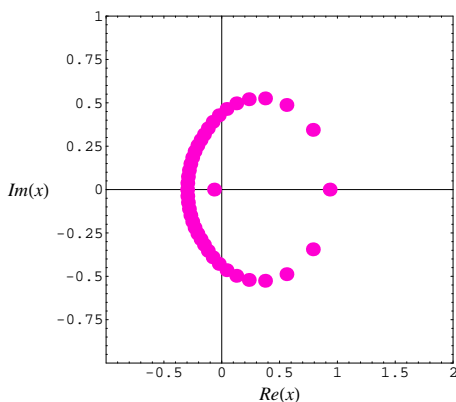
In order to study  $G_{n,q}^{(h)}(x)$ , we must understand the structure of the  $(h, q)$ -extension of  $q$ -analogue of Genocchi polynomials  $G_{n,q}^{(h)}(x)$ . By numerical investigation, we examine properties of the figures, look for patterns, and make open problems. Here is the list of the first the  $(h, q)$ -Euler polynomials  $G_{n,q}^{(h)}(x)$ .

$$\begin{aligned} G_{1,q}^{(h)}(x) &= \frac{q^{hx}(1+q)}{1+q^h}, \\ G_{2,q}^{(h)}(x) &= -\frac{2(1+q)(q^{hx} - q^{(1+h)x} + q^{1+h+hx} - q^{h+(1+h)x})}{(-1+q)(1+q^h)(1+q^{1+h})}, \dots \end{aligned}$$

We investigate the beautiful zeros of the  $G_{n,q}^{(h)}(x)$  by using a computer. We plot the zeros of the  $(h, q)$ -extension of  $q$ -analogue of Genocchi polynomials  $G_{n,q}^{(2)}(x)$  for  $n = 10, 20, 30, 60, q = \frac{1}{5}$ , and  $x \in \mathbb{C}$ . (Figure 1). We plot the zeros of the  $G_{n,q}^{(h)}(x)$  for  $n = 40, q = \frac{1}{5}, h = 2, 3, 4, 5$ , and  $x \in \mathbb{C}$ . (Figures 2, 3, 4, and 5). We observe a remarkably regular structure of the complex roots of  $G_{n,q}^{(h)}(x)$ . We hope to verify a remarkably regular structure of the complex roots of the  $G_{n,q}^{(h)}(x)$  (Table 1). Next, we calculate an approximate solution satisfying  $G_{n,q}^{(h)}(x), x \in \mathbb{R}$ . The results are given in Table 2.

Figure 1: Zeros of  $G_{n,q}^{(2)}(x)$  for  $n = 10, 20, 30, 60, q = 1/5$ **Table 1.** Numbers of real and complex zeros of  $G_{n,q}^{(h)}(x)$ 

degree $n$	$h = 2$		$h = 3$	
	real zeros	complex zeros	real zeros	complex zeros
2	1	0	1	0
3	2	0	2	0
4	1	2	1	2
5	2	2	2	2
6	1	4	1	4
7	2	4	2	4
8	1	6	1	6
9	2	6	2	6
10	1	8	1	8
11	2	8	2	8

Figure 2: Zeros of  $G_{40,q}^{(2)}(x)$ Figure 3: Zeros of  $G_{40,q}^{(3)}(x)$ Figure 4: Zeros of  $G_{40,q}^{(4)}(x)$ Figure 5: Zeros of  $G_{40,q}^{(5)}(x)$ **Table 2.** Approximate solutions of  $G_{n,1/5}^{(h)}(x) = 0, x \in \mathbb{R}$ 

$n$	$h = 2$	$h = 3$
2	0.0194183	0.00395756
3	-0.0865271, 0.109903	-0.0419744, 0.0467265
4	0.210809	0.112746
5	-0.172576, 0.303527	-0.125933, 0.181708
6	0.38639	0.247597
7	-0.211514, 0.46049	-0.175576, 0.308997
8	0.527205	0.365857
9	-0.230443, 0.587741	-0.205166, 0.418522
10	0.643079	0.467428
11	-0.2358, 0.694006	-0.223993, 0.512998

Figure 6 presents the distribution of real zeros of  $G_{n,1/5}^{(2)}(x)$  for  $11 \leq n \leq 50$ .

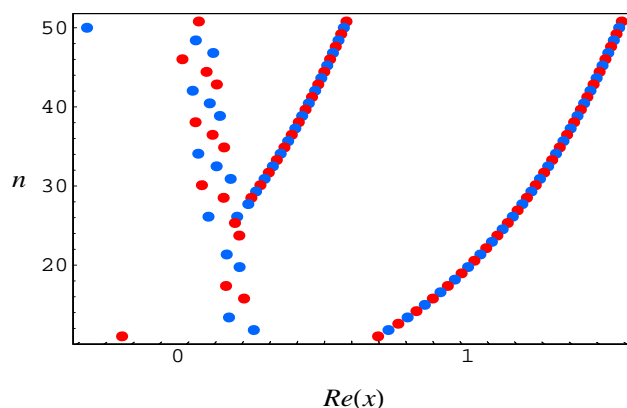


Figure 6: Plot of real zeros of  $G_{n,1/5}^{(2)}(x)$ ,  $11 \leq n \leq 50$

#### 4. Directions for Further Research

In general, how many roots does  $G_{n,q}^{(h)}(x)$  have? Prove or disprove:  $G_{n,q}^{(h)}(x)$  has  $n - 1$  distinct solutions. Find the numbers of complex zeros  $C_{G_{n,q}^{(h)}(x)}$  of  $G_{n,q}^{(h)}(x)$ ,  $Im(x) \neq 0$ . Prove or give a counterexample: *Conjecture:* Since  $n - 1$  is the degree of the polynomial  $G_{n,q}^{(h)}(x)$ , the number of real zeros  $R_{G_{n,q}^{(h)}(x)}$  lying on the real plane  $Im(x) = 0$  is then  $R_{G_{n,q}^{(h)}(x)} = n - 1 - C_{G_{n,q}^{(h)}(x)}$ , where  $C_{G_{n,q}^{(h)}(x)}$  denotes complex zeros. See Table 1 for tabulated values of  $R_{G_{n,q}^{(h)}(x)}$  and  $C_{G_{n,q}^{(h)}(x)}$ . Find the equation of envelope curves bounding the real zeros lying on the plane, and the equation of a trajectory curve running through the complex zeros on any one of the arcs. In Figures 1, 2, 3, 4, and 5,  $G_{n,q}^{(h)}(x)$ ,  $x \in \mathbb{C}$ , has  $Im(x) = 0$  reflection symmetry. This translates to the following open problem. Prove or disprove:  $G_{n,q}^{(h)}(x)$ ,  $x \in \mathbb{C}$ , has  $Im(x) = 0$  reflection symmetry. These figures give mathematicians an unbounded capacity to create visual mathematical investigations of the behavior of the roots of the  $G_{n,q}^{(h)}(x)$ . Moreover, it is possible to create a new mathematical ideas and analyze them in ways that generally are not possible by hand. The author has no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of  $G_{n,q}^{(h)}(x)$  to appear in mathematics and physics. For related topics the interested reader is referred to [3], [4], [5].

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**Lacunary strong almost  $A$  – convergence with respect to a sequence of Orlicz functions****\*Ayhan ESI & \*\*Ayşegül GOKHAN**

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**Abstract:** The purpose of this paper is to introduce and study some sequence spaces which are defined by combining the concepts of a sequence of Orlicz functions, infinite matrix and lacunary convergence. We also examine some topological properties of these sequence spaces and establish some elementary connections between them.

**Mathematics subject classification(2000):** Primary 46A45; 40H05; Secondary 40A05;40D25.

**Key words and phrases:** Lacunary sequence, Orlicz function, almost convergence.

**1.Introduction**

By a lacunary sequence  $\theta = (k_r)$ ,  $r = 0, 1, 2, \dots$ , where  $k_0 = 0$ , we shall mean an increasing sequence of non-negative integers  $h_r = k_r - k_{r-1} \rightarrow \infty$  as  $r \rightarrow \infty$ . The intervals determined by  $\theta$  are denoted by  $I_r = (k_{r-1}, k_r]$  and the ratio  $\frac{k_r}{k_{r-1}}$  will be denoted by  $q_r$ . The space of lacunary strongly convergent sequence  $N_\theta$  was defined by Freedman et al. [3] as follows:

$$N_\theta = \left\{ x = (x_k) : \lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{i \in I_r} |x_i - L| = 0, \text{ for some } L \right\}.$$

Recall [4,5] that an Orlicz function is a function  $M : [0, \infty) \rightarrow [0, \infty)$  which is continuous, non-decreasing and convex with  $M(0) = 0$ ,  $M(x) > 0$  for  $x > 0$  and  $M(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . If convexity of  $M$  is replaced by subadditivity, then this function is called a modulus function (see, Ruckle [11]).

An Orlicz function  $M$  is said to satisfy  $\Delta_2$  –condition for all values of  $u$ , if there exists a constant  $T > 0$ , such that  $M(2u) \leq TM(u)$  ( $u \geq 0$ ). The  $\Delta_2$  –condition is equivalent to  $M(Lu) \leq TLM(u)$ , for all values of  $u$  and for  $L > 1$ .

An Orlicz function  $M$  can be always be represented (see [5]) in the integral form  $M(x) = \int_0^x q(t)dt$ , where  $q$  known as the kernel of  $M$ , is right differentiable for  $t \geq 0$ ,  $q(t) > 0$  for  $t > 0$ ,  $q$  is non-decreasing and  $q(t) \rightarrow \infty$  as  $t \rightarrow \infty$ .

**Remark.** An Orlicz function satisfies the inequality  $M(\lambda u) \leq \lambda M(u)$  for all  $\lambda$  with  $0 < \lambda < 1$ .

Lindenstrauss and Tzafriri [6] used the idea of Orlicz functions to construct Orlicz sequence space,

$$l_M = \left\{ x = (x_i) : \sum_i M\left(\frac{|x_i|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The sequence space  $l_M$  with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_i M\left(\frac{|x_i|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz Sequence Space. The space  $l_M$  is closely related to the space  $l_p$ , which is an Orlicz sequence space with  $M(x) = x^p$  for  $1 \leq p < \infty$ .

Recently, Esi [1] have introduced and examined some properties of three sequence spaces defined by using an Orlicz function, which generalized the well-known Orlicz sequence space  $l_M$  and strongly summable sequence spaces  $[C, 1, p]$ ,  $[C, 1, p]_o$  and  $[C, 1, p]_\infty$ . It may be noted here that the space of strongly summable sequences were discussed by Maddox [7].

A generalization of Orlicz sequence space is due to Woo [13]. Let  $M = (M_i)$  be a sequence of Orlicz functions. Define the sequence space  $l(M_i)$  by

$$l(M_i) = \left\{ x = (x_i) : \sum_i M_i\left(\frac{|x_i|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

and equip this space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_i M_i\left(\frac{|x_i|}{\rho}\right) \leq 1 \right\}.$$

The space  $l(M_i)$  is a Banach space and is called a modular sequence space. The space  $l(M_i)$  also generalizes the concept of modular sequence space introduced earlier by Nakano [9], who considered the space  $l(M_i)$  when  $M_i(x) = x^{\alpha_i}$ , where  $1 \leq \alpha_i < \infty$  for  $i \geq 1$ .

Recently, Esi and Et [2] have introduced and examined some properties of three sequence spaces defined by using a sequence of Orlicz functions, which generalized the well-known Orlicz sequence space  $l_M$  and strongly invariant  $A$ -summable sequence spaces

$[A_\sigma, p]$ ,  $[A_\sigma, p]_o$  and  $[A_\sigma, p]_\infty$ . It may be noted here that the space of strongly invariant  $A$  – summable sequences were discussed by Savaş [12].

A sequence space  $E$  is said to be solid (or normal) if  $(\lambda_i x_i) \in E$ , whenever  $(x_i) \in E$  for all sequences  $(\lambda_i)$  of scalars such that  $|\lambda_i| \leq 1$  for all  $i \in N$ .

A sequence space  $E$  is said to be monotone if  $E$  contains the canonical preimages of all its step spaces.

**Lemma.** A sequence space  $E$  is solid implies  $E$  is monotone.

Let  $A = (a_{ik})_{i,k \geq 1}$  be an infinite matrix of complex numbers. We write  $Ax = (A_{im}(x))$  if  $A_{im}(x) = \sum_{k \geq 1} a_{ik} x_{k+m}$  converges for each  $i, m$ . (This assumption is made throughout the paper).

Let  $M = (M_i)$  be a sequence of Orlicz functions and  $p = (p_i)$  be any sequence of strictly positive real numbers. Now we define the following sequence spaces.

$$w^\theta(M, A, p) = \left\{ x = (x_k) : \lim_r h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x-L)|}{\rho} \right) \right]^{p_i} = 0, \text{ uniformly in } m, \text{ for some } \rho > 0, L \right\},$$

$$w_o^\theta(M, A, p) = \left\{ x = (x_k) : \lim_r h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x)|}{\rho} \right) \right]^{p_i} = 0, \text{ uniformly in } m, \text{ for some } \rho > 0 \right\}$$

and

$$w_\infty^\theta(M, A, p) = \left\{ x = (x_k) : \sup_{r,m} h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x)|}{\rho} \right) \right]^{p_i} < \infty, \text{ for some } \rho > 0 \right\}.$$

If  $x \in w^\theta(M, A, p)$ , we said that the sequence  $x$  is lacunary strongly almost  $A$  – convergent to the number  $L$  with respect to the sequence of Orlicz functions  $M = (M_i)$ .

Some sequence spaces are obtained by specializing  $\theta = (k_r)$ ,  $M = (M_i)$ ,  $A = (a_{ik})_{i,k \geq 1}$   $m$  and  $p = (p_i)$ . Some examples are below:

If we take  $p_i = 1$  for all  $i \in N$ , then we write the spaces  $w^\theta(M, A)$ ,  $w_o^\theta(M, A)$  and  $w_\infty^\theta(M, A)$  in place of the spaces  $w^\theta(M, A, p)$ ,  $w_o^\theta(M, A, p)$  and  $w_\infty^\theta(M, A, p)$ .

When  $A = I$  unit matrix, the spaces  $w^\theta(M, A, p)$ ,  $w_o^\theta(M, A, p)$  and  $w_\infty^\theta(M, A, p)$  reduce the spaces  $w^\theta(M, p)$ ,  $w_o^\theta(M, p)$  and  $w_\infty^\theta(M, p)$  respectively, which are defined as

$$w^\theta(M, p) = \left\{ x = (x_k) : \lim_r h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|x_{i+m} - L|}{\rho} \right) \right]^{p_i} = 0, \text{ uniformly in } m, \text{ for some } \rho > 0, L \right\},$$

$$w_o^\theta(M, p) = \left\{ x = (x_k) : \lim_r h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|x_{i+m}|}{\rho} \right) \right]^{p_i} = 0, \text{ uniformly in } m, \text{ for some } \rho > 0 \right\}$$

and

$$w_\infty^\theta(M, p) = \left\{ x = (x_k) : \sup_{r,m} h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|x_{i+m}|}{\rho} \right) \right]^{p_i} < \infty, \text{ for some } \rho > 0 \right\}.$$

If  $x \in w^\theta(M, p)$ , we said that the sequence  $x$  is lacunary strongly almost convergent to the number  $L$  with respect to the sequence of Orlicz functions  $M = (M_i)$ .

If  $\theta = (2^r)$ ,  $M_i = M$  for all  $i \in N$ ,  $A = I$  unit matrix and  $m = 0$ , the spaces the spaces  $w^\theta(M, A, p)$ ,  $w_o^\theta(M, A, p)$  and  $w_\infty^\theta(M, A, p)$  reduce to  $w(M, p)$ ,  $w_o(M, p)$  and  $w_\infty(M, p)$  which were defined and studied by Parashar and Choudhary [10].

If  $\theta = (2^r)$ ,  $M_i(x) = x$  for all  $i \in N$ ,  $A = I$  unit matrix and  $m = 0$ , the spaces the spaces  $w^\theta(M, A, p)$ ,  $w_o^\theta(M, A, p)$  and  $w_\infty^\theta(M, A, p)$  reduce to  $[C, 1, p]$ ,  $[C, 1, p]_o$  and  $[C, 1, p]_\infty$  which were defined and studied by Maddox [7].

## 2. Main Results

We have

**Theorem 2.1.** For any sequence of Orlicz functions  $M = (M_i)$  and a bounded sequence  $p = (p_i)$  strictly positive real numbers,  $w^\theta(M, A, p)$ ,  $w_o^\theta(M, A, p)$  and  $w_\infty^\theta(M, A, p)$  are linear spaces over the set of complex numbers  $\mathbb{C}$ .

**Proof.** We shall prove the result only for  $w_o^\theta(M, A, p)$ . The others can be treated similarly. Let  $x, y \in w_o^\theta(M, A, p)$  and  $\lambda, \mu \in \mathbb{C}$ . In order to prove the result we need to find some  $\rho_3$  such that

$$\lim_r h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(\lambda x + \mu y)|}{\rho_3} \right) \right]^{p_i} = 0, \text{ uniformly in } m.$$

Since  $x, y \in w_o^\theta(M, A, p)$ , there exist  $\rho_1 > 0$  and  $\rho_2 > 0$  such that

$$\lim_r h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x)|}{\rho_1} \right) \right]^{p_i} = 0, \text{ uniformly in } m$$

and

$$\lim_r h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(y)|}{\rho_2} \right) \right]^{p_i} = 0, \text{ uniformly in } m.$$

Define,  $\rho_3 = \max(2|\lambda|\rho_1, 2|\mu|\rho_2)$ . Since  $M_i$  is non-decreasing and convex for all  $i \in N$ ,

$$\begin{aligned} & h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(\lambda x + \mu y)|}{\rho_3} \right) \right]^{p_i} \\ & \leq h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(\lambda x)|}{\rho_3} + \frac{|A_{im}(\mu y)|}{\rho_3} \right) \right]^{p_i} \leq h_r^{-1} \sum_{i \in I_r} 2^{-p_i} \left[ M_i \left( \frac{|A_{im}(x)|}{\rho_1} \right) + M_i \left( \frac{|A_{im}(y)|}{\rho_2} \right) \right]^{p_i} \\ & \leq K h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x)|}{\rho_1} \right) \right]^{p_i} + K h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(y)|}{\rho_2} \right) \right]^{p_i} \rightarrow 0 \text{ as } r \rightarrow \infty, \text{ uniformly in } \end{aligned}$$

$m$ , where  $K = \max(1, 2^{H-1})$ ,  $H = \sup_i p_i < \infty$ . Therefore  $\lambda x + \mu y \in w_o^\theta(M, A, p)$ .

**Theorem 2.2.** For any sequence of Orlicz functions  $M = (M_i)$  and a bounded sequence  $p = (p_i)$  strictly positive real numbers,  $w_o^\theta(M, A, p)$  is a paranormed linear space, paranormed by

$$f(x) = \inf \left\{ \rho^{\frac{p_r}{H}} : \left( h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x)|}{\rho} \right) \right]^{p_i} \right)^{\frac{1}{H}} \leq 1, \begin{matrix} r = 1, 2, \dots \\ m = 1, 2, \dots \end{matrix} \right\}$$

where  $H = \max(1, \sup_i p_i)$ .

**Proof.** It is easy to see that  $f(x) = f(-x)$ . Since  $M_i(0) = 0$  for all  $i \in N$ , we get  $\inf \left\{ \rho^{\frac{p_r}{H}} \right\} = 0$  for  $x = 0$ . Conversely, suppose that  $f(x) = 0$ , then

$$\inf \left\{ \rho^{\frac{p_r}{H}} : \left( h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x)|}{\rho} \right) \right]^{p_i} \right)^{\frac{1}{H}} \leq 1, \begin{matrix} r = 1, 2, \dots \\ m = 1, 2, \dots \end{matrix} \right\} = 0.$$

This implies that for a given  $\varepsilon > 0$ , there exists some  $\rho_\varepsilon$  ( $0 < \rho_\varepsilon < \varepsilon$ ) such that

$$\left( h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x)|}{\rho_\varepsilon} \right) \right]^{p_i} \right)^{\frac{1}{H}} \leq 1.$$

Thus

$$\left( h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x)|}{\varepsilon} \right) \right]^{p_i} \right)^{\frac{1}{H}} \leq \left( h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x)|}{\rho_\varepsilon} \right) \right]^{p_i} \right)^{\frac{1}{H}} \leq 1 \text{ for each } r \text{ and } m.$$

Suppose that  $x_{k_j} \neq 0$  for some  $j$ . This implies that  $A_{im}(x_{k_j}) \neq 0$  for some  $j$ . Let  $\varepsilon \rightarrow 0$ .

Then,  $\frac{|A_{im}(x_{k_j})|}{\varepsilon} \rightarrow \infty$ . It follows that  $\left( h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x_{k_j})|}{\varepsilon} \right) \right]^{p_i} \right)^{\frac{1}{H}} \rightarrow \infty$ , which is

contradiction. Therefore  $x_{k_j} = 0$  for some  $j$ .

Next let  $\rho_1 > 0$  and  $\rho_2 > 0$  be such that

$$h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x)|}{\rho_1} \right) \right]^{p_i} \leq 1$$

and

$$h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(y)|}{\rho_2} \right) \right]^{p_i} \leq 1.$$

Let  $\rho = \rho_1 + \rho_2$ . Then we have

$$\begin{aligned} h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x+y)|}{\rho} \right) \right]^{p_i} &\leq \left( \frac{\rho_1}{\rho_1 + \rho_2} \right)^H h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x)|}{\rho_1} \right) \right]^{p_i} \\ &\quad + \left( \frac{\rho_2}{\rho_1 + \rho_2} \right)^H h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(y)|}{\rho_2} \right) \right]^{p_i} \leq 1. \end{aligned}$$

Since the  $\rho$ 's are non-negative, we have

$$f(x+y) = \inf \left\{ \rho^{\frac{p_r}{H}} : \left( h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x+y)|}{\rho} \right) \right]^{p_i} \right)^{\frac{1}{H}} \leq 1, \begin{matrix} r=1,2,\dots \\ m=1,2,\dots \end{matrix} \right\}$$

$$\begin{aligned} &\leq \inf \left\{ \rho_1^{\frac{p_r}{H}} : \left( h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x)|}{\rho_1} \right) \right]^{p_i} \right)^{\frac{1}{H}} \leq 1, \right. \\ &\quad \left. r = 1, 2, \dots, m = 1, 2, \dots \right\} \\ &+ \inf \left\{ \rho_2^{\frac{p_r}{H}} : \left( h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(y)|}{\rho_2} \right) \right]^{p_i} \right)^{\frac{1}{H}} \leq 1, \right. \\ &\quad \left. r = 1, 2, \dots, m = 1, 2, \dots \right\} \\ &= f(x) + f(y). \end{aligned}$$

Finally, we prove scalar multiplication is continuous. Let  $\lambda$  be any complex number. By definition,

$$f(\lambda x) = \inf \left\{ \rho^{\frac{p_r}{H}} : \left( h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(\lambda x)|}{\rho} \right) \right]^{p_i} \right)^{\frac{1}{H}} \leq 1, \right. \\ \left. r = 1, 2, \dots, m = 1, 2, \dots \right\}.$$

Then,

$$f(\lambda x) = \inf \left\{ (|\lambda|t)^{\frac{p_r}{H}} : \left( h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(\lambda x)|}{t} \right) \right]^{p_i} \right)^{\frac{1}{H}} \leq 1, \right. \\ \left. r = 1, 2, \dots, m = 1, 2, \dots \right\} \text{ where } t = \rho|\lambda|^{-1}.$$

Since  $|\lambda|^{p_r} \leq \max(1, |\lambda|^H)$  ( $H = \sup_r p_r$ ), we have

$$f(\lambda x) \leq \max(1, |\lambda|^H)^{\frac{1}{H}} \inf \left\{ t^{\frac{p_r}{H}} : \left( h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(\lambda x)|}{t} \right) \right]^{p_i} \right)^{\frac{1}{H}} \leq 1, \right. \\ \left. r = 1, 2, \dots, m = 1, 2, \dots \right\} \text{ which}$$

converges to zero as  $x$  converges to zero in  $w_o^\theta(M, A, p)$ . Now suppose that  $\lambda_n \rightarrow 0$  and  $x$  is fixed in  $w_o^\theta(M, A, p)$ . For arbitrary  $\varepsilon > 0$ , let  $R$  be a positive integer such that

$$h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(\lambda x)|}{\rho} \right) \right]^{p_i} < \left( \frac{\varepsilon}{2} \right)^H, \text{ for some } \rho > 0, r > R \text{ and all } m \in N.$$

This implies that

$$\left( h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(\lambda x)|}{\rho} \right) \right]^{p_i} \right)^{\frac{1}{H}} < \frac{\varepsilon}{2}, \text{ for some } \rho > 0, r > R \text{ and all } m \in N.$$

Let  $0 < |\lambda| < 1$ . For all  $i \in N$ , using the convexity of  $M_i$ , for  $r > R$  and all  $m \in N$ , we get

$$h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(\lambda x)|}{\rho} \right) \right]^{p_i} < h_r^{-1} \sum_{i \in I_r} \left[ |\lambda| M_i \left( \frac{|A_{im}(x)|}{\rho} \right) \right]^{p_i} < \left( \frac{\varepsilon}{2} \right)^H.$$

Since  $M_i$  is continuous everywhere in  $[0, \infty)$  for all  $i \in N$ , then for  $r \leq R$ ,

$g(s) = h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(sx)|}{\rho} \right) \right]^{p_i}$  is continuous at zero. So there is  $0 < \delta < 1$  such that  $|g(s)| < \left( \frac{\varepsilon}{2} \right)^H$  for  $0 < s < \delta$ . Let  $K$  be such that  $|\lambda_n| < \delta$  for  $n > K$ . Then for  $n > K$ ,  $r \leq R$  and all  $m \in N$ ,

$$\left( h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(\lambda_n x_k)|}{\rho} \right) \right]^{p_i} \right)^{\frac{1}{H}} < \frac{\varepsilon}{2}.$$

Thus

$$\left( h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(\lambda_n x_k)|}{\rho} \right) \right]^{p_i} \right)^{\frac{1}{H}} < \varepsilon, \text{ for } n > K \text{ and all } r \text{ and } m, \text{ so that } f(\lambda x) \rightarrow 0$$

as  $\lambda \rightarrow 0$ . This completes the proof.

**Theorem 2.3.** For any sequences of Orlicz functions  $M = (M_i)$  and  $N = (N_i)$  which satisfies the  $\Delta_2$ -condition for all  $i \in N$  and a bounded sequence  $p = (p_i)$  strictly positive real numbers, we have  $w^\theta(M, A, p) \subset w^\theta(MoN, A, p)$ .

**Proof.** Let  $x \in w^\theta(M, A, p)$ . Then there exists  $\rho > 0$  such that

$$h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x - L)|}{\rho} \right) \right]^{p_i} \rightarrow 0, \text{ as } r \rightarrow \infty, \text{ uniformly in } m, \text{ for some } L.$$

Let  $\varepsilon > 0$  and choose  $\delta$  with  $0 < \delta < 1$  such that  $M_i(u) < \varepsilon$  for  $0 \leq u \leq \delta$  and all  $i \in N$ . Let

$$B_{im} = M_i \left( \frac{|A_{im}(x - L)|}{\rho} \right)$$

and let

$$h_r^{-1} \sum_{i \in I_r} [N_i(B_{im})]^{p_i} = h_r^{-1} \sum_{\substack{i \in I_r \\ B_{im} \leq \delta}} [N_i(B_{im})]^{p_i} + h_r^{-1} \sum_{\substack{i \in I_r \\ B_{im} > \delta}} [N_i(B_{im})]^{p_i}.$$

By the Remark, we have



$$h_r^{-1} \sum_{\substack{i \in I_r \\ B_{im} \leq \delta}} [N_i(B_{im})]^{p_i} \leq (N_i(1))^H h_r^{-1} \sum_{\substack{i \in I_r \\ B_{im} \leq \delta}} (B_{im})^{p_i} \leq (N_i(2))^H h_r^{-1} \sum_{\substack{i \in I_r \\ B_{im} \leq \delta}} (B_{im})^{p_i} \quad \dots(1)$$

where  $H = \sup_i p_i < \infty$ . For  $B_{im} > \delta$

$$B_{im} < \frac{B_{im}}{\delta} \leq 1 + \frac{B_{im}}{\delta},$$

since  $N_i$  is non-decreasing and convex for all  $i \in N$ , so

$$N_i(B_{im}) < N_i\left(1 + \frac{B_{im}}{\delta}\right) < \frac{1}{2} N_i(2) + \frac{1}{2} N_i\left(2 \frac{B_{im}}{\delta}\right).$$

Since  $N_i$  satisfies  $\Delta_2$ -condition for all  $i \in N$ , so

$$N_i(B_{im}) < \frac{1}{2} T \frac{B_{im}}{\delta} N_i(2) + \frac{1}{2} T \frac{B_{im}}{\delta} N_i(2) = T \frac{B_{im}}{\delta} N_i(2).$$

Hence

$$h_r^{-1} \sum_{\substack{i \in I_r \\ B_{im} > \delta}} [N_i(B_{im})]^{p_i} \leq \max\left(1, \frac{T}{\delta} N_i(2)\right)^H h_r^{-1} \sum_{\substack{i \in I_r \\ B_{im} > \delta}} (B_{im})^{p_i} \quad \dots(2)$$

where  $H = \sup_i p_i < \infty$ . By (1) and (2), we have  $x \in w^\theta(MoN, A, p)$ . Thus  $w^\theta(M, A, p) \subset w^\theta(MoN, A, p)$ .

By using the method of the proof of Theorem 2.3. it is not hard to see that  $w_o^\theta(M, A, p) \subset w_o^\theta(MoN, A, p)$  and  $w_\infty^\theta(M, A, p) \subset w_\infty^\theta(MoN, A, p)$ .

Taking  $M_i(x) = x$  for all  $i \in N$  in Theorem 2.3. we have the following result.

**Corollary 2.4.** For any sequence of Orlicz functions  $M = (M_i)$  which satisfies the  $\Delta_2$ -condition for all  $i \in N$  and a bounded sequence  $p = (p_i)$  strictly positive real numbers, we have  $w^\theta(A, p) \subset w^\theta(M, A, p)$ ,  $w_o^\theta(A, p) \subset w_o^\theta(M, A, p)$  and  $w_\infty^\theta(A, p) \subset w_\infty^\theta(M, A, p)$ .

In the following theorem, we prove the inclusions  $w(M, A, p) \subset w^\theta(M, A, p)$  and  $w^\theta(M, A, p) \subset w(M, A, p)$  for any sequence of Orlicz functions  $M = (M_i)$  under certain restrictions on lacunary sequence  $\theta = (k_r)$ .

**Theorem 2.5.** Let  $M = (M_i)$  be any sequence of Orlicz functions.

(a) Let  $\theta = (k_r)$  be a lacunary sequence with  $\liminf_r q_r > 1$ . Then

$$w(M, A, p) \subset w^\theta(M, A, p),$$

where

$$w(M, A, p) = \left\{ x = (x_k) : \lim_n n^{-1} \sum_{i=1}^n \left[ M_i \left( \frac{|A_{im}(x-L)|}{\rho} \right) \right]^{p_i} = 0, \text{ uniformly in } m, \text{ for some } \rho > 0, L \right\}.$$

**(b)** Let  $\theta = (k_r)$  be a lacunary sequence with  $\limsup_r q_r < \infty$ . Then  $w^\theta(M, A, p) \subset w(M, A, p)$ .

**Proof. (a).** Suppose that  $\liminf_r q_r > 1$ . Then there exists  $\delta > 0$  such that

$q_r = \frac{k_r}{k_{r-1}} \geq 1 + \delta$  for all  $r \geq 1$ . Then, for  $x \in w(M, A, p)$ , we write

$$\begin{aligned} C_{r,m} &= h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x)|}{\rho} \right) \right]^{p_i} \\ &= h_r^{-1} \sum_{i=1}^{i_r} \left[ M_i \left( \frac{|A_{im}(x-L)|}{\rho} \right) \right]^{p_i} - h_r^{-1} \sum_{i=1}^{i_{r-1}} \left[ M_i \left( \frac{|A_{im}(x-L)|}{\rho} \right) \right]^{p_i} \\ &= k_r h_r^{-1} \left( k_r^{-1} \sum_{i=1}^{i_r} \left[ M_i \left( \frac{|A_{im}(x-L)|}{\rho} \right) \right]^{p_i} \right) - k_{r-1} h_r^{-1} \left( k_{r-1}^{-1} \sum_{i=1}^{i_{r-1}} \left[ M_i \left( \frac{|A_{im}(x-L)|}{\rho} \right) \right]^{p_i} \right). \end{aligned}$$

Since  $h_r = k_r - k_{r-1}$ , we have  $\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta}$  and  $\frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}$ .

The terms  $k_r^{-1} \sum_{i=1}^{i_r} \left[ M_i \left( \frac{|A_{im}(x-L)|}{\rho} \right) \right]^{p_i}$  and  $k_{r-1}^{-1} \sum_{i=1}^{i_{r-1}} \left[ M_i \left( \frac{|A_{im}(x-L)|}{\rho} \right) \right]^{p_i}$  both converge

to zero uniformly in  $m$  and it follows that  $C_{r,m} \rightarrow 0$  as  $r \rightarrow \infty$ , uniformly in  $m$ . That is,

$x \in w^\theta(M, A, p)$ . Thus  $w(M, A, p) \subset w^\theta(M, A, p)$ .

**(b).** Suppose that  $\limsup_r q_r < \infty$ . Then, there exists  $B > 0$  such that  $q_r < B$  for all  $r \geq 1$ . Let  $x \in w^\theta(M, A, p)$  and  $\varepsilon > 0$ . Then there exists  $R > 0$  such that for every  $j \geq R$  and all  $m \in N$ ,

$$C_{j,m} = h_j^{-1} \sum_{i \in I_j} \left[ M_i \left( \frac{|A_{im}(x)|}{\rho} \right) \right]^{p_i} < \varepsilon.$$

We can also find  $C > 0$  such that  $C_{j,m} < C$  for all  $j \in N$ . Now let  $n$  be any integer with  $k_{r-1} < n \leq k_r$ , where  $r > R$ . Then, for all  $m \in N$

$$\begin{aligned}
 n^{-1} \sum_{i=1}^n \left[ M_i \left( \frac{|A_{im}(x-L)|}{\rho} \right) \right]^{p_i} &\leq k_{r-1}^{-1} \sum_{i=1}^{k_r} \left[ M_i \left( \frac{|A_{im}(x-L)|}{\rho} \right) \right]^{p_i} \\
 &= k_{r-1}^{-1} \left( \sum_{i \in I_1} \left[ M_i \left( \frac{|A_{im}(x-L)|}{\rho} \right) \right]^{p_i} + \sum_{i \in I_2} \left[ M_i \left( \frac{|A_{im}(x-L)|}{\rho} \right) \right]^{p_i} + \dots + \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x-L)|}{\rho} \right) \right]^{p_i} \right) \\
 &= k_1 k_{r-1}^{-1} k_1^{-1} \sum_{i \in I_1} \left[ M_i \left( \frac{|A_{im}(x-L)|}{\rho} \right) \right]^{p_i} + (k_2 - k_1) k_{r-1}^{-1} (k_2 - k_1)^{-1} \sum_{i \in I_2} \left[ M_i \left( \frac{|A_{im}(x-L)|}{\rho} \right) \right]^{p_i} \\
 &\quad + \dots + (k_R - k_{R-1}) k_{r-1}^{-1} (k_R - k_{R-1})^{-1} \sum_{i \in I_R} \left[ M_i \left( \frac{|A_{im}(x-L)|}{\rho} \right) \right]^{p_i} \\
 &\quad + \dots + (k_r - k_{r-1}) k_{r-1}^{-1} (k_r - k_{r-1})^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x-L)|}{\rho} \right) \right]^{p_i} \\
 &= k_1 k_{r-1}^{-1} C_{1,m} + (k_2 - k_1) k_{r-1}^{-1} C_{2,m} + \dots + (k_R - k_{R-1}) k_{r-1}^{-1} C_{R,m} + \dots + (k_r - k_{r-1}) k_{r-1}^{-1} C_{r,m} \\
 &\leq k_R k_{r-1}^{-1} (\sup_{j \geq 1} C_{j,m}) + (k_r - k_R) k_{r-1}^{-1} (\sup_{j \geq R} C_{j,m}) \\
 &< C k_R k_{r-1}^{-1} + \varepsilon B.
 \end{aligned}$$

Since  $k_{r-1} \rightarrow \infty$  as  $n \rightarrow \infty$ , it follows that

$$n^{-1} \sum_{i=1}^n \left[ M_i \left( \frac{|A_{im}(x-L)|}{\rho} \right) \right]^{p_i} \rightarrow 0, \text{ uniformly in } m. \text{ Therefore } x \in w(M, A, p). \text{ This}$$

completes the proof.

**Theorem 2.6.** Let  $M = (M_i)$  be any sequence of Orlicz functions and let  $\theta = (k_r)$  be a lacunary sequence with  $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$ . Then  $w(M, A, p) = w^\theta(M, A, p)$ .

**Proof.** It follows from Theorem 2.5.(a) and (b).

**Theorem 2.7.** Let  $M = (M_i)$  be any sequence of Orlicz functions. The space  $w_\infty^\theta(M, A, p)$  is solid.

**Proof.** Let  $x \in w_\infty^\theta(M, A, p)$ . Then, for some  $\rho > 0$

$$\sup_{r,m} h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x)|}{\rho} \right) \right]^{p_i} < \infty. \quad \dots(3)$$

Let  $(\lambda_i)$  be a sequence of scalars with  $|\lambda_i| \leq 1$  for all  $i \in N$ . Then the result follows from (3), Remark and the following inequality

$$h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(\lambda x)|}{\rho} \right) \right]^{p_i} \leq h_r^{-1} \sum_{i \in I_r} \left[ |\lambda_i| M_i \left( \frac{|A_{im}(x)|}{\rho} \right) \right]^{p_i} \leq h_r^{-1} \sum_{i \in I_r} \left[ M_i \left( \frac{|A_{im}(x)|}{\rho} \right) \right]^{p_i}.$$

The following result follows from Theorem 2.7. and the Lemma .

**Corollary 2.8.** Let  $M = (M_i)$  be any sequence of Orlicz functions. The space  $w_\infty^\theta(M, A, p)$  is monotone.

**Theorem 2.9.** Let  $M = (M_i)$  be any sequence of Orlicz functions and let  $0 < p_i \leq q_i$  and  $\left(\frac{q_i}{p_i}\right)$  be bounded. Then  $w^\theta(M, A, q) \subset w^\theta(M, A, p)$ .

**Proof.** Let  $x \in w^\theta(M, A, q)$ . Write  $w_{i,m} = \left[ M_i \left( \frac{|A_{im}(x)|}{\rho} \right) \right]^{q_i}$  and  $\lambda_i = \frac{q_i}{p_i}$  so that  $0 < \lambda < \lambda_i \leq 1$  ( $\lambda$  constant) for each  $i \in N$ . We define the sequences  $(u_{i,m})$  and  $(v_{i,m})$  as follows:

Let  $u_{i,m} = w_{i,m}$  and  $v_{i,m} = 0$  if  $w_{i,m} \geq 1$  and let  $u_{i,m} = 0$  and  $v_{i,m} = w_{i,m}$  if  $w_{i,m} < 1$ . Then, it is clear that for all  $i \in N$  we have  $w_{i,m} = u_{i,m} + v_{i,m}$  and  $w_{i,m}^{\lambda_i} = u_{i,m}^{\lambda_i} + v_{i,m}^{\lambda_i}$ . Now it follows that  $u_{i,m}^{\lambda_i} \leq u_{i,m} \leq w_{i,m}$  and  $v_{i,m}^{\lambda_i} \leq v_{i,m}^{\lambda}$ . Therefore

$$h_r^{-1} \sum_{i \in I_r} w_{i,m}^{\lambda_i} = h_r^{-1} \sum_{i \in I_r} (u_{i,m}^{\lambda_i} + v_{i,m}^{\lambda_i}) \leq h_r^{-1} \sum_{i \in I_r} w_{i,m} + h_r^{-1} \sum_{i \in I_r} v_{i,m}^{\lambda}.$$

Since  $0 < \lambda < 1$ , for each  $r \geq 1$ , by using Hölder's Inequality, we have

$$\begin{aligned} h_r^{-1} \sum_{i \in I_r} v_{i,m}^{\lambda} &= \sum_{i \in I_r} (h_r^{-1} v_{i,m})^{\lambda} (h_r^{-1})^{1-\lambda} \leq \left( \sum_{i \in I_r} \left[ (h_r^{-1} v_{i,m})^{\lambda} \right]^{\frac{1}{\lambda}} \right)^{\lambda} \left( \sum_{i \in I_r} \left[ (h_r^{-1})^{1-\lambda} \right]^{\frac{1}{1-\lambda}} \right)^{1-\lambda} \\ &= \left( h_r^{-1} \sum_{i \in I_r} v_{i,m} \right)^{\lambda} \end{aligned}$$

and thus

$$h_r^{-1} \sum_{i \in I_r} w_{i,m}^{\lambda_i} \leq h_r^{-1} \sum_{i \in I_r} w_{i,m} + \left( h_r^{-1} \sum_{i \in I_r} v_{i,m} \right)^{\lambda}.$$

Therefore  $x \in w^\theta(M, A, p)$ .

We have the following result which follows from the above result.

**Corollary 2.10.** Let  $M = (M_i)$  be any sequence of Orlicz functions.

(a). Let  $0 < \inf_i p_i \leq p_i \leq 1$ , then  $w^\theta(M, A, p) \subset w^\theta(M, A)$ .

(b). Let  $1 \leq p_i \leq \sup_i p_i < \infty$ , then  $w^\theta(M, A) \subset w^\theta(M, A)$ ,

where

$$w^\theta(M, A) = \left\{ x = (x_k) : \lim_r h_r^{-1} \sum_{i \in I_r} M_i \left( \frac{|A_{im}(x - L)|}{\rho} \right) = 0, \text{ uniformly in } m, \text{ for some } \rho > 0, L \right\}.$$

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